BANKRUPTCY, LIMITED LIABILITY AND THE

MODIGLIANI-MILLER THEOREM*

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1. Introduction

The famous proposition by Modigliani and Miller (1958) that the value of a firm is independent of its debt-equity ratio has a firm place in the theory of finance. This paper gives necessary and sufficient conditions for the validity of this theorem when there is a positive probability that either the firm or an individual who borrows to invest in the firm goes bankrupt.

Usually the theorem is proved under the assumption that neither the firm nor the individual can default. Under this assumption all debt is riskless. If capital markets are perfect, individual borrowing and firm borrowing are perfect substitutes.

This approach is unsatisfactory. Whether the firm can go bankrupt depends on its debt-equity ratio. If the debt-equity ratio is large enough, there will always be a positive probability of bankruptcy. The M-M Theorem is designed to analyze the effects of changes in the debt-equity ratio. The analysis is incomplete if it neglects the effects of changes in the debt-equity ratio on the probability of bankruptcy.

The assumption that no individual can default is even less appealing. If an agent is very optimistic about a firm's prospects, he may desire to borrow so heavily in order to invest in the firm that he faces a positive probability of bankruptcy, even though the firm itself does not default. The traditional statement of the M-M Theorem does not cover this situation.

The strength of the M-M Theorem is its independence of agent's beliefs and preferences. The assumption that no individual can go bankrupt rules out all those beliefs and preferences which make people wish to run a risk of bankruptcy. It reintroduces a dependence on beliefs and preferences which is the more objectionable because the precise nature of the restrictions involved is not known.

If one allows for the possibility of individual or firm bankruptcy, the traditional proof of the M-M Theorem is incomplete. In order to show that the value of a levered firm cannot exceed the value of an unlevered firm with the same returns, it is no longer sufficient to argue that the returns on the levered firm's equity can be replicated by a margin investment in the unlevered firm. In addition, one must establish the possibility of arbitrage for margin investments in the levered firm.

This class of securities has been overlooked in past discussions of the M-M Theorem, which considered only margin investments in the unlevered firm. Implicitly, it has always been taken for granted that the margin investment in a firm with low leverage allows the investor to replicate the returns on the equity of a firm with high leverage. But previous studies have failed to consider this class of securities explicitly.

This oversight does not affect the validity of the M-M Theorem when neither the individual nor the firm can go bankrupt. But the failure to take account of margin investments in a levered firm does invalidate attempts by Stiglitz (1969) and Merton (1974) to extend the M-M Theorem to the case of bankruptcy. With bankruptcy, the usual arbitrage operations fail for almost all margin investments in levered firms.

In this paper, I shall assume that when an agent borrows on margin, his liability is limited to the amount of collateral that he puts up. Under this

arrangement, the structure of returns on both the margin investment and the margin loan depends on the proportions in which the collateral contains a firm's bonds and equity. For instance, if the firm goes bankrupt, a collateral that contains only equity is worthless, whereas a collateral that includes some bonds may still earn some return, because the bonds have a privileged claim on the firm's remaining assets.

The main result of this paper shows that with individual or firm bankruptcy the M-M Theorem is valid, if and only if all portfolios that are used as collateral for individual borrowing contain a firm's bonds and equity in the same proportions in which the firm has issued them. If an individual can borrow on margin to invest in the firm's equity only, the M-M Theorem is no longer valid.

This result vindicates the intuition of Lintner (1962), Smith (1970) and Stiglitz (1972), who suggested that the M-M Theorem cannot be extended to the case of bankruptcy. Because the earlier authors did not prove their contention rigorously, they failed to see that there exists one condition under which individual and firm borrowing are perfect substitutes so that the M-M Theorem holds even with bankruptcy. But it should be realized that this condition is too restrictive to be of much practical importance.

The paper refutes the proposition by Stiglitz (1972, 1975), that "the value of the firm decreases as it issues more bonds because there is a divergence in the estimation of the chances of bankruptcy between the lender and the borrower." A close reading reveals that his result is based upon a form of market segmentation that is without economic merit and cannot be

maintained under perfect capital markets.[1] Differences of opinion among agents

are neither sufficient nor necessary to upset the M-M Theorem when there is

bankruptcy.

Stiglitz' contention is the more surprising because in an earlier paper

he had claimed to have shown -- regardless of agents' beliefs -- that "if a

firm has a positive probability of going bankrupt, and an individual can borrow

using those securities as collateral ... the value of the firm is invariant

to the debt-equity ratio" (Stiglitz 1969).

