MAXIMUM LIKELIHOOD ESTIMATION OF DISEQUILIBRIUM MODELS

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1. Introduction

The explicit treatment of disequilibrium is a recent phenomenon in economic theory and in econometrics. The overwhelming bulk of economic theory deals with markets in which buyers' intentions are represented by a demand function and sellers' intentions by a supply function[1]; these functions represent the various quantities that buyers and sellers intend to buy and sell respectively at alternative prices and it is usually posited that the presence of an auctioneer or a similar mechanism results in the establishment of a market clearing price, i.e., one at which demand and supply are equal. Since equality of demand and supply means that nobody has an incentive to change his behavior, this state is one of equilibrium.

Both microtheory and macrotheory have traditionally relied on this conceptualization, the practical and for econometric estimation relevant consequence of which is the fact that endogenous variable values that are actually observed satisfy both the demand and supply function. Thus, given the demand function

$$D_t = D(p_t, x_{1t}) + u_{1t} \qquad (1\text{-}1)$$

and the supply function

$$S_t = S(p_t, x_{2t}) + u_{2t} \qquad (1\text{-}2)$$

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[1] It should be obvious that even if the supply function is not defined as in the case of monopoly, the traditional treatment of such markets still falls in the equilibrium framework.

where p_t is the price in the tth period, x_{1t}, x_{2t} vectors of exogenous variables in the tth period and u_{1t}, u_{2t} the (unobservable) error terms, then, if in the tth period the actually observed price-quantity pair is (p_t^*, q_t^*), it must be true that

$$D(p_t^*, x_{1t}) + u_{1t} = q_t^* = S(p_t^*, x_{2t}) + u_{2t} \qquad (1\text{-}3)$$

In recent years several macro as well as micro-models have departed from the traditional insistence that prices always clear all markets. Articulate and persuasive macroeconomic formulations dealing with the determination of aggregate employment and income are due to Barro and Grossman [2], Korliras [8], and others. An important microeconomic model devoted to analyzing the demand and supply of housing starts is due to Fair and Jaffee [5]. Much earlier models of certain agricultural markets, due to Suits [12] and Suits and Koizumi [14] were recently recognized [6] to be in fact disequilibrium models of a certain type. There are numerous individual differences among these various models. In the macro-literature, for example, the classical demand and supply functions based on utility maximization by consumers and profit maximization by producers are referred to as "notional" demand and supply functions; if, due to price rigidities, some form of rationing occurs and it becomes impossible for economic actors to obtain price-quantity pairs satisfying these functions, they become replaced by "effective" demand and supply functions which incorporate the idea of such rationing. The distinction between such notional and effective demand and supply functions is generally absent in the microapplications. Our present purpose is not, however, to discuss all such differences in a detailed manner, but rather to focus on the econometric consequences of one important common feature among all the genuine disequilibrium models, namely that they incorporate in the specification of the model a "minimum" condition. The simplest case in which such a minimum condition occurs is in a single-market case, the demand and supply functions for which are given by (1-1) and (1-2). If prices are rigid and not assumed to adjust so as to clear the market, the observed quantity q_t must satisfy

$$q_t = \min(D_t, S_t) \tag{1-4}$$

and the simplest disequilibrium model imaginable thus consists of (1-1), (1-2) and (1-4). Such a min condition is present in all disequilibrium models that we consider genuine disequilibrium models. It has been more or less customary to refer to models incorporating some form of price dynamics such as those containing a partial adjustment equation

$$p_t - p_{t-1} = \gamma(p_t^e - p_{t-1}) \tag{1-5}$$

where p_t^e is the equilibrium price, as disequilibrium models; we shall specifically rule out from consideration such models unless they explicitly contain the min condition which is necessary to close the model, i.e., without which one cannot determine what quantity will be observed at prices that are not equilibrium prices.

In Section 2 we describe some simple single-market disequilibrium models and derive the likelihood function for some illustrative examples. In Section 3 we point out a potential hazard which is that the resulting likelihood function may not satisfy the usual regularity conditions under which maximum likelihood estimates are consistent and asymptotically normal. In Section 4 we introduce several simple models in which two interrelated markets may be in disequilibrium and discuss the problem of multiple solutions. The likelihood function for such a simple model is derived in Section 5 and, finally some suggestions are offered for hypothesis testing in Section 6. Some further details on the subject of

Section 4 are contained in an Appendix.

