

TESTING HYPOTHESES IN DISEQUILIBRIUM MODELS

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1. Introduction

It is only recently that econometric demand and supply models have explicitly recognized the possibility that the observed price-quantity pairs need not represent equilibrium values.¹ In such models it is characteristically assumed that the quantity observed represents the short side of the market, i.e., that

$$D_t = D(x_t^D, p_t) + u_{1t} \quad (1-1)$$

$$S_t = S(x_t^S, p_t) + u_{2t} \quad (1-2)$$

and

$$Q_t = \min(D_t, S_t) \quad (1-3)$$

where Q_t is the observed quantity, p_t price, x_t^D , x_t^S vectors of exogenous variables and u_{1t} , u_{2t} error terms. If (1-1) to (1-3) are assumed to represent the full specification of the model p_t has to be assumed to be exogenous; otherwise one may adjoin an additional relation such as

$$p_t - p_{t-1} = \gamma(D_t - S_t) + u_{3t} \quad (1-4)$$

in which case price becomes an endogenous variable.

Previous studies have concentrated on the question of how properly to estimate the parameters of such systems and specifically on deriving the

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¹Models of this type have been analyzed in [1], [7], [8], [9], [12], [14]. Empirical results appear to have been obtained only by Fair and Jaffee [7] and by Fair and Kelejian [8] in a model of the housing market and by Goldfeld and Quandt [9] in a model of the market for watermelons.

appropriate likelihood functions. A question considered only peripherally up to now is that of testing the hypothesis that the market is in equilibrium against the alternative that it is in disequilibrium. If the market is in equilibrium, it is represented by the system of equations

$$Q_t = D(x_t^D, p_t) + u_{1t} \quad (1-5)$$

$$Q_t = S(x_t^S, p_t) + u_{2t} \quad (1-6)$$

in which Q_t and p_t are the endogenous variables; thus if equilibrium holds (1-5) and (1-6) replace (1-1) to (1-3) or (1-1) to (1-4), as the case may be.

The various studies concerned with estimating the parameters of markets in disequilibrium generally distinguish two cases: (1) the case in which it is known for which observations $D_t < S_t$ and for which the reverse is true, and (2) the case in which such information is not available. In the former instance there is obviously no meaningful equilibrium hypothesis that requires testing; either $D_t = S_t$ for all values of t in which case the market is in equilibrium at all times or $D_t \neq S_t$ for some t in which case the market is a disequilibrium market. Thus the only interesting case from the point of view of hypothesis testing is the one in which sample points cannot be classified a priori according to whether $D_t < S_t$ or $D_t \geq S_t$; this is the case that will be considered in the rest of the paper. In Section 2 we consider some specific models; in Section 3 we describe various possible test procedures and in Section 4 we describe some sampling experiments with the test statistics and empirical results based on the Fair-Jaffee model of housing starts.

2. Some Specific Models

The demand and supply functions will throughout be taken to be

$$D_t = a_0 + a_1 p_t + a_2 z_{1t} + a_3 z_{2t} + u_{1t} \quad (2-1)$$

$$S_t = a_4 + a_5 p_t + a_6 z_{3t} + a_7 z_{4t} + u_{2t} \quad (2-2)$$

where z_{1t}, \dots, z_{4t} are exogenous.² For convenience we shall abbreviate these as

$$D_t = a_1 p_t + b_{1t} + u_{1t} \quad (2-3)$$

$$S_t = a_5 p_t + b_{2t} + u_{2t} \quad (2-4)$$

which define b_{1t} and b_{2t} . There are at least two ways in which different assumptions may be made about the price equation. We may write it in general as

$$p_t = p_{t-1} + a_8 (D_\tau - S_\tau) + u_{3t} \quad (2-5)$$

where (1) τ is either t or $t-1$, and (2) where u_{3t} is either a non-degenerate random variable or $u_{3t} \equiv 0$ for all t . In any event we shall assume that nondegenerate errors are jointly normally distributed; hence either $(u_{1t}, u_{2t}) \sim N(0, \Sigma_1)$ or $(u_{1t}, u_{2t}, u_{3t}) \sim N(0, \Sigma_2)$ where

$$\Sigma_1 = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix} \quad \Sigma_2 = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_2^2 & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_3^2 \end{bmatrix}$$

The probability density function of the observable random variables and hence the likelihood function are different for each of the four possible cases that arise by choosing one of the two possible assumptions for τ and one of the two possible ones for u_{3t} . This strongly suggests that few uniform

²We simply wish to ensure that both functions, taken in the context of an equilibrium model, are overidentified. The precise number of exogenous variables is irrelevant.

procedures will exist that can be applied to test the hypothesis of equilibrium, irrespective of what other subsidiary assumptions are made.³ For the sake of brevity we shall not derive the pdf and the likelihood function for all four cases. We shall concentrate on the case in which $\tau = t$ and u_{3t} is not identically 0; alluding to other cases only as appears necessary to establish particular points. On this assumption (2-5) now becomes

$$p_t = p_{t-1} + a_8(D_t - S_t) + u_{3t} \quad (2-6)$$

and the disequilibrium model consists of (2-3), (2-4) and (2-6) with $Q_t = \min(D_t, S_t)$.⁴ The corresponding equilibrium model consists of (2-3) and (2-4) with $Q_t = D_t = S_t$ and (2-6) is "missing" altogether.

In the disequilibrium model in question Q_t and p_t are the only observable random variables. Letting $g(D_t, S_t, p_t)$ represent the joint pdf of D_t , S_t , and p_t it is shown in [9] that the joint pdf of Q_t and p_t ,

³In some models it may not even be possible to write down the "equilibrium version" of the model in a unique fashion. Consider the Suits model of the watermelon market analyzed in [9], consisting of

$$q_t = b_1 z_{1t} + b_2 + u_{1t}, \quad x_t = b_3 p_t + b_4 q_t + b_5 z_{2t} + b_6 + u_{2t}, \quad p_t = b_7 z_{3t} + b_8 y_t + b_9 + u_{3t}$$

and $y_t = \min(q_t, x_t)$, where q_t is the (unobservable) crop, x_t is intended (ex ante) harvest, y_t is actual harvest, p_t price and the z_t exogenous variables. The model is a disequilibrium model because the observed (p_t, y_t) pair fails to satisfy either the first equation (if $q_t > x_t$) or the second (if $q_t < x_t$). A corresponding model that is always in equilibrium can then be specified as either (1) having $q_t \equiv y_t$, with the second equation missing altogether, or as (2) having $x_t \equiv y_t$, with q_t in the second equation replaced from the first.

⁴It is debatable whether the model with $u_{3t} \neq 0$ is or is not intrinsically more interesting than the model in which $u_{3t} \equiv 0$. It might be argued that in the latter case, assuming that $a_8 > 0$, we can deduce whether $D_t > S_t$ or not by observing the sign of the price change. However, from the hypothesis tester's point of view the very existence of (2-6) (with or without an error term) must be questionable and so the alternative hypothesis including an Equ. (2-6) and without an error term must be considered potentially viable. For this reason we shall treat the case of $u_{3t} \equiv 0$, peripherally in the text and in some more detail in Appendix 2. Argument in favor of not neglecting this case becomes even weightier if the price equation is specified to include exogenous variables on the right hand side (in addition to $D_t - S_t$).