In the following section, the basic model is introduced. Section 3 analyzes

the breakdown of the M-M Theorem under bankruptcy. Section 4 proves the M-M

Theorem under the additional condition that all loan collaterals contain a

firm's debt and equity in the proportions in which it issues them.

2. The Returns on Margin Lending and Borrowing

Following Stiglitz (1969), I use a two-period model of the capital

market. In the first period, firms' shares and bonds and individuals'

bonds are traded, and the equilibrium interest rates and share prices are

determined. In the second period, the firm earns a gross return X, which

I assume to be distributed with strictly positive density on $[X_{min},\infty)$, where

$X_{min} \geq 0$. This gross return is distributed to bondholders and stockholders

according to the following rules:

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[1]In his model, there are two groups of agents, optimists and pessimists.
The optimists invest all their wealth in the firm's equity, <u>valued to make
the rate of return equal to that on the riskless asset</u>. The pessimists are
indifferent between the risky firm's bonds and the riskless asset. It follows
that the optimists, being more optimistic than the pessimists, must prefer the
risky bond to the riskless asset, and hence to the equity. Yet, Stiglitz
assumes that the optimists invest in the equity and not in the risky bond. Of
course, if one group invests <u>all</u> its wealth into one asset, that asset need not
satisfy a marginal equality condition at all. A correct analysis of his model,
along the lines of Section 4 of this paper shows that the M-M Theorem need
not be upset.

The returns are used first to pay bondholders. In the first period, the firm has issued B bonds with a nominal value of $1 each and a contractual gross return r. The contractual repayment to bondholders in the second period is rB. If the gross return X falls short of this obligation to bondholders, the firm goes bankrupt and its gross return is divided evenly among the bondholders. The return on a dollar invested in the firm's bonds is:

(1) $\quad S = \min(r, \frac{X}{B})$.

If the firm goes bankrupt, its stockholders earn nothing. If the firm does not go bankrupt, its net returns X - rB are distributed evenly among the stockholders. If the value of the firm's equity is E, the returns per dollar invested in the firm's equity are:

(2) $\quad d = \max(\frac{X-rB}{E}, 0)$.

The value of the firm, V, is defined as:

(3) $\quad V = E + B$.

Individuals can borrow on a limited liability basis, using the firms' securities as collateral. I shall assume that only one firm's securities serve as collateral for any one loan contract. The individual's liability is limited to the smaller of his repayment obligation or the return on his collateral.

The assumption that only one firm's securities serve as collateral for any one loan does not prevent the individual from borrowing money to invest in different firms; it merely implies that in order to invest in different firms, the individual takes out different loans, so that the collateral for one loan does not protect another loan.

A loan for investment in a firm is characterized by two parameters a and k . For every dollar borrowed, the individual puts up k dollars of his own money. A loan of one dollar contributes to a total investment of 1+k dollars that serves as collateral for the loan. The parameter a indicates the share of the firm's bonds in the collateral. The collateral for a one-dollar loan consists of a(1+k) bonds and (1-a)(1+k) worth of stock. The return on this collateral is:

$$(1+k) [(1-a)d + as] .$$

Suppose that the contractual repayment next period on a one-dollar loan is $\bar{r}(a,k)$. The borrower pays this amount when it is less than the return on the collateral. If the return on the collateral is less than the contractual repayment, the borrower simply forfeits the collateral. The return to the lender of investing in an (a,k) -loan is:

$$(4) \quad \bar{s}(a,k) = \min\{\bar{r}(a,k), (1+k)[(1-a)d + as]\} .$$

When he forfeits the collateral, the borrower receives nothing. Otherwise, he receives the excess of the return on the collateral over the contractual repayment to the lender. Since his own contribution to the levered investment in the firm consists of k dollars for every dollar that he borrows, the return on a margin investment that is partly financed by an (a,k) -loan is:

$$(5) \quad \bar{d}(a,k) = \frac{1}{k} \max\{(1+k)[(1-a)d + as] - \bar{r}(a,k),0\} .$$

Individuals evaluate portfolios in terms of the income patterns over the different states of nature that they provide. In particular, loans subject to the risk of bankruptcy are evaluated in terms of their pattern of returns without regard to the identity of the borrower. A firm and an individual can borrow at the same conditions if they issue the same security. This assumption

implies the following:

Lemma 1: (a) In equilibrium, any (a,k)-loan market with active borrow-
ing and lending satisfies: $\bar{r}(a,k) \lesseqgtr r$ as $(1+k)a \gtreqless 1$.
(b) For (a,k)-loan markets with active borrowing and lending, the equilibrium
interest $\bar{r}(a,k)$ decreases with a and k .