2. Single-Market Disequilibrium Models[2]

The simplest disequilibrium model in a single market can be characterized by (1-1), (1-2) and (1-4) which we repeat here for convenience:

$$D_t = D(p_t, x_{1t}) + u_{1t} \tag{2-1}$$

$$S_t = S(p_t, x_{2t}) + u_{2t} \tag{2-2}$$

$$q_t = \min(D_t, S_t) \tag{2-3}$$

It is clear that in the present model p_t is fully rigid and hence exogenous: the model permits no endogenous price adjustment of any kind. Maximum likelihood estimation requires that we first derive the probability density function (pdf) of the observable random variable q_t. The nature of this derivation will clearly be contingent on what additional assumptions we make concerning information available to the investigator. In particular, although D_t and S_t themselves are unobservable, we may have information (for each observation $t = 1,\ldots,T$) that $D_t < S_t$ or that $D_t \geqq S_t$; alternatively we may even lack this knowledge. Denote by $g(D_t, S_t)$ the joint density function of D_t and S_t, which is clearly obtainable in general from the joint pdf of u_{1t} and u_{2t}. Denote further by $g(D_t, S_t | D_t < S_t)$ the joint conditional pdf of D_t and S_t, conditional on the event $D_t < S_t$. If we have no knowledge of whether $D_t < S_t$ or not, we can write the pdf of q_t as

$$h(q_t) = f(q_t | D_t < S_t)\Pr\{D_t < S_t\} + f(q_t | D_t \geqq S_t)\Pr\{D_t \geqq S_t\} \tag{2-4}$$

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[2]For detailed development of several such models see [6], [9] and [12].

where $f(q_t|\cdot)$ is the obvious conditional pdf. Since $f(q_t|D_t<S_t) = \int_{q_t}^{\infty} g(q_t,S_t|D_t<S_t)dS_t = (1/\Pr\{D_t<S_t\}) \int_{q_t}^{\infty} g(q_t,S_t)dS_t$ we can write (2-4) as

$$h_t(q_t) = \int_{q_t}^{\infty} g(q_t,S_t)dS_t + \int_{q_t}^{\infty} g(D_t,q_t)dD_t \qquad (2\text{-}5)^3$$

whence the likelihood function is

$$L = \prod_{t=1}^{T} h_t(q_t) \qquad (2\text{-}6)$$

If, on the contrary, one assumed prior knowledge as to what periods were periods of excess demand and what were periods of excess supply, the pdf of q_t would become

$$h_t(q_t) = \begin{cases} f(q_t|D_t<S_t)\Pr\{D_t<S_t\} & \text{if } D_t < S_t \\ f(q_t|D_t\geqq S_t)\Pr\{D_t\geqq S_t\} & \text{if } D_t \geqq S_t \end{cases} \qquad (2\text{-}7)$$

and the appropriate likelihood function is

$$L = \prod_{T_1} f(q_t|D_t<S_t)\Pr\{D_t<S_t\} \prod_{T_2} f(q_t|D_t\geqq S_t)\Pr\{D_t\geqq S_t\} \qquad (2\text{-}8)$$

where T_1 and T_2 are the index sets for which $D_t < S_t$ and $D_t \geqq S_t$ are true respectively.

A much more interesting and realistic set of models is established if one explicitly introduces a price adjustment equation. Standard microtheory suggests that price is rising when there is an excess demand and conversely. A significant variety of models may be obtained by varying the precise manner in which this assumption is introduced. In general one might posit

[3] We introduce an index t on the functional form itself, since y_t is not identically distributed over the sample due to the fact that the exogenous variables have different values for different t.

$$p_t - p_{t-1} = \gamma_1(D_{\tau_1} - S_{\tau_1}) + \gamma_2' x_{3\tau_2} + u_{3t} \qquad (2\text{-}9)$$

vector of corresponding coefficients, and where τ_1 and τ_2 may be either equal to t or $t-1$ and where u_{3t} may either be a degenerate random variable ($u_{3t} \equiv 0$ for all t) or not. It is not worth while to treat each possible case here; for detail the reader is referred to [12]. It is sufficient to observe that in the case that in some sense is most plausible, i.e., when $\tau_1 = t$ and u_{3t} is not identically zero, one can analogously to (2-6) and (2-8) derive likelihood functions that correspond to the assumption of not having or having prior information as to the classification of the sample between excess demand and excess supply periods.