$h(Q_t, p_t)$, can be written as⁵

$$h(Q_t, p_t) = \int_{Q_t}^{\infty} g(Q_t, S_t, p_t) dS_t + \int_{Q_t}^{\infty} g(D_t, Q_t, p_t) dD_t \quad (2-7)$$

Equation (2-7) can be obtained more explicitly by integrating

$$g(D_t, S_t, p_t) = \frac{|1+a_8(a_6-a_1)|}{(2\pi)^{3/2} |\Sigma_2|^{1/2}} \exp \left\{ -\frac{1}{2} (u_{1t} \ u_{2t} \ u_{3t}) \Sigma_2^{-1} \begin{pmatrix} u_{1t} \\ u_{2t} \\ u_{3t} \end{pmatrix} \right\} \quad (2-8)$$

where we substitute for u_{1t} , u_{2t} and u_{3t} from (2-3), (2-4) and (2-6) and where $|1+a_8(a_5-a_1)|$ is the Jacobian of the transformation. Denoting by σ^{ij} the ij th element of Σ_2^{-1} , the exponent in (2-8) can be written as

$$\begin{aligned} & \sigma^{11} (D_t - a_1 p_t - b_{1t})^2 + \sigma^{22} (S_t - a_5 p_t - b_{2t})^2 + \sigma^{33} (p_t - a_8 D_t + a_8 S_t - b_{3t})^2 + \\ & + 2\sigma^{12} (D_t - a_1 p_t - b_{1t}) (S_t - a_5 p_t - b_{2t}) + 2\sigma^{13} (D_t - a_1 p_t - b_{1t}) (p_t - a_8 D_t + a_8 S_t - b_{3t}) + \\ & + 2\sigma^{23} (S_t - a_5 p_t - b_{2t}) (p_t - a_8 D_t + a_8 S_t - b_{3t}) \end{aligned} \quad (2-9)$$

where for the sake of uniform notation we write p_{t-1} as b_{3t} . Collecting terms alternately on powers of D_t and S_t and replacing S_t or alternately D_t by Q_t , (2-9) can be written as

$$A_1 D_t^2 + 2A_{2t} D_t + A_{3t} = \frac{(D_t - \mu_{1t})^2}{\omega_1} + B_{1t} \quad (2-10)$$

where

$$A_1 = \sigma^{11} + a_8^2 \sigma^{33} - 2a_8 \sigma^{13}$$

$$\begin{aligned} A_{2t} &= (a_1 p_t + b_{1t}) (\sigma^{13} a_8 - \sigma^{11}) + (Q_t - a_5 p_t - b_{2t}) (\sigma^{12} - a_8 \sigma^{23}) \\ &+ (a_8 Q_t + p_t - b_{3t}) (\sigma^{13} - a_8 \sigma^{33}) \end{aligned}$$

⁵This pdf will be conditional on p_{t-1} ; however, for simplicity, we neglect this complication and treat p_{t-1} as if it were exogenous.

$$\begin{aligned}
A_{3t} &= \sigma^{11}(a_1 p_t + b_{1t})^2 + \sigma^{22}(Q_t - a_5 p_t - b_{2t})^2 + \sigma^{33}(a_8 Q_t + p_t - b_{3t})^2 \\
&+ 2\sigma^{12}(Q_t - a_5 p_t - b_{2t})(-a_1 p_t - b_{1t}) + 2\sigma^{13}(-a_1 p_t - b_{1t})(a_8 Q_t + p_t - b_{3t}) \\
&+ 2\sigma^{23}(Q_t - a_5 p_t - b_{2t})(a_8 Q_t + p_t - b_{3t})
\end{aligned}$$

$$\mu_{1t} = -A_{2t}/A_1$$

$$\omega_1^2 = 1/A_1$$

$$B_{1t} = (A_{3t}A_1 - A_{2t}^2)/A_1 \quad (2-11)$$

or as

$$A_4 S_t^2 + 2A_{5t} S_t + A_{6t} = \frac{(S_t - \mu_{2t})^2}{\omega_2^2} + B_{2t} \quad (2-12)$$

where

$$A_4 = \sigma^{22} + a_8^2 \sigma^{33} + 2a_8 \sigma^{23}$$

$$\begin{aligned}
A_{5t} &= (a_5 p_t + b_{2t})(-\sigma^{22} - a_8 \sigma^{23}) + (p_t - b_{3t} - a_8 Q_t)(a_8 \sigma^{33} + \sigma^{23}) \\
&+ (Q_t - a_1 p_t - b_{1t})(\sigma^{12} + a_8 \sigma^{13})
\end{aligned}$$

$$\begin{aligned}
A_{6t} &= \sigma^{11}(Q_t - a_1 p_t - b_{1t})^2 + \sigma^{22}(a_5 p_t + b_{2t})^2 + \sigma^{33}(p_t - a_8 Q_t - b_{3t})^2 \\
&+ 2\sigma^{12}(Q_t - a_1 p_t - b_{1t})(-a_5 p_t - b_{2t}) + 2\sigma^{13}(Q_t - a_1 p_t - b_{1t})(p_t - a_8 Q_t - b_{3t}) \\
&+ 2\sigma^{23}(-a_5 p_t - b_{2t})(p_t - a_8 Q_t - b_{3t})
\end{aligned}$$

$$\mu_{2t} = A_{5t}/A_4$$

$$\omega_2^2 = 1/A_4$$

$$B_{2t} = (A_{6t}A_4 - A_{5t}^2)/A_4 \quad (2-13)$$

Substituting in (2-8) gives

$$h(Q_t, p_t) = \frac{|1 + a_8(a_5 - a_1)|}{2\pi |\Sigma_2|^{1/2}} \left[\frac{e^{-B_{1t}/2}}{A_1^{1/2}} (1 - \Phi(\ell_{1t})) + \frac{e^{-B_{2t}/2}}{A_4^{1/2}} (1 - \Phi(\ell_{2t})) \right] \quad (2-14)$$

where $\phi(x) = \int_{-\infty}^x \frac{1}{(2\pi)^{1/2}} \exp\{-\frac{y^2}{2}\} dy$ and where

$$x_{1t} = A_1^{1/2}(Q_t + A_{2t}/A_1), \quad x_{2t} = A_4^{1/2}(Q_t + A_{5t}/A_4) \quad (2-15)$$

The corresponding disequilibrium likelihood function is

$$L = \prod_t h(Q_t, p_t) \quad (2-16)$$

If equilibrium holds, the pdf is the standard simultaneous equation pdf

$$h_e(Q_t, p_t) = \frac{|a_5 - a_1|}{2\pi |\Sigma_1|^{1/2}} \exp\{-\frac{1}{2}(u_{1t} \quad u_{2t}) \Sigma_1^{-1} \begin{pmatrix} u_{1t} \\ u_{2t} \end{pmatrix}\} \quad (2-17)$$

where u_{1t} and u_{2t} are replaced by using (2-3) and (2-4).

The nature of the problems inherent in testing the null hypothesis of equilibrium will depend on how the disequilibrium mechanism is formulated.⁶ If the latter is specified so that (2-14) is the right density function, disequilibrium involves four more parameters ($\sigma_{12}, \sigma_{13}, \sigma_3^2, a_8$) than equilibrium. The question is whether the hypothesis of equilibrium is nested; i.e., if H^d is the set of distributions of the endogenous variables in the case of disequilibrium and H^e the corresponding set in equilibrium, is $H^e \subset H^d$? The question might be answered by examining the behavior of (2-14) for large values of a_8 . It can be argued that large values of a_8 imply rapid adjustment of price; hence, in the limit as $a_8 \rightarrow \infty$, the price adjusts instantaneously to demand-supply differences and thus equilibrium holds at all times. The argument obtains its justification from observing that, disregarding the time-varying character of b_{1t} , b_{2t} , u_{1t} , u_{2t} and combining (2-3), (2-4) and (2-6)

⁶One alternative not considered here is the case of a market which is in equilibrium some of the time and in disequilibrium at other times. Such a formulation is rather appealing since the disequilibrium model discussed above allows equilibrium values of the variables only with probability zero -- an implausibly rigid requirement on the basis of casual empirical observations. A possible statistical model which allows for both states (at various times) is a mixture model with (unknown) mixing parameter λ and pdf given by $\lambda h(Q_t, p_t) + (1-\lambda)h_e(Q_t, p_t)$. The properties of such models remain to be explored.

yields a difference equation in p_t with the solution

$$p_t = \left[\frac{1}{1+a_8(a_5-a_1)} \right]^t (p_0 - p_e) + p_e \quad (2-18)$$

where p_e is the equilibrium price. Thus if $a_8 \rightarrow \infty$, $p_t \rightarrow p_e$ for all t . It is such considerations that make Fair and Jaffee [7] suggest that we should test for equilibrium by examining if $1/a_8$ is significantly different from zero.