Proof: (a) If $(1+k)a \leq 1$ and $\bar{r}(a,k) \leq r$, the firm's bond dominates the
(a,k)-bond in all states of nature, so there is no supply of (a,k)-loans. If
$(1+k)a \geq 1$ and $\bar{r}(a,k) \geq r$, a portfolio holding the firm's equity and
bonds in proportions $(1+k)(1-a)$ and $(1+k)a-1$ dominates the (a,k)-levered
investment in all states of nature, so that there is no demand for (a,k)-loans.
(b) If $\bar{r}(a,k)$ increases in a or k, the bond with the lower a or k is
strictly dominated in all states of nature.   Q.E.D.

When $(1+k)a \leq 1$, the proceeds of the loan exceed the value of the
firm's bonds in the collateral. When the firm goes bankrupt, the lender sal-
vages less than if he had directly invested in the firm's bonds. A higher
contractual return must compensate him for this loss of security. As a
consequence, individual default is strictly more likely than the firm's default.
If on the other hand, $(1+k) \geq 1$, the lender obtains more security at the
cost of a contractual return that is lower than what he could earn in the
firm's bonds.

Securities with $(1+k)a = 1$ satisfy $\bar{s}(a,k) = s$ and $\bar{d}(a,k) = d$,
independent of X . They do not enlarge the set of available patterns of
returns. Whether such securities exist is of no consequence to the economy.
Therefore, I shall neglect them for the rest of this paper.

There are then two basic types of margin loans markets: Markets for
(a,k)-loans with $(1+k)a \leq 1$, and markets for (a,k)-loans with $(1+k)a \geq 1$.
Tables 1 and 2 list the returns on securities in these two types of loan

Table 1

Returns on Securities of the First Type: $(1+k)a \leq 1$

X	$[0,rB]$	$\left[rB, rB+E\ \dfrac{\bar{r}(a,k)-(1+k)ar}{(1+k)(1-a)}\right]$	$\left[rB+E\ \dfrac{\bar{r}(a,k)-(1+k)ar}{(1+k)(1-a)}, \infty\right)$
$\bar{s}(a,k)$	$(1+k)a\dfrac{X}{B}$	$(1+k)\left[(1-a)\dfrac{X-rB}{E} + ar\right]$	$\bar{r}(a,k)$
$\bar{d}(a,k)$	0	0	$\dfrac{1}{k}\left\{(1+k)\left[(1-a)\dfrac{X-rB}{E} + ar\right] \dot{-} \bar{r}(a,k)\right\}$
$\dfrac{d\bar{s}(a,k)}{dX}$	$(1+k)\dfrac{a}{B}$	$(1+k)\dfrac{1-a}{E}$	0
$\dfrac{d\bar{d}(a,k)}{dX}$	0	0	$\dfrac{1+k}{k}\ \dfrac{1-a}{E}$

Table 2

Returns on Securities of the Second Type: $(1+k)a \geq 1$

X	$[0, \frac{\bar{r}(a,k)}{(1+k)a}B]$	$[\frac{\bar{r}(a,k)}{(1+k)a}B, rB]$	$[rB, \infty]$
$\bar{s}(a,k)$	$(1+k)a\frac{X}{B}$	$\bar{r}(a,k)$	$\bar{r}(a,k)$
$\bar{d}(a,k)$	0	$\frac{1}{k}[(1+k)a\frac{X}{B} - \bar{r}(a,k)]$	$\frac{1}{k}\{(1+k)[(1-a)\frac{X-rB}{E} + ar] - \bar{r}(a,k)\}$
$\frac{d\bar{s}(a,k)}{dX}$	$(1+k)\frac{a}{B}$	0	0
$\frac{d\bar{d}(a,k)}{dX}$	0	$\frac{1+k}{k}\frac{a}{B}$	$\frac{1+k}{k}\frac{1-a}{E}$

markets for different intervals of values of the gross return X. Also listed
are the derivatives of returns with respect to · X. These derivatives exhibit
discontinuities at the points of individual and firm default.

The kink in returns at the point of the firm's default, $X = rB$, is the
most striking instance of the fact that the firm's interest burden, rB,
affects the returns on margin lending and borrowing. In the following section
it will be shown that the interest burden rB increases with the firm's
indebtedness. The effects of this change cannot in general be counteracted
by individuals' arbitrage operations.