The demand-supply models discussed above are appropriate for studies such as the Fair-Jaffee study of the housing market [5]. A quite different notion of equilibrium may emerge, however, in different contexts. The essence of the Suits models ([13], [14]) of various agricultural markets (such as watermelons or onions) is that the crop is determined by exogenous (weather) and other predetermined variables (past prices) but that farmers' intentions to harvest depend on current wages and prices; hence it is possible that part of the crop remains unharvested. This type of model is dealt with formally and the likelihood function is derived under alternative assumptions in [6]. Let q_t denote the crop, x_t the intended (ex ante) harvest, p_t the price, z_{1t}, z_{2t}, z_{3t} sets of exogenous variables and y_t the actual amount harvested. The relations describing the market are[4]

$$q_t = b_1 z_{1t} + b_2 + u_{1t} \qquad (2\text{-}10)$$

$$x_t = b_3 p_t + b_4 q_t + b_5 z_{2t} + b_6 + u_{2t} \qquad (2\text{-}11)$$

[4] There is a further, unsatisfactory, complication in Suits' treatment of the onion market in that the price which enters (2-12) is the annual average price but the price entering (2-11) is the price in the third calendar quarter; clearly related to the former but in an unspecified manner.

$$p_t = b_7 z_{3t} + b_8 y_t + b_9 + u_{3t} \qquad (2\text{-}12)$$

$$y_t = \min(q_t, x_t) \qquad (2\text{-}13)$$

Equation (2-10) is simply the determinant of the crop. Equation (2-11) expresses the notion that the intended harvest depends on the price, the crop and on other exogenous variables.[5] Equation (2-12) is an ordinary demand function and (2-13) expresses the obvious requirement that the actual harvest not be greater than the crop. There are basically two possibilities: (1) q_t happens to be very large in which case it is likely that $x_t < q_t$; then $y_t = x_t$ and the endogenous variables p_t , x_t are determined from (2-11) and (2-12) or (2) q_t is small in which case $q_t = y_t$, p_t is determined from (2-12) alone and (2-11) plays no role. The likelihood function for this model is derived in [6] on each of the following three alternative assumptions: (1) the crop is an unobserved variable and the investigator has no prior knowledge as to when $q_t > x_t$ and $q_t \leq x_t$; (2) the crop is unobserved as in (1) but the investigator at least knows in what periods part of the crop was left unharvested; (3) the crop is actually observed. The three alternative likelihood functions are quite different and thus a disequilibrium model can vary econometrically substantially depending on what is assumed about the investigator's level of information.

An interesting possibility in disequilibrium models is that the model is internally inconsistent.[6] To demonstrate this rewrite (2-10) through (2-13) in a simplified form by suppressing all exogenous variables and constants. Thus

[5] The version of the model is a reasonably faithful rendition of Suits' original model; it may, however, be criticized on several grounds, among them the possible inappropriateness of making harvest intentions depend on the actual crop.

[6] I am indebted to T. Amemiya for this observation and the following demonstration.

$$q_t = u_{1t} \tag{2-14}$$

$$x_t = b_3 p_t + b_4 u_{1t} + u_{2t} \tag{2-15}$$

$$p_t = b_8 y_t + u_{3t} \tag{2-16}$$

$$y_t = \min(q_t, x_t) \tag{2-17}$$

This version of the model is equivalent to having error terms with nonzero means. If $q_t < x_t$, then $p_t = b_8 u_{1t} + u_{3t} \equiv p^A$ and if $b_3 > 0$, then $p_t > \frac{1}{b_3}[(1-b_4)u_{1t} - u_{2t}] \equiv p^B$ and $p^A > p^B$ implies that $p_t = p^A$. If $x_t < q_t$, then $p_t = (b_4 b_8 u_{1t} + b_8 u_{2t} + u_{3t})/(1-b_3 b_8) \equiv p^C$ and $p^B > p^C$. Inconsistency would arise if simultaneously $p^A > p^B > p^C$ for then the left hand inequality implies $p_t = p^A$ and the right hand inequality implies $p_t = p^C$, a contradiction. Fortunately in the present case it is economically meaningful to posit $b_3 > 0$ and $b_8 < 0$ and these two conditions guarantee that $p^A > p^C$ implies $p^C > p^B$ and $p^A < p^C$ implies $p^B > p^C$ and thus the solution is determinate on economically reasonable assumptions. In general, however, there is no guarantee that such a model is consistent and we conclude that particular care must be employed to verify in each individual case that a disequilibrium model is consistent.