Setting $\sigma_{23} = \sigma_{13} = \sigma_3^2 = 0$ and $a_8 = \infty$ does not yield a proper density function but we can examine the limit of the sequence of density functions as $a_8 \rightarrow \infty$ (with $\sigma_{23} = \sigma_{13} = 0$). It is shown in Appendix 1 that this limit is (2-17); hence in the present case the hypothesis is nested, but only in this limiting sense. We note, moreover, that the limit is (2-17) without any particular assumption about σ_3^2 except that it is not identically zero; thus the test of equilibrium requires only a test of whether $1/a_8$ is zero.

Alternatively, we may wish to specify a version of the disequilibrium model in which $u_{3t} \equiv 0$. In that case it can be shown easily that (2-14) has precisely the same general form as in the previous example, except of course that the definitions of A_1, \dots, A_{6t} are different and Σ_2 is replaced by Σ_1 . Although in that case (2-14) has only one more parameter than (2-17) (a_8), the disequilibrium pdf is a univariate one, since p_t is a linear combination of D_t and S_t ; moreover, as is shown in Appendix 2, the hypothesis of equilibrium is not nested even in the limiting sense since the limit of the disequilibrium pdf as $a_8 \rightarrow \infty$ is not (2-17). In such cases special procedures may be necessary for hypothesis testing. Finally, the disequilibrium model may lack a price equation altogether and consist of only (1-1), (1-2), (1-3). The parameters in the equilibrium model are identical, one for one, with those in the disequilibrium model and the corresponding pdf's are obviously not subsets

of one another; moreover, it is easy to show that even the estimation problem is burdened with special difficulties, since without improving special a priori restrictions the likelihood function corresponding to this model is unbounded and fails to satisfy the usual regularity conditions.⁷

3. Some Alternative Tests of Equilibrium

In the present section we briefly discuss the advantages and disadvantages of several possible test procedures.

Tests based on a_8 . The intuitive argument, condensed in (2-18) and employed by Fair and Jaffee, suggests that $1/a_8$ should be examined for significant deviations from zero, where the significance of the maximum likelihood estimates may be judged by appealing to asymptotic normality. One reason for the possible inappropriateness of this procedure was pointed out in the previous section and consists of the fact that $a_8 = \infty$ need not be stochastically equivalent to the equilibrium model. Another, and more practical reason, is that even on this intuitive basis $1/a_8 = 0$ is the appropriate test only if (2-5) is specified as (2-6). One might write it as

$$p_t = p_{t-1} + a_8(D_{t-1} - S_{t-1}) \quad (3-1)$$

being an alternative discrete time approximation to $\dot{p} = a_8(D(p) - S(p))$. The solution of the difference equation in p_t becomes

$$p_t = [1 - a_8(a_5 - a_1)]^t (p_0 - p_s) + p_s \quad (3-2)$$

where p_s is the stationary solution, and the corresponding test is on whether $a_8(a_5 - a_1) = 1$. Thus at least two elements of misspecification may be present in tests based on a_8 . It should be noted, however, in any event that the

⁷For a proof in a similar case see [9] .

model based on (3-1) does not become a proper equilibrium model when

$a_8(a_6 - a_1) = 1$; the condition $|1 - a_8(a_5 - a_1)| < 1$ merely ensures convergence but not that $D_t = S_t$.

Tests Based on $\Pr\{D_t < S_t\}$. It is clear that if the disequilibrium model is estimated, we can also compute estimates of the probability that $D_t < S_t$.

Using (1-1) and (1-2) we can write this probability, if $u_{3t} \equiv 0$, as

$\Pr\{D(x_t^D, p_t) - S(x_t^S, p_t) < u_{2t} - u_{1t}\}$ irrespective of whether p_t is determined exogenously or from (2-6). If estimates \hat{D}_t , \hat{S}_t and $\hat{\sigma}^2$ are obtained, where the latter is the estimated variance of the distribution of $u_{2t} - u_{1t}$, the estimate of the required probability is

$$\hat{\Pr}\{D_t < S_t\} = 1 - \int_{-\infty}^{\hat{D}_t - \hat{S}_t} \frac{1}{\sqrt{2\pi} \hat{\sigma}} e^{-u^2/2\hat{\sigma}^2} du \quad (3-3)$$

In equilibrium $D_t = S_t$ and also $\hat{D}_t = \hat{S}_t$; hence if in fact the disequilibrium model were estimated from equilibrium data, we would expect the estimated $\Pr\{D_t < S_t\}$ to be approximately 0.5. This suggests a test of the hypothesis that the true probability is 0.5. The obvious difficulty with the proposed test is that the successive values of $\hat{\Pr}\{D_t < S_t\}$ are not independent. If they were, they would be treated as observations from an approximately normal distribution with mean 0.5 and variance $0.25/n$ and we could use the Kolmogorov-Smirnov test to compare the sampling distributions of $(\hat{\Pr}\{D_t - S_t\} - 0.5)/\sqrt{0.25/n}$ with $N(0,1)$.

Tests Based on Embedding Procedures. If $f_1(y)$ and $f_2(y)$ are two pdf's for the random variable y , representing nonnested hypotheses, Cox [4], [5] has suggested a test procedure based on estimating the compound density $f(y) = kf_1(y)^\lambda f_2(y)^{1-\lambda}$, where λ is a (new) parameter to be estimated and $k = 1 / \int f_1(y)^\lambda f_2(y)^{1-\lambda} dy$, by maximizing the corresponding likelihood function. Inferences about the two hypotheses are to be made on the basis of λ , with a

λ -value close to 1 confirming $f_1(y)$ and close to 0 confirming $f_2(y)$. A somewhat similar embedding procedure is explored in Quandt [13] where the compound pdf is taken to be $\lambda f_1(y) + (1-\lambda)f_2(y)$. Either of the two approaches could be employed in testing for equilibrium in nonnested cases such as when $u_{3t} \equiv 0$ in (2-6) and the pdf's corresponding to the disequilibrium and equilibrium models would be weighted with λ and $1-\lambda$ respectively. The maximum likelihood estimate of λ and the estimate of its asymptotic variance would then be used to decide between the two hypotheses.

In principle these procedures are acceptable and some have been examined in sampling experiments as well as in concrete models, although not in the disequilibrium context, and found to produce reasonable results. Embedding procedures have disadvantages, however, and will not be dealt with further in the present paper. Cox's embedding technique requires the numerical evaluation of an integral every time the likelihood function is evaluated in a numerical optimization algorithm. The procedure based on forming a convex combination of pdf's will, if either hypothesis is strongly confirmed, lead to maximum likelihood estimates of λ at the end points of the permissible (0,1) interval with consequent difficulties for the numerical evaluation of the asymptotic covariance matrix.

Tests Based on the Likelihood Ratio. It is intuitively appealing to consider the likelihood ratio $\lambda = L_e/L_d$ where L_e and L_d denote respectively the maximum of the equilibrium and disequilibrium likelihood functions. In the principal case of interest here, namely when the disequilibrium model consists of (2-1), (2-2) and (2-6) the hypothesis of equilibrium is a nested one only in a limiting sense and the likelihood ratio may not be bounded by unity as in the standard case. The usual asymptotic theory for testing $-2\log\lambda$ will be theoretically inappropriate but may be adequate as a practical matter. Some empirical

evidence (Epple [6]) suggests that even in nonnested cases λ will be smaller than one if the null hypothesis is in fact false in most instances and a model selection criterion based on whether λ is less than or greater than one provides excellent discrimination.