3. The Breakdown of Modigliani-Miller Arbitrage under Bankruptcy

This section shows how the usual proof of the M-M Theorem breaks down in
the presence of bankruptcy. Following Stiglitz (1969), I take firm behaviour
as given exogenously. Each firm specifies and maintains a given debt-equity
ratio $z = B/E$.

An economy is a specification of agents, their preferences, beliefs, and
endowments, and a list of firms $i = 1,2,...,N$, with debt-equity ratios z_1,
$z_2,...,z_N$ and gross returns $X_i \varepsilon [X_{i_{min}} ,\infty)$, $i = 1,2,...,N$.

The market conditions of an economy are the values of the firms' equities,
$E_1, E_2,...,E_N$, firms' interest rates $r_1, r_2,...,r_N$ are interest schedules
$r_i(a,k)$, for the (a,k)-loan market connected with the ith firm, $i = 1,2,...,N$.

The M-M Theorem is usually proved as a corollary to the following:

Proposition (*): Consider two economies U and W that are identical
except for firms' debt-equity ratios. For any equilibrium of economy U, there
exists an equilibrium of economy W, such that the patterns of returns over
states of nature that are available at the equilibrium market conditions
and are not dominated by another portfolio are identical in both economies.

Proposition (*) asserts the irrelevance of the firm's financial policy. This conclusion breaks down when limited liability borrowing with a risk of individual bankruptcy is introduced. The following lemma shows that the firm's interest burden depends on its financial policy.

Lemma 2: For any two economies U and W satisfying Proposition (*), the equilibrium conditions satisfy:

$$r_i^U B_i^U \gtrless r_i^W B_i^W \quad \text{as} \quad B_i^U \lessgtr B_i^W, \quad i = 1, 2, \ldots, N.$$

Proof: Suppose that for any firm $r^U B^U \geq r^W B^W$ and $B^U < B^W$. It follows that $r^U > r^W$. By inspection of (1), then, $s^U > s^W$ for all values of X. By Proposition (*), there exists a portfolio in W that replicates the return s^U on the firm's bond in economy U. By inspection of (1) and Tables 1 and 2, this portfolio can only contain margin loans whose returns have a single kink at $X = r^U B^U$. But then, these margin loans have $(1+k)a = B^W/B^U > 1$ and $\bar{r}^W(a,k) = r^U > r^W$, by inspection of (4) and (1). From Lemma 1.a, it follows that economy W is not in equilibrium, a contradiction to Proposition (*). Q.E.D.

The firm's interest burden depends on its financial policy. The changes in the interest burden affect the returns on securities connected with the firm. Some of these effects can be neutralized through agents' portfolio revisions. However, this is not true for changes in the returns on limited liability lending where the collateral contains no bonds.

Theorem 3: Any two economies U and W that differ only in the ith firm's debt-equity ratio and allow limited liability borrowing and lending on pure equity collateral with a risk of individual bankruptcy do not satisfy Proposition (*).

Proof: Without loss of generality, assume $B^U > B^W$ (dropping subscripts), and suppose that Proposition (*) holds. Replicate $s^U(0,k*)$ by a portfolio

of margin loans and firm loans in economy W:[2]

$$s^U(0,k^*) = \sum_{j \epsilon J} c_j s^W(a_j, k_j) + c_o s^W \; ; \; c_o, c_j \geq 0 \; ; \; c_o + \sum_{j \epsilon J} c_j = 1 \; .$$

For $X \epsilon (r^U B^U, \; r^U B^U + E^U \bar{r}^U(0,k^*)/(1+k^*))$, this implies

$$(6) \quad (1+k^*)\frac{X-r^U B^U}{E^U} = \sum_{j \epsilon J_o} c_j(1+k_j) \; [(1-a_j)\frac{X-r^W B^W}{E^W} + a_j r^W]$$

$$+ \sum_{j \epsilon J_1} c_j \bar{r}^W(a_j, k_j) + c_o r^W \; .$$

Differentiating (6) with respect to X, one has:

$$(1+k^*)\frac{1}{E^U} = \sum_{j \epsilon J_o} c_j(1+k_j)\frac{1-a_j}{E^W} \; .$$

Substituting for $\sum_{j \epsilon J_o} c_j(1+k_j)(1-a_j)/E^W$ in (6) and rearranging terms, one has:

$$(1+k^*)\frac{r^W B^W - r^U B^U}{E^U} = \sum_{j \epsilon J_1} c_j \bar{r}^W(a_j, k_j) + [c_o + \sum_{j \epsilon J_o} c_j(1+k_j)a_j]r^W \; .$$

All the terms on the right hand side are nonnegative, so that $r^W B^W \geq r^U B^U$. By Lemma 2, it follows that $B^W \leq B^U$, in contradiction to the original assumption.

Q.E.D.

An increase in the firm's debt-equity ratio increases the sensitivity of $s(0,k^*)$ to changes in X and decreases the level of $s(0,k^*)$ in the area of individual bankruptcy. The individual agent cannot neutralize both these effects at the same time.

The breakdown of Proposition (*) is not confined to limited liability loans on a pure equity collateral. In fact, the only loan contracts for which Proposition (*) is valid in the presence of bankruptcy are those whose collateral contains the firm's debt and equity in the same proportion in which the firm issues them.

The proof of the following theorem is relegated to the appendix because it is rather tedious:

<u>Theorem 4</u>: For almost all economies U which satisfy $X_{i_{min}} = 0$ and have an active market in the $(a*,k*)$-loan connected with the ith firm, where $a* = z_i^U/(1+z_i^U)$ and $(1+k*)a* = 1$, some I, there does not exist an economy W with $z_i^W = z_i^U$, such that economies U and W satisfy Proposition $(*)$.

The existence of an economy W in which the returns $s^U(a*,k*)$ and $d^U(a*,k*)$ can be replicated when $a* \neq z^U/(1+z^U)$ imposes restrictions on the equilibrium interest rates of economy U which are seldom satisfied.

As a rule, one cannot expect the M-M Theorem to hold unless all individual borrowing and lending is restricted to margin loans whose collateral contains the firm's debt and equity in the same proportions in which the firm issues them.

This is a very restrictive condition. For instance, consider two agents who expect the firm to do so well that in their opinion the firm's equity dominates the bond. If they disagree about precisely how well the firm will do, there is still room for a bet in the form of an (a,k)-loan from the less optimistic to the more optimistic agent. Since both agents are significantly more optimistic than the market, they agree that the loan collateral should not contain any of the firm's bonds, so that $a = 0$.

In general, a collateral with $a < z/(1+z)$, $[a > z/(1+z)]$ is desirable for a borrower and a lender who are both more optimistic (pessimistic) about the firm than the market as a whole; they agree that the collateral should contain more equity (bonds) than the market portfolio. The loan contract itself allows them to bet on their remaining differences of opinion.

To some extent this result vindicates Stiglitz' intuition that differences of belief between agents are important in the breakdown of the M-M Theorem.

But contrary to his assertion, the M-M Theorem does not break down because the pessimists do not want to buy more of the firm's bonds and force its value down as the debt-equity ratio rises. Indeed, it is easy to show that the levered firm must always be _more_ valuable than the unlevered firm, because the set of available patterns of return is strictly larger when the firm is levered.[3]

As the firm begins to borrow, its value increases initially. Eventually, as the debt-equity ratio grows out of bounds, the equity becomes irrelevant and the debt becomes like unlevered equity, so that the value of the 100% levered firm is equal to that of the unlevered firm.

4. The Characterization of the Modigliani-Miller Economy

The condition that loan collaterals hold firms' securities in the proportions in which firms issue them is sufficient as well as necessary for the validity of the M-M Theorem.

I shall call an _M-M economy_ any economy in which all margin purchases of the ith firm are restricted to (a_i^*, k)-contracts such that $a_i^* = z_i/(1+z_i)$, $i = 1, 2, \ldots, N$.

The returns on such _M-M loans_ with $a^* = z/(1+z)$ are given as:

$$(7) \quad s^*(k) = \min[r^*(k), (1+k) X/V],$$

where $s^*(k) \equiv \bar{s}(z/(1+z), k)$, $r^*(k) \equiv \bar{r}(z/(1+z), k)$. Similarly, one has

$$(8) \quad d^*(k) = \frac{1}{k}\max\left[(1+k)\frac{X}{V} - r^*(k), 0\right],$$

for the return on the margin investment that is financed by an M-M loan. It is

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[3] For the unlevered firm, $a = 0$ on all margin loans. The $(0,k)$-margin contract on the unlevered firm is replicated by the $(z/(1+z),k)$-contract on the levered firm.

important that s*(k), d*(k) do not have kinks at X = rB , the point of

the firm's bankruptcy.

The following Lemma reveals an important property of M-M economies:

Lemma 5: At the equilibrium market conditions of any M-M economy, there

exists an equilibrium allocation in which all securities issued by firms are

held in loan collaterals and no unlevered investment in any firm takes place.

Proof: Suppose first that r*(1/z) ≠ r . For r*(1/z) >(<)r , there is

no unlevered investment in the firm's bonds (equity). In the equilibrium of

the M-M economy, unlevered investments in bonds and equity are made in the

ratio z . Hence, for r*(1/z) ≠ r , no unlevered investment takes place at all.