Tests Based on Posterior Odds. Cox has suggested in [5] that computing the posterior odds may be an effective way of discriminating between two competing nonnested hypotheses. Denote by $L(y|\theta, M)$ the likelihood of the sample y conditional on the parameter vector θ and the model M . Then the posterior probabilities of the two models in question are

$$p(M_i | y) = \frac{1}{p(y)} \int L(y|\theta_i, M_i) \pi_{1i}(\theta_i | M_i) \pi_2(M_i) d\theta_i \quad (3-4)$$

where π_1 and π_2 are the prior density functions and $p(y)$ is the (marginal) probability of the sample. The posterior odds of M_1 against M_2 are

$$\frac{p(M_1 | y)}{p(M_2 | y)} = \frac{\int L(y|\theta_1, M_1) \pi_{11}(\theta_1 | M_1) \pi_2(M_1) d\theta_1}{\int L(y|\theta_2, M_2) \pi_{12}(\theta_2 | M_2) \pi_2(M_2) d\theta_2} \quad (3-5)$$

It is reasonable to assume diffuse priors over the parameters; in that case, however (3-5) is invalid because the normalizing constants for θ_1 and θ_2 are not comparable if π_{11} and π_{12} are improper prior densities. A sensible alternative is to employ Barnard's OAAAA (obviously arbitrary and always admissible) procedure ([2], [5]) and to compute the ratio of the mean likelihoods which approximates

$$\frac{\int L(y|\theta_1, M_1) d\theta_1}{\int L(y|\theta_2, M_2) d\theta_2} \quad (3-6)$$

If the number of parameters in either model is sizeable (and in the present case this number may easily be as large as 15), the evaluation of (3-5) will involve a many-dimensional numerical quadrature.

4. Monte Carlo Experiments and the Fair-Jaffee Model

Introduction. Some limited computer experiments were performed to test the null hypothesis of equilibrium (with pdf given by (2-17)) against the disequilibrium model consisting of (2-3), (2-4), (2-6) and the min condition (1-3), (with pdf given by (2-7)). We shall compare the quality of the estimates from the correctly specified and the misspecified models and consider tests based on a_8 , tests based on $\hat{\Pr}\{D_t < S_t\}$, likelihood ratio tests and tests based on the ratio of mean likelihood as an approximation to the posterior odds. In the standard experiments the true values of the parameters were $a_0 = 64.0$, $a_1 = -8.0$, $a_2 = 1.0$, $a_3 = 0.75$, $a_4 = -10.00$, $a_5 = 10.0$, $a_6 = 0.6$, $a_7 = -1.5$, $a_8 = 0.1$ and the true covariance matrices were

$$\Sigma_1 = \begin{bmatrix} 0.1 & 0.05 & 0 \\ 0.05 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}$$

for the disequilibrium model and

$$\Sigma_2 = \begin{bmatrix} 0.1 & 0.05 \\ 0.05 & 0.1 \end{bmatrix}$$

for the equilibrium model. The exogenous variables were identical in repeated samples of an experiment and were generated in most cases from the uniform distribution over the ranges (10,20) for z_1 , (10,30) for z_2 , (20,40) for z_3 and (5,15) for z_4 . These ranges are denoted as the "standard" ones. Additional features of the experiments in terms of which the various experiments differed from one another are given in Table 1 and consist of variations in sample size, covariance matrices, exogenous variables and values of a_8 . Equation (2-6) requires an observation on p_t at time 0; we selected p_0 by setting the exogenous variables at the midpoints of their respective

ranges, assuming zero error terms and that p_0 was an equilibrium price. For each experiment we alternately assumed that the true model was the disequilibrium model and the equilibrium model; for each state of the truth we estimated both models and computed the relevant test statistics. In most experiments 50 replications were performed; in some, for reasons of economy, only 25 replications were used.

The computations based on data generated from the equilibrium model allow an examination of the test statistics when the null hypothesis is in fact true; in the alternate case the test statistics may be used to judge the power of the tests. Some experiments differ among each other only in the value of a_8 ; it should be obvious that in these cases computations based on the assumption that the equilibrium model represented the truth did not have to be repeated. If either estimation procedure failed for any of a large number of possible reasons, the replication was discarded altogether.⁸ Numerical optimization was performed by the Davidon-Fletcher-Powell algorithm and the quadratic hill-climbing algorithm [10]. The disequilibrium model was reparametrized so that we estimated $1/a_8$ rather than a_8 directly; asymptotic variances were estimated by the diagonal elements of the negative inverse of the Hessian of the loglikelihood function. Denoting the maximum likelihood estimate of $1/a_8$ by $1/\hat{a}_8$, the estimate of its asymptotic variance by s_{1/\hat{a}_8}^2 , the relevant test statistic will be denoted as $\tilde{a}_8 = (1/\hat{a}_8)/s_{1/\hat{a}_8}$. The likelihood ratio is formed by placing the maximum likelihood value from the equilibrium model in the numerator and will be denoted by λ .

The most important summary results are displayed in Tables 2, 3, 4 and 5. Tables 2, 3 and 4 contain certain summary statistics concerning the goodness of the estimates when the estimating model is correctly specified and misspecified

⁸For more detailed analysis of failures see the discussion of Table 5.

respectively, and other summary statistics concerning the proposed test procedures. Table 5 contains a summary of computational failures.

Effect of Misspecification on Parameter Estimates. Misspecification is present when the data are generated from the equilibrium model and the disequilibrium model is estimated and conversely. Summary measures of the effect of such misspecification are contained in rows 1 through 5 of Tables 2 and 3 as well as in Table 4. The two models have eleven parameters in common (a_0 through a_7 , σ_1^2 , σ_2^2 and σ_{12}). Rows 1 and 2 give the number of instances that the mean square error (MSE) and mean absolute deviation (MAD) respectively are smaller when estimated by the equilibrium model than by the disequilibrium model.⁹ Row 3 displays the midspread, a robust measure of dispersion. Rows 4 and 5 respectively contain the median and the mean of the eleven ratios for each case obtained by dividing for each parameter the MAD of the disequilibrium estimate by the MAD of the equilibrium estimate.

The results in these five rows of Tables 2 and 3 are completely unambiguous and as expected. When the truth is equilibrium, the equilibrium estimates have smaller MSE's in 9 or 10 out of 11 possible instances; when the truth is disequilibrium, the equilibrium estimates never have smaller MSE's and MAD's except in a few instances in Cases 3 and 4 in which the value of a_8 is sufficiently large that we may expect the disequilibrium model to mimic the behavior of the equilibrium model. The midspreads generally indicate smaller dispersion for the estimates involving no misspecification. The mean and median ratios in rows 4 and 5 are comparable and tell a consistent story. Considering the medians, when the truth is equilibrium, the MAD's for the disequilibrium estimates are on the average 6 to 19 percent larger than those

⁹If the parameter estimates could be assumed to be approximately normally distributed, the MSE would suffice. In the present case it seemed desirable to display the more robust measure given by MAD.

of the corresponding equilibrium estimates. On the other hand, when the truth is disequilibrium, the MAD's of the equilibrium estimates are on the average 10 to 40 times as large as those of the disequilibrium estimates, except again for Cases 3 and 4. In these Cases the mean and median ratios are larger, but as expected, are monotone in the value of a_8 . When a_8 is at its largest value of 3.0, the percentage inferiority of the equilibrium estimates is roughly the same as that of the disequilibrium estimates in the case in which the truth is equilibrium.

Additional summary statistics are displayed in Table 4. These statistics represent severe condensation of the information contained in the replications of the experiments and are employed only for the sake of brevity. First, a crude but useful statistic is the fraction of times that a parameter estimate is smaller than its true value; ceteris paribus the closer this fraction is to 0.5 the less the bias can be expected to be. In the first four columns we display, for each state of the truth and each estimating method and each experiment, the mean square deviation (MSD) of these fractions from 0.5 for the set of parameters a_0 through a_7 . Secondly, for consistent estimates the ratio of the mean square error to the mean of the estimated asymptotic variances will tend to be close to unity. In the second four columns we display the MSD's of these ratios from 1.0 for the set of parameters a_0 through a_7 . With two exceptions all mean square deviations for the correctly specified estimating technique are smaller than for the corresponding misspecified technique. Moreover, when the truth is disequilibrium, the inferiority of the equilibrium estimates is much worse than in the converse situation; exactly as the measures in Tables 2 and 3 suggest. We note that an increase in sample size reduces the MSD's for the correctly specified estimating technique and increases them for the misspecified one and that in Cases 3 and

4 (which tend to resemble equilibrium models because of the high value of a_8) the inferiority of the equilibrium estimates is generally less marked when disequilibrium is in fact the truth than in the other cases. In summary, (a) both estimating techniques are quite sensitive to misspecification; (b) when the truth is disequilibrium and for small values of a_8 , the assumption that the data may have been generated by an equilibrium model is a much more serious misspecification than the converse; (c) as a_8 assumes larger values the disequilibrium model begins to behave much more like the equilibrium model, even in relatively small samples of 60 observations; (d) estimates are generally good and improve with sample size from the correctly specified but not from the misspecified technique.