Assume next that r*(1/z) = r ; this implies s*(1/z) = s and d*(1/z) = d ,

by inspection of (1), (2), (7), (8). Agents are indifferent between the

firm's bond and the (z/(1+z), 1/z)-loan and between equity and the (z/(1+z),1/z)-

margin investment. If there is an equilibrium with unlevered holdings of b

bonds and e equity, consider the allocation in which these bondholders and

equityholders make use of the (z/(1+z),1/z)-loan market instead. This

creates an additional supply of b (z/(1+z),1/z)-loans and an additional

supply of e dollars of own funds for margin investments, which generate a

demand for e z margin loans. In the equilibrium of the M-M economy, b = e z ,

so that the equilibrium of the (z/(1+z),1/z)-loan market is not upset. The

value of the additional collateral is b+e , generating investments in b

bonds and e holdings of equity to replace the original unlevered investments

in the firm. Hence, the new allocation is an equilibrium allocation.   Q.E.D.

Lemma 5 allows a very simple proof of

Theorem 6: Consider two M-M economies U and W that are identical except

for firms' debt-equity ratios. For any equilibrium of economy U , there

exists an equilibrium of economy W , such that the pattern of returns over

states of nature that are available at the equilibrium market conditions are identical in both economies. The values of all firms are the same in the two equilibria.

Proof: Let E_i^U, r_i^U, $r*_i^U(k)$, $i = 1,2,\ldots,N$, be the equilibrium market conditions for economy U. Without loss of generality, assume $r_i^U = r*_i^U(1/z_i^U)$ for all i.[4]

Consider the following market conditions for economy W:

$$(9) \begin{cases} E_i^W = \dfrac{1+z_i^U}{1+z_i^W} E_i^U \\[2mm] r_i^W = r*_i^U(1/z_i^W) \\[2mm] r*_i^W(k) = r*_i^U(k) \text{ for all } k \end{cases} \qquad i = 1,2,\ldots,N .$$

By inspection of (1), (2), (7), (8), the market conditions (9) imply, for all i

$$V_i^W = V_i^U$$

$$s_i^W = s*_i^U(1/z_i^W) \ ; \ d_i^W = d*_i^U(1/z_i^W)$$

$$s_i^U = s*_i^W(1/z_i^U) \ ; \ d_i^U = d*_i^W(1/z_i^U)$$

$$s*_i^W(k) = s*_i^U(k) \ ; \ d*_i^W(k) = d*_i^U(k) \text{ for all } k .$$

At the market conditions (9), the available patterns of return and the values of all firms in economy W are the same as in the equilibrium of economy U. Every agent then chooses the same pattern of returns over states of nature as in economy U.

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[4]If $r*_i^U(1/z_i^U) \neq r_i^U$, neither the $(z_i^U/(1+z_i^U), 1/z_i^U)$-loan market nor the market for direct investments in the ith firm is active, by Lemma 1 and the argument in the proof of Lemma 5. Then, economy U remains in equilibrium if the rate on $(z_i^U/(1+z_i^U),1/z_i^U)$-loans is set equal to the rate charged to the ith firm .

Consider the equilibrium allocation in economy U in which there is no unlevered investment in firms. Then, for all k, agents in economy W demand the same amounts of margin loans and investments with the margin rate $k/(1+k)$ as in economy U. Hence, all the margin loan markets are in equilibrium in economy W. Furthermore, the value of all collaterals is V^U, the same as in economy U. Since $V^W = V^U$ and all collaterals hold firms' bonds and equities in the proportions in which the firms issue them, the markets for the firms' securities are also in equilibrium. This completes the proof of Theorem 6.
                                                                                Q.E.D.

Lemma 5 provides an important insight into the functioning of the M-M economy. In the M-M economy, all matters of portfolio choice can be left to the markets for borrowing and lending on margin. Direct investment in a firm is irrelevant to the equilibrium of the economy because it can be replicated through the $(z/(1+z), 1/z)$-loan market. In effect, firms' securities are only demanded for collateral purposes.

Debt-equity ratios are then irrelevant because collateral compositions adjust to any changes in them. The markets for a firm's debt and equity affect the general equilibrium only through the overall value of the firm, but not through its debt-equity ratio.