Tests Based on \tilde{a}_8 . If the data are generated from the equilibrium model and the disequilibrium model is estimated, it may be expected to mimic the former by assigning as estimates large values to a_8 and thus relatively low values to the \tilde{a}_8 statistic. The behavior of this quantity is displayed in rows 6, 7 and 8 of Tables 2 and 3. From Table 2 it is evident that the distribution of \tilde{a}_8 is not well approximated by the normal distribution; if it were, the figures in row 8 should be approximately 0.025 and are, on the average, an order of magnitude greater. Hence, using \tilde{a}_8 for testing the hypothesis of equilibrium will tend to lead to a very high probability of Type I error. On the other hand, the experimentally ascertained power of the \tilde{a}_8 test when the hypothesis is false is quite satisfactory; using 1.96 as the critical value leads to rejection of the false hypothesis in essentially all instances except (a) Case 2 when the true error variances are relatively large and (b) Case 4 in which the true value of a_8 is so large as to make that case resemble the equilibrium model. Even in these cases the power exceeds 0.88. We conclude that (a) the test based on \tilde{a}_8 is not fully satis-

factory as it involves too high a probability of Type I error, but (b) gives satisfactory inferences when the null hypothesis is false.

Tests Based on $-2\log\lambda$. Rows 9, 10 and 11 in Tables 2 and 3 contain measures of the behavior of $-2\log\lambda$ when the null hypothesis is true and false respectively. It is shown in Appendix 1 that the null hypothesis is a nested one in a limiting sense (as $a_g \rightarrow \infty$); it is not surprising, therefore, that when the null hypothesis is in fact true, the maximum of the equilibrium likelihood occasionally exceeds the maximum of the disequilibrium likelihood, even though the parameters of the former are a subset of those of the latter. This occurs in about a quarter of all cases on the average and thus there is simply no question of $-2\log\lambda$ having a χ^2 distribution under the null hypothesis. It is nevertheless interesting to observe that the right tail of the sample cumulative distribution fits $\chi^2(1)$ reasonably well. This is consistent with the figures in row 10 of Table 2 which give an average probability of Type I error of 0.075 (instead of the theoretically ideal 0.05) if the critical value of $\chi^2(1)$ (at the 0.05 level) is employed. From Table 3 we ascertain that using this critical level gives excellent power when the hypothesis is false. First, there is no instance in which $-2\log\lambda$ is negative; secondly, the powers are 100 percent in every case except Case 4 (which is "closest" in structure to an equilibrium model).

Tests Based on $\hat{\Pr}\{D_t < S_t\}$. Because of the manner in which the data were generated from the disequilibrium model, approximately one half of the data points in each replication of each case corresponded to excess demand. The mean $\hat{\Pr}\{D_t < S_t\}$ over the sample is very close to 0.5 and the grand mean of these means over all replications is negligibly different from 0.5 for all experiments. Obviously the same observation holds for the situation in which the data come from the equilibrium model. What is different between the

"equilibrium truth" and "disequilibrium truth" cases is the sampling variance of the estimated $\hat{\Pr}\{D_t < S_t\}$ over the sample points of a replication. The qualitative behavior of these estimated sampling variances is as expected: when equilibrium is the truth the variances are much smaller than in the alternative case but are still 2 to 4 times as large as they would be if the binomial approximation were valid. Row 12 in Tables 2 and 3 displays the average over the replications of the Kolmogorov-Smirnov statistic resulting from comparing the sample distribution of $(\hat{\Pr}\{D_t < S_t\} - 0.5)/(0.5/\sqrt{n})$ with $N(0,1)$. The hypothesis that the fit is acceptable is rejected in every case; in spite of the somewhat smaller Kolmogorov-Smirnov statistics in the case in which the truth is equilibrium, the procedure must be considered unworkable.

Test Based on the Ratio of Mean Likelihoods. We use the ratio of the means of the likelihood functions as an approximation to the posterior odds. For both the equilibrium and disequilibrium likelihood functions 300 random points were generated in a hyperrectangle in the (11 or 13 dimensional) parameter space centered on the maximum likelihood estimate, with width in each direction equal to six estimated standard errors for the parameter in question. Generated points at which the covariance matrix of the implied pdf was not positive definite were discarded. Attempts to use standard variance reduction techniques for numerical quadrature (see e.g. Schreider [15]) were both expensive and ineffective on the whole and were not employed. The accuracy of the results reported is low; however, the ratios are typically so much larger or smaller than unity that the fraction of cases in which the ratio is greater than or smaller than unity is robust with respect to errors in the numerator or denominator of several orders of magnitude. In some replications, repeated overflows prevented the computation of the ratios and these were omitted from the summary statistics in Row 13; in two cases the failures

were so numerous that no summary statistic is reported at all. The conclusions From Row 13 are (a) if the null hypothesis is true, the fraction of cases in which the approximation to the posterior odds favors the equilibrium model is 0.6 or larger except when the true residual variances are large in which case it is not even one half; thus the probability of Type I error is, as in previous tests, quite high. (b) If the null hypothesis of equilibrium is false, the fraction of cases in which the odds favor equilibrium is negligibly small, except in Cases 3 and 4 which have structure resembling the equilibrium model; moreover, the fraction is monotone in the true value of a_8 .

Error Analysis. Table 5 displays the frequency in which a replication had to be discarded as a result of various possible conditions. The most frequent causes are singularity or non-negative definiteness in the Hessian of the log-likelihood function for the disequilibrium model and essentially all failures are due to problems with this model. Difficulties of this type may be encountered often when derivatives are taken by numerical differencing and are particularly common to likelihood functions based on pdf's that resemble random mixtures of normals. The overall failure rate is 32 percent of the replications generated and cannot be considered satisfactory. It is reasonably certain, however, that use of analytic derivatives would have improved the success rate.

The Fair-Jaffee Model Revisited. In [7] Fair and Jaffee have proposed a model of housing starts consisting of the demand and supply functions

$$Q_t = a_0 + a_1x_{1t} + a_2x_{2t} + a_3x_{3t} + u_{1t} \quad (4-1)$$

$$Q_t = a_4 + a_5x_{1t} + a_6x_{4t} + a_7x_{5t} + a_8x_{6t} + u_{2t} \quad (4-2)$$

where Q_t is the observed quantity of housing starts in month t , x_{1t} a time trend, x_{2t} a measure of the stock of houses in existence in month t , x_{3t} the mortgage rate lagged two months, x_{4t} the 6-month moving average of

flow of private deposits into savings and loan associations and mutual savings banks, x_{5t} the 3-month moving average of borrowings by savings and loan associations from the Federal Home Loan Bank and x_{6t} the mortgage rate lagged one month. They recognized the disequilibrium nature of their model but estimated it with ad hoc techniques. In order to translate their model into the framework of the present paper the following changes were undertaken:

(1) Q_t in (4-1) is replaced by the (unobserved) variable D_t and Q_t in (4-2) is replaced by the (unobserved) variable S_t .

(2) To make the model genuinely simultaneous, the lagged mortgage rates x_{3t} and x_{6t} are replaced by the current mortgage rate, call it p_t .

(3) The min condition $Q_t = \min(D_t, S_t)$ and the price adjustment equation $p_t = p_{t-1} + \gamma(D_t - S_t) + u_{3t}$ are added to the system.

(4) The autocorrelation of the error terms was ignored.