In certain situations, one may want to consider additional restrictions on margin borrowing and lending, which specify which of the possible M-M loan markets are allowed to open. In this case, Theorem 6 continues to hold, if and only if the following additional conditions are satisfied for all i:

a: The $(z_i^W/(1+z_i^W), 1/z_i^U)$-loan market is open in economy W.

b: The $(z_i^U/(1+z_i^U), 1/z_i^W)$-loan market is open in economy U.

c: For all k, the $(z_i^W/(1+z_i^W), k)$-loan market is open and active in economy W, if and only if the $(z_i^U/(1+z_i^U), k)$-loan market is open and active in economy U.

The proof of this assertion is left to the reader. Conditions (a), (b) and (c) ensure essentially that the restrictions imposed on margin borrowing and lending do not destroy the arbitrage operations used to prove Theorem 6.

5. Concluding Remarks

Traditionally, the M-M Theorem seemed to rest firmly on the assumption of perfect capital markets which ensured that individuals and firms could borrow at the same rate if they offered debt of the same quality. In the presence of bankruptcy, however, the M-M Theorem rests on capital market imperfections which prohibit borrowing and lending on margin with collaterals containing debt and equity in proportions other than those in which the firm issues them. In general, the M-M Theorem is invalid under bankruptcy because the debt-equity ratio of a firm affects the structure of returns on margin contracts.

The result does not affect the validity of the M-M Theorem with Arrow-Debreu Securities or in a mean-variance framework (Stiglitz 1969). With Arrow-Debreu Securities the set of available patterns of returns is a priori independent of firms' debt-equity ratios.

On the other hand, the separation theorem in mean-variance analysis presumes that individuals can borrow and lend without limits in the riskless asset. This assumes away the problem of individual bankruptcy. Individuals have finite net worth at the time of their portfolio choice, but they are able to accomodate arbitrarily large negative returns after the dice have been rolled.

In the model analyzed in this paper, individuals can choose among a multiplicity of loan contracts, so that there is a much wider scope for individual than for firm borrowing. This is patently unrealistic. As a next step, it is therefore necessary to study an intertemporal model of individual and firm

borrowing. In such a framework, the future value of a security is determined not only by the firm's returns, but also by the future market price of the security. To the extent that uncertainty about future market prices merely reflects the uncertainty about later returns to the firm, the analysis of the one-period model can be generalized to the intertemporal context.[5] It seems likely, however, that future market prices are subject to additional uncertainty concerning general market conditions, especially the demand for securities. This additional uncertainty will then create a bias in favour of firm rather than individual borrowing.[6]

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[5]In the absence of bankruptcy, this is shown by Stiglitz (1974).

[6]This point is due to Lintner (1962).

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Appendix: Proof of Theorem 4

Consider the case $(1+k^*)a^* < 1$ and suppose there exists an economy W, such that U and W satisfy Proposition (*) and the return $\bar{s}^U(a^*,k^*)$ can be replicated in economy W. Clearly, the portfolio that replicates $\bar{s}^U(a^*,k^*)$ cannot contain equity or margin investments.

The return $\bar{s}^U(a^*,k^*)$ has two kinks, one at $X = r^U B^U$, the other at

$$X = r^U B^U + E^U \frac{\bar{r}^U(a^*,k^*) - (1+k^*)a^* \, r^U}{(1+k^*)(1-a^*)} \quad .$$

The returns on a margin loan of the first type in economy W have two kinks, at $X = r^W B^W$ and at

$$X = r^W B^W + E^W \frac{\bar{r}^W(a,k) - (1+k)a \, r^W}{(1+k)(1-a)} \quad .$$

From Lemma 2, it follows that at least one of the kinks of a margin loan of the first type does not coincide with one of the kinks of $\bar{s}^U(a^*,k^*)$.

Hence, more than one security is needed to replicate $\bar{s}^U(a^*,k^*)$ in economy W. The portfolio that replicates $\bar{s}^U(a^*,k^*)$ contains two subsets of securities whose returns have single kinks at $X = r^U B^U$ and at

$$X = r^U B^U + E^U \frac{\bar{r}^U(a^*,k^*) - (1+k^*)a^* \, r^U}{(1+k^*)(1-a^*)} \quad , \text{ respectively.}$$

The securities in either one of these two subsets must all have the same returns. If not, one dominates the others for all values of X, by inspection of Tables 1 and 2, where one makes use of the fact that all have their kink at the same point. By Proposition (*), the returns on this security can be replicated in economy U. But then, there exists a portfolio in economy U that dominates lending in the (a^*,k^*)-loan market, contrary to the assumption that this market is active.