The resulting disequilibrium model (assuming zero covariances between all error terms as did previous estimations of the model) as well as the corresponding equilibrium model (consisting of the amended forms of (4-1) and (4-2) only) were both estimated. The estimates and the absolute values of the ratios of the estimates to the asymptotic standard errors, as obtained from the negative inverse of the Hessian of the loglikelihood function, are displayed in columns 3 and 4 of Table 6 (for purposes of comparison some of the original Fair-Jaffee (F-J) estimates are displayed in columns 1 and 2). The disequilibrium estimates of the supply function are fairly close to the F-J Directional Method I. The estimates of the coefficients of the demand function have contrary to the F-J results, a negative sign for the time trend and a positive one for the stock of houses although the effect of the mortgage rate one demand is similar to the F-J Directional Method II. Essentially all results are significant and more so than by either F-J method. Most importantly, the two

hypothesis tests most successful in the Monte Carlo experiments unambiguously require rejection of the equilibrium hypothesis: $1/\hat{\gamma}$ divided by its asymptotic standard error is 3.33 and $-2\log\lambda = 488$. It is interesting to note finally that $\hat{\gamma} = 0.034$ implying a very slow adjustment in price. The equivalent of Equation (2-18) becomes $p_t = .979^t(p_0 - p_e) + p_e$ implying that (without further external shocks) it would take 32 months for a discrepancy between the initial mortgage rate and the equilibrium rate to be cut in half.

5. Conclusions

It has been shown that there does not exist a uniformly best procedure for testing the hypothesis that a market is in equilibrium against the alternative that it is not. The various test procedures may depend heavily on the precise specification of both the equilibrium version of the model and on its disequilibrium counterpart. A particular difficulty is that the equilibrium hypothesis is not likely in most models to be a nested one. Sampling experiments with a simple demand-supply model with a price adjustment equation have examined the effectiveness of several test procedures. Tests based on the magnitude of the coefficient of excess demand in the price equation and on the likelihood ratio appear to give excellent power but probabilities of Type I error that are too large. All proposed test procedures behave correctly in the qualitative sense and the quality of the estimates from an equilibrium model that represents a misspecification are very much worse than from the corresponding (correctly specified) equilibrium model. This result underscores the importance of effectively discriminating between the two types of formulations. Finally, the Fair-Jaffee model of housing starts was re-estimated and the null hypothesis that the demand-supply system for housing is in equilibrium was conclusively rejected.

Table 1

Characteristics of Experiments

<u>Case</u>	<u>No. of Replications</u>	<u>Sample Size</u>	<u>α</u>	<u>Covariance Matrices</u>	<u>Exogenous Variables</u>
1	50	60	.1	Σ_1, Σ_2	Uniform over standard ranges
2	25	60	.1	$10\Sigma_1, 10\Sigma_2$	Uniform over standard ranges
3	25	60	1.5	Σ_1, Σ_2	Uniform over standard ranges
4	50	60	3.0	Σ_1, Σ_2	Uniform over standard ranges
5	50	120	.1	Σ_1, Σ_2	Uniform over standard ranges
6	50	60	.1	Σ_1, Σ_2	Uniform over range equal to 1.5 x standard ranges

Table 2

Date Generated From Equilibrium Model

<u>Row</u>		<u>Case 1</u>	<u>Case 2</u>	<u>Case 5</u>	<u>Case 6</u>
	Number of Parameters for Which Equilibrium Model Has Smaller				
1	MSE	10	9	10	10
2	MAD	10	9	10	10
3	Midspread	9	7	7	5
	(MAD Disequilibrium Estimate) ÷ (MAD Equilibrium Estimate)				
4	Median	1.190	1.151	1.062	1.138
5	Mean	1.233	1.260	1.049	1.138
6	Fraction of \tilde{a}_8 Within 1.0 of Origin	0.20	0.36	0.08	0.16
7	Fraction of \tilde{a}_8 Within 2.0 of Origin	0.58	0.60	0.40	0.64
8	Fraction of $\tilde{a}_8 \geq 1.96$	0.26	0.20	0.24	0.20
9	Fraction of $-2\log\lambda \leq 0$	0.28	0.28	0.24	0.24
10	Fraction of $-2\log\lambda \geq \chi^2_{.05}(1)$	0.08	0.08	0.10	0.04
11	Fraction of $-2\log\lambda \geq \chi^2_{.05}(2)$	0.02	0.08	0.0	0.02
12	Mean Kolmogorov-Smirnov Statistic for Test of $\Pr\{\hat{D} < \hat{S}\} \sim N(.5, .25/n)$	0.369	0.310	0.406	0.399
13	Fraction of Odds Favoring Equilibrium Model	0.625	0.440	-	0.600

Table 3

Data Generated From Disequilibrium Model

<u>Row</u>		<u>Case 1</u>	<u>Case 2</u>	<u>Case 3</u>	<u>Case 4</u>	<u>Case 5</u>	<u>Case 6</u>
	Number of Parameters for Which Equilibrium Model Has Smaller						
1	MSE	0	0	0	4	0	0
2	MAD	0	0	1	2	0	0
3	Midsread	0	0	4	1	0	0
	(MAD Disequilibrium Estimate) ÷ (MAD Equilibrium Estimate)						
4	Median	0.051	0.100	0.584	0.953	0.023	0.031
5	Mean	0.110	0.128	0.709	0.803	0.077	0.083
6	Fraction of \tilde{a}_8 Within 1.0 of Origin	0.0	0.0	0.0	0.0	0.0	0.0
7	Fraction of \tilde{a}_8 Within 2.0 of Origin	0.02	0.12	0.0	0.04	0.0	0.0
8	Fraction of $\tilde{a}_8 \geq 1.96$	0.98	0.88	1.00	0.92	1.00	1.00
9	Fraction of $-2\log\lambda \leq 0$	0.0	0.0	0.0	0.0	0.0	0.0
10	Fraction of $-2\log\lambda \geq \chi^2_{.05}(1)$	1.00	1.00	1.00	0.96	1.00	1.00
11	Fraction of $-2\log\lambda \geq \chi^2_{.05}(2)$	1.00	1.00	1.00	0.94	1.00	1.00
12	Mean Kolmogorov-Smirnov Statistic for Test of $\Pr\{\hat{D} < \hat{S}\} \sim N(.5, .25/u)$	0.412	0.306	0.468	0.471	0.452	0.441
13	Fraction of Odds Favoring Equilibrium Model	0.021	-	0.440	0.562	0.0	0.0

Table 4

Summary Measures of Goodness

	Mean Square Deviation							
	From 0.5 of Fraction of Estimates < True Value			From 1.0 of Mean Square Error ÷ Mean Asymptotic Variance				
	Data Generated by		Equ. Model	Data Generated by		Equ. Model		
	Disequ. Model	Estimated from	Disequ. Model	Estimated from	Disequ. Model	Equ. Model		
		Disequ. Model	Disequ. Model	Disequ. Model	Disequ. Model	Equ. Model		
		Model	Model	Model	Model	Model		
Case 1	0.0056	0.1538	0.0124	0.0117	0.1290	35.2232	0.2349	0.0604
Case 2	0.0168	0.0872	0.0096	0.0088	0.2668	0.2200	0.7366	0.0551
Case 3	0.0104	0.1000	-	-	0.1185	4.9942	-	-
Case 4	0.0102	0.1020	-	-	0.1240	1.0556	-	-
Case 5	0.0048	0.1403	0.0014	0.0044	0.0375	206.9570	0.0456	0.0171
Case 6	0.0058	0.1696	0.0048	0.0030	0.0735	57.2386	0.6464	0.0807

Table 5

Error Analysis

<u>Case</u>	<u>1</u>		<u>2</u>		<u>3</u>		<u>4</u>		<u>5</u>		<u>6</u>	
	<u>Disequ.</u>	<u>Equ.</u>	<u>Disequ.</u>	<u>Equ.</u>	<u>Disequ.</u>	<u>Equ.</u>	<u>Disequ.</u>	<u>Equ.</u>	<u>Disequ.</u>	<u>Equ.</u>	<u>Disequ.</u>	<u>Equ.</u>
<u>True Model</u>												
Failure in Optimizing Disequilibrium Model	0.13	0.06	0.10	0.08	0	0	0	0	0.13	0.02	0.19	0.09
Failure in Optimizing Equilibrium Model	0	0	0.05	0	0	0	0	0	0	0	0	0
Hessian in Disequilibrium Model Singular	0.22	0.13	0.10	0.05	0	0	0	0	0	0.03	0.13	0.10
Hessian in Equilibrium Model Singular	0	0	0	0	0	0	0	0	0	0	0	0
Hessian in Disequilibrium Model Not Negative Definite	0.06	0.26	0.05	0.19	0.07	0.25	0.07	0.25	0.06	0.15	0.05	0.17
Hessian in Equilibrium Model Not Negative Definite	0	0	0.08	0	0	0	0	0	0	0	0	0
Total Cases Generated	85	90	40	37	27	64	62	62	62	62	79	79

Table 6

Results for the Fair-Jaffee Model

	Fair-Jaffee Directional Method I	Fair-Jaffee Directional Method II	Equilibrium Model	Disequilibrium Model
a_0	193.16 (3.10)	328.43 (6.06)	347.43 (8.12)	427.58 (12.93)
a_1	6.78 (2.01)	3.94 (1.69)	6.14 (3.12)	-16.87 (17.58)
a_2	-0.055 (1.93)	-0.032 (1.63)	-0.048 (.290)	0.153 (17.53)
a_3	-0.241 (2.27)	-0.471 (5.73)	-0.435 (5.50)	-0.504 (8.65)
a_4	-40.84 (1.29)	-75.87 (1.74)	-65.33 (2.86)	0.194 (0.01)
a_5	-0.236 (3.12)	-0.332 (2.71)	-0.338 (6.35)	-0.219 (3.95)
a_6	0.048 (6.20)	0.047 (4.32)	0.067 (12.21)	0.054 (9.54)
a_7	0.033 (2.76)	0.012 (0.62)	0.053 (7.80)	0.056 (7.30)
a_8	0.116 (2.69)	0.190 (2.74)	0.220 (6.30)	0.124 (3.14)
$1/\gamma$	-	-	-	29.36 (3.33)
σ_1^2	76.38	65.45	452.04 (3.68)	28.75 (1.92)
σ_2^2	57.61	47.06	109.86 (6.07)	105.77 (6.66)
σ_3^2	-	-	-	33.33 (7.90)
log L	-	-	-1100.76	-856.76

APPENDIX 1

THE LIMIT OF $h(Q_t, p_t)$ as $a_8 \rightarrow \infty$

Consider the special case in which Σ_2 is diagonal. Then, letting σ_1^2 , σ_2^2 , σ_3^2 be the (diagonal) elements of Σ_2 ,

$$A_1 = \frac{1}{\sigma_1^2} + \frac{a_8^2}{\sigma_3^2} \quad (A-1)$$

$$A_{2t} = -(a_1 p_t + b_{1t}) \frac{1}{\sigma_1^2} - a_8 (a_8 Q_t + p_t - b_{3t}) \frac{1}{\sigma_3^2} \quad (A-2)$$

$$A_{3t} = \frac{(a_1 p_t + b_{1t})^2}{\sigma_1^2} + \frac{(Q_t - a_5 p_t - b_{2t})^2}{\sigma_2^2} + \frac{(a_8 Q_t + p_t - b_{3t})^2}{\sigma_3^2} \quad (A-3)$$

$$A_4 = \frac{1}{\sigma_2^2} + \frac{a_8^2}{\sigma_3^2} \quad (A-4)$$

$$A_{5t} = -(a_5 p_t + b_{2t}) \frac{1}{\sigma_2^2} + a_8 (p_t - b_{3t} - a_8 Q_t) \frac{1}{\sigma_3^2} \quad (A-5)$$

$$A_{6t} = \frac{(Q_t - a_1 p_t - b_{1t})^2}{\sigma_1^2} + \frac{(a_5 p_t + b_{2t})^2}{\sigma_2^2} + \frac{(p_t - a_8 Q_t - b_{3t})^2}{\sigma_3^2} \quad (A-6)$$

We shall examine the behavior of (2-14) as $a_8 \rightarrow \infty$. In order to do so, we require the limits as $a_8 \rightarrow \infty$ of

$$\psi_1 = \frac{|1 + a_8(a_5 - a_1)|}{2\pi(|\Sigma_2| A_1)^{1/2}}$$

$$\psi_2 = \frac{|1 + a_8(a_5 - a_1)|}{2\pi(|\Sigma_2| A_4)^{1/2}}$$

and of B_{1t} , ℓ_{1t} , B_{2t} , ℓ_{2t} .

But ψ_1 and ψ_2 can be written as

$$\psi_1 = \frac{|1+a_8(a_5-a_1)|}{2\pi\sigma_2(\sigma_3^2+a_8\sigma_1^2)^{1/2}} = \frac{|\frac{1}{a_8} + a_5-a_1|}{2\pi\sigma_2(\frac{\sigma_3^2}{a_8} + \sigma_1^2)^{1/2}}$$

and

$$\psi_2 = \frac{|1+a_8(a_5-a_1)|}{2\pi\sigma_1(\sigma_3^2+a_8\sigma_2^2)^{1/2}} = \frac{|\frac{1}{a_8} + a_5-a_1|}{2\pi\sigma_1(\frac{\sigma_3^2}{a_8} + \sigma_2^2)^{1/2}}$$

and hence $\lim_{a_8 \rightarrow \infty} \psi_1 = \lim_{a_8 \rightarrow \infty} \psi_2 = \frac{|a_5-a_1|}{2\pi\sigma_1\sigma_2}$.

Substituting from (A-1) to (A-6) in (2-11) and (2-13), we can write

$$B_{1t} = \frac{\left(k_{1t} + \frac{(a_8 Q_t + p_t - b_{3t})^2}{\sigma_3^2}\right) \left(\frac{1}{\sigma_1^2} + \frac{a_8^2}{\sigma_3^2}\right) - \left(k_{2t} - \frac{a_8(a_8 Q_t + p_t - b_{3t})}{\sigma_3^2}\right)^2}{\frac{1}{\sigma_1^2} + \frac{a_8^2}{\sigma_3^2}}$$

where k_{1t} , k_{2t} do not depend on a_8 . Simplifying the above expression yields

$$B_{1t} = \frac{k_{3t} + k_{4t}/a_8 + k_{5t}/a_8^2}{1/\sigma_3^2 + 1/\sigma_1^2 a_8^2}$$

where k_{3t} , k_{4t} , k_{5t} do not depend on a_8 . Hence $\lim_{a_8 \rightarrow \infty} B_{1t} = \sigma_3^2 k_{3t}$ which

may be verified to be equal to $(Q_t - a_1 p_t - b_{1t})^2 / \sigma_1^2 + (Q_t - a_5 p_t - b_{2t})^2 / \sigma_2^2$. By

similar considerations it also follows that $\lim_{a_8 \rightarrow \infty} B_{2t} = \lim_{a_8 \rightarrow \infty} B_{1t}$. Substituting (A-1), (A-2), (A-4) and (A-5) in (2-15) yields

$$\begin{aligned}
l_{1t} &= \left(\frac{a_8^2}{\sigma_3^2} + \frac{1}{\sigma_1^2} \right)^{1/2} \left(Q_t + \frac{k_{7t} - a_8(a_8 Q_t + p_t - b_{3t})/\sigma_3^2}{1/\sigma_1^2 + a_8^2/\sigma_3^2} \right) \\
&= \left(\frac{1}{\sigma_3^2} + \frac{1}{a_8^2 \sigma_1^2} \right)^{1/2} \left(\frac{Q_t + \frac{k_{7t}}{a_8} - \frac{p_t - b_{3t}}{\sigma_3^2}}{\frac{1}{a_8^2 \sigma_1^2} + \frac{1}{\sigma_3^2}} \right)
\end{aligned}$$

where k_{7t} does not depend on a_8 . Hence $\lim_{a_8 \rightarrow \infty} l_{1t} = -\frac{p_t - b_{3t}}{\sigma_3}$ and by similar considerations $\lim_{a_8 \rightarrow \infty} l_{2t} = \frac{p_t - b_{3t}}{\sigma_3} = -\lim_{a_8 \rightarrow \infty} l_{1t}$.