Therefore, there is no loss of generality in assuming that the portfolio

that replicates $\bar{s}^U(a^*,k^*)$ contains just two margin loans with single kinks at

$$X = r^U B^U \quad \text{and at} \quad X = r^U B^U + E^U \frac{\bar{r}^U(a^*,k^*) - (1+k^*)a^* \ r^U}{(1+k^*)(1-a^*)} \quad .$$

Formally, one has:

(A.1) $\bar{s}^U(a^*,k^*) = c \ \bar{s}^W(a_1,k_1) + (1-c) \ \bar{s}^W(a_2,k_2) \ ; \quad c \ \varepsilon(0,1) \ .$

For large X , this implies:

(A.2) $\bar{r}^U(a^*,k^*) = c \ \bar{r}^W(a_1,k_1) + (1-c) \ \bar{r}^W(a_2,k_2) \ .$

$\bar{s}^W(a_1,k_1)$ has a single kink at $X = r^U B^U$, so that:

(A.3) $r^U B^U = \dfrac{\bar{r}^W(a_1,k_1) B^W}{(1+k_1)a_1} \ .$

(A.1) implies that $\dfrac{d \ \bar{s}^U(a^*,k^*)}{dX} = c \ \dfrac{d \ \bar{s}^W(a_1,k_1)}{dX} + (1-c) \ \dfrac{d \ \bar{s}^W(a_2,k_2)}{dX}$

for all X for which $d \ \bar{s}^U/dX$ exists. Therefore,

(A.4) $\dfrac{(1+k^*)a^*}{B^U} = c \ \dfrac{(1+k_1)a_1}{B^W} + (1-c) \ \dfrac{(1+k_2)a_2}{B^W}$

(A.5) $\dfrac{(1+k^*)(1-a^*)}{E^U} = (1-c) \ \dfrac{(1+k_2)a_2}{B^W} \ .$

Equations (A.2)-(A.5) determine the parameters c, a_1, k_1, a_2, k_2 of the portfolio that replicates $\bar{s}^U(a^*,k^*)$. But the last four parameters occur only in the form $(1+k_1)a_1$ and $(1+k_2)a_2$. Thus, the system of <u>four</u> equations (A.2)-(A.5) is overdetermined in the <u>three</u> parameters c, $(1+k_1)a_1$, $(1+k_2)a_2$, unless the market conditions ensure that only three of the equations are independent.

Consider therefore the restrictions placed on the market conditions by (A.2)-(A.5). First, note that Proposition (*) implies that $\bar{r}^W(a_1,k_1) = r^U$; otherwise

the replication of $\bar{s}^W(a_1, k_1)$ in economy U violates Lemma 1. From (A.3) then:

(A.6) $\quad (1+k_1)a_1 = B^W/B^U$.

Next, use (A.4)-(A.6) to solve for c:

(A.7) $\quad c = (1+k^*) \, B^U \left[\dfrac{a^*}{B^U} - \dfrac{1-a^*}{E^U} \right]$.

When $a^* < z^U/(1+z^U)$, the term in brackets is negative, implying a contradiction to (A.1). In this case, $\bar{s}^U(a^*, k^*)$ can never be replicated in economy W .

Suppose therefore that $a^* > z^U/(1+z^U)$ and rewrite (A.3) and (A.5):

(A.8) $\quad (1+k_2)a_2 = \dfrac{(1+k^*)(1-a^*)\,B^W}{(1-c)\,E^U}$

(A.9) $\quad \bar{r}^W(a_2, k_2) = [\bar{r}^U(a^*, k^*) - c\,r^U]/(1-c)$.

Finally, note that the return $\bar{s}^W(a_2, k_2)$ must be replicated in economy U by an (a_2', k_2')-loan which satisfies:

(A.10) $(1+k_2')\,a_2' = (1+k_2)a_2$

(A.11) $\bar{r}^U(a_2', k_2') = \bar{r}^W(a_2, k_2)$.

It has now been shown that the validity of Proposition (*) for economy U and another economy W implies the existence in economy U of an (a_2', k_2')-loan contract as specified by (A.7), (A.8) and (A.10), bearing the contractual interest $\bar{r}^U(a_2', k_2')$ given by (A.7), (A.9) and (A.11). However, the set of economies U which have an equilibrium at the given rate $\bar{r}^U(a_2', k_2')$ on the given (a_2', k_2')-loan is of measure zero. This completes the proof for the case $(1+k^*)a^* < 1$.

The argument for the case $(1+k^*)a^* > 1$ is left to the reader. It concentrates on the replication of $\bar{d}^U(a^*,k^*)$ in economy W, but is otherwise the same as the one given for $(1+k^*)a^* < 1$.