Hence

$$\begin{aligned}
\lim_{a_8 \rightarrow \infty} h(Q_t, p_t) &= \frac{|a_5 - a_1|}{2\pi\sigma_1\sigma_2} \exp\left\{-\frac{1}{2} \left[\frac{(Q_t - a_1 p_t - b_{1t})^2}{\sigma_1^2} + \frac{(Q_t - a_5 p_t - b_{2t})^2}{\sigma_2^2} \right]\right\} \left[\Phi\left(-\frac{p_t - b_{3t}}{\sigma_3}\right) + \Phi\left(\frac{p_t - b_{3t}}{\sigma_3}\right) \right] \\
&= \frac{|a_5 - a_1|}{2\pi\sigma_1\sigma_2} \exp\left\{-\frac{1}{2} \left[\frac{(Q_t - a_1 p_t - b_{1t})^2}{\sigma_1^2} + \frac{(Q_t - a_5 p_t - b_{2t})^2}{\sigma_2^2} \right]\right\} \quad (A-7)
\end{aligned}$$

which is the same as (2-17).

APPENDIX 2

THE CASE OF $u_{3t} \equiv 0$ AND THE LIMIT OF $h(Q_t)$ AS $a_8 \rightarrow \infty$

Assume that u_{3t} in (2-6) is identically zero. In that event D_t , S_t and p_t do not have a nonsingular joint distribution and we cannot proceed as in (2-7). However, the pdf $h(Q_t)$ can be obtained from the nonsingular joint pdf of D_t , S_t as

$$h(Q_t) = \int_{Q_t}^{\infty} g(Q_t, S_t) dS_t + \int_{Q_t}^{\infty} g(D_t, Q_t) dD_t \quad (A-8)$$

Equation (A-8) is derived explicitly as follows. Substituting (2-6) into (2-3) and (2-4) yields

$$u_{1t} = (1-a_1 a_8) D_t + a_1 a_8 S_t + c_{1t} \quad (A-9)$$

$$u_{2t} = -a_5 a_8 D_t + (1+a_5 a_8) S_t + c_{2t} \quad (A-10)$$

where $c_{1t} = -a_1 p_{t-1} - b_{1t}$ and $c_{2t} = -a_5 p_{t-1} - b_{2t}$. Hence

$$g(D_t, S_t) = \frac{|1+a_8(a_5-a_1)|}{2\pi |\Sigma_1|^{1/2}} \exp\left\{-\frac{1}{2}(u_{1t}, u_{2t}) \Sigma_1^{-1} \begin{pmatrix} u_{1t} \\ u_{2t} \end{pmatrix}\right\} \quad (A-11)$$

where in (A-11) we replace u_{1t} , u_{2t} from (A-9) and (A-10) and where

$|1+a_8(a_5-a_1)|$ is the Jacobian of the transformation from (u_{1t}, u_{2t}) to (D_t, S_t) .

For simplicity we now assume, analogously to Appendix 1 that Σ_1 is diagonal.

The exponent in (A-11) can then be written as

$$\frac{1}{\sigma_1^2} \left[(1-a_1 a_8) D_t + a_1 a_8 S_t + c_{1t} \right]^2 + \frac{1}{\sigma_2^2} \left[-a_5 a_8 D_t + (1+a_5 a_8) S_t + c_{2t} \right]^2 \quad (A-12)$$

Collecting terms alternately on powers of D_t and S_t , (A-12) can be written as

$$A_1 D_t^2 + 2A_{2t} D_t + A_{3t} = \frac{(D_t - \mu_{1t})^2}{\omega_1^2} + B_{1t}$$

and

$$A_4 S_t^2 + 2A_{5t} S_t + A_{6t} = \frac{(S_t - \mu_{2t})^2}{\omega_2^2} + B_{2t}$$

where

$$A_1 = \frac{(1 - a_1 a_8)^2}{\sigma_1^2} + \frac{a_5^2 a_8^2}{\sigma_2^2}$$

$$A_{2t} = \frac{1}{\sigma_1^2} (1 - a_1 a_8) (a_1 a_8 Q_t + c_{1t}) - \frac{1}{\sigma_2^2} a_5 a_8 [(1 + a_5 a_8) Q_t + c_{2t}]$$

$$A_{3t} = \frac{1}{\sigma_1^2} (a_1 a_8 Q_t + c_{1t})^2 + \frac{1}{\sigma_2^2} [(1 + a_5 a_8) Q_t + c_{2t}]^2$$

$$A_4 = \frac{a_1^2 a_8^2}{\sigma_1^2} + \frac{(1 + a_5 a_8)^2}{\sigma_2^2}$$

$$A_{5t} = \frac{1}{\sigma_1^2} a_1 a_8 [(1 - a_1 a_8) Q_t + c_{1t}] + \frac{1}{\sigma_2^2} (1 + a_5 a_8) (-a_5 a_8 Q_t + c_{2t})$$

$$A_{6t} = \frac{1}{\sigma_1^2} [(1 - a_1 a_8) Q_t + c_{1t}]^2 + \frac{1}{\sigma_2^2} (1 - a_5 a_8 Q_t + c_{2t})^2$$

$$\mu_{1t} = -A_{2t}/A_1$$

$$\omega_1^2 = 1/A_1$$

$$B_{1t} = A_{3t} - A_{2t}^2/A_1$$

$$\mu_{2t} = -A_{5t}/A_4$$

$$\omega_2^2 = 1/A_4$$

$$B_{2t} = A_{6t} - A_{5t}^2/A_4$$

Making the obvious substitutions, (A-11) becomes

$$h(Q_t) = \frac{|1+a_8(a_5-a_1)|}{(2\pi)^{\frac{1}{2}}|\Sigma_1|^{\frac{1}{2}}A_1^{\frac{1}{2}}} \exp\{-\frac{B_{1t}}{2}\}(1-\phi(\ell_{1t})) +$$

$$+ \frac{|1+a_8(a_5-a_1)|}{(2\pi)^{\frac{1}{2}}|\Sigma_1|^{\frac{1}{2}}A_4^{\frac{1}{2}}} \exp\{-\frac{B_{2t}}{2}\}(1-\phi(\ell_{2t})) \quad (A-13)$$

where

$$\ell_{1t} = A_1^{\frac{1}{2}}(Q_t + A_{2t}/A_1)$$

$$\ell_{2t} = A_4^{\frac{1}{2}}(Q_t + A_{5t}/A_4)$$

We now proceed as in Appendix 1. We first evaluate the limits of the analogues to ψ_1 and ψ_2 and find immediately that

$$\lim_{a_8 \rightarrow \infty} \psi_1 = \lim_{a_8 \rightarrow \infty} \psi_2 = \frac{|a_5 - a_1|}{(2\pi)^{\frac{1}{2}}\sigma_1\sigma_2(a_1^2/\sigma_1^2 + a_5^2/\sigma_2^2)^{\frac{1}{2}}} \quad (A-14)$$

We next evaluate $\lim_{a_8 \rightarrow \infty} B_{1t}$ and $\lim_{a_8 \rightarrow \infty} B_{2t}$. After tedious manipulation we find

$$\lim_{a_8 \rightarrow \infty} B_{1t} = \lim_{a_8 \rightarrow \infty} B_{2t} = \frac{[Q_t(a_5 - a_1) + a_5 c_{1t} - a_1 c_{2t}]^2}{\sigma_1^2 \sigma_2^2 (a_1^2/\sigma_1^2 + a_5^2/\sigma_2^2)} \quad (A-15)$$

Finally we obtain

$$\lim_{a_8 \rightarrow \infty} \ell_{1t} = - \lim_{a_8 \rightarrow \infty} \ell_{2t} = - \frac{Q_t(a_1/\sigma_1^2 + a_5/\sigma_2^2)}{(a_1^2/\sigma_1^2 + a_5^2/\sigma_2^2)^{\frac{1}{2}}} = \ell_t \quad (A-16)$$

which defines ℓ_t .

Combining (A-14), (A-15) and (A-16) and noting that $\phi(-\ell_t) = 1 - \phi(\ell_t)$ we obtain

$$\lim_{a_8 \rightarrow \infty} h(Q_t) = \frac{|a_5 - a_1|}{(2\pi)^{\frac{1}{2}}\sigma} e^{-\frac{1}{2\sigma^2}[Q_t(a_5 - a_1) + a_5 c_{1t} - a_1 c_{2t}]^2} \quad (A-17)$$

where $\sigma = \sigma_1 \sigma_2 (a_1^2/\sigma_1^2 + a_2^2/\sigma_2^2)^{1/2}$. Equation (A-17) is clearly not equal to (2-17), the pdf for the corresponding equilibrium model.

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