3. Irregular Likelihood Functions

For simplicity we now return to the model given by (2-1), (2-2) and (2-3). We assume that u_{1t} and u_{2t} are normally distributed with mean zero and variances σ_1^2, σ_2^2 and moreover that u_{1t} and u_{2t} are independent of one another as well as of their own lagged values. The joint pdf of D_t, S_t is

$$g(D_t, S_t) = \frac{1}{2\pi\sigma_1\sigma_2} \exp\left\{-\frac{1}{2}\left[\frac{(D_t - D(p_t, x_{1t}))^2}{\sigma_1^2} + \frac{(S_t - S(p_t, x_{2t}))^2}{\sigma_2^2}\right]\right\} \tag{3-1}$$

and the pdf of the observable random variable q_t is

$$h(q_t) = \frac{1}{\sqrt{2\pi}\,\sigma_1} \exp\left\{-\frac{(q_t - D(p_t, x_{1t}))^2}{2\sigma_1^2}\right\} \left(1 - \Phi\left(\frac{q_t - S(p_t, x_{2t})}{\sigma_2}\right)\right) +$$

$$\frac{1}{\sqrt{2\pi}\,\sigma_2} \exp\left\{-\frac{(q_t - S(p_t, x_{2t}))^2}{2\sigma_2^2}\right\} \left(1 - \Phi\left(\frac{q_t - D(p_t, x_{1t})}{\sigma_1}\right)\right) \qquad (3\text{-}2)$$

where $\Phi(\ )$ is the cumulative standard normal integral. The expression in (3-2) is the sum of two products and can, for simplicity, also be written as $\alpha_t\beta_t + \gamma_t\delta_t$. The likelihood function is

$$L = \prod_{t=1}^{T} (\alpha_t\beta_t + \gamma_t\delta_t) \qquad (3\text{-}3)$$

We shall now show that this likelihood function is unbounded in parameter space. The functions $D(p_t,x_{1t})$ and $S(p_t,x_{2t})$ depend on unknown parameters which are the object of estimation. We can always choose the parameters in $D(p_t,x_{1t})$ in such a manner that for some t , say $t=k$, $q_k = D(p_k,x_{1k})$ and $q_t < D(p_t,x_{1t})$ for all $t \neq k$. Choose for the parameters of $S(p_t,x_{2t})$ any set of values for which $S(p_t,x_{2t})$ is finite. Now consider the sequence of points obtained in parameter space by letting $\sigma_1 \to 0$ and the corresponding values of (3-3). Since the exponent part of α_k is identically zero, α_k itself increases without bound. All β_t and γ_t are nonzero. Since $(q_t - D(p_t,x_{1t}))/\sigma_1 = 0$ for $t=k$ and goes to $-\infty$ for $t \neq k$, $\lim \delta_k = 1/2$ and $\lim \delta_t = 1$ for all $t \neq k$; hence in (3-3) we have all terms bounded strictly away from zero and one term becoming unbounded and therefore the likelihood function becomes unbounded. It follows that any successful attempt to locate the global maximum of the likelihood function is guaranteed to produce inconsistent estimates due to this breakdown of the usual boundedness requirement.[7]

[7]The reader may note that the nature of the unboundedness and the argument employed to prove it are rather similar to the case of the likelihood function for a pdf which is a λ-weighted random mixture of normals with arbitrary unknown variances.

Prior information to the effect that $\sigma_1 = c\sigma_2$, c a known constant, is sufficient for avoiding this difficulty although such prior information may not always be available. An almost identical form of unboundedness arises for somewhat more complicated reasons in the likelihood function for the Suits model based on the minimal set of prior information, i.e., when neither q_t nor the sample partition is known. It is therefore essential to examine the likelihood function with care prior to estimation in order to verify that such unboundedness does not occur. It is fortunate that there are also important and more realistic case, such as when a price adjustment equation (2-9) is present, when the regularity conditions are not violated.

4. Many Market Disequilibrium Models

For the sake of simplicity we shall restrict outselves to cases in which there are two markets that may be in disequilibrium and are linked together. Even so, the number of possible models is enormous and we shall further restrict ourselves to some relatively simple cases.

The first model we discuss is a modified version of a model of a planned economy proposed by Richard Portes [10]. This modification is undertaken primarily to permit the diagrammatic treatment in Figure 1. For ease of notation we dispense with the time subscript t in the first part of the present section. The variables $z_1,\ldots,z_5$ denote (sets of) exogenous variables and $u_1,\ldots,u_6$ unobservable random errors. The endogenous variables in the model are c^d and c^s (consumption demand and supply of consumer goods), ℓ^d and ℓ^s (demand for and supply of labor), c (actual amount of consumption), ℓ (actual amount of labor supplied) and y (national income or product). An initial version of the model might be

$$c^d = \alpha_1 y + \alpha_2 z_1 + u_1 \qquad (4\text{-}1)$$

$$c^s = \alpha_3 y + \alpha_4 z_2 + u_2 \tag{4-2}$$

$$c = \min(c^d, c^s) \tag{4-3}$$

$$\ell^d = \alpha_5 z_3 + u_3 \tag{4-4}$$

$$\ell^s = \alpha_6 c + u_4 \tag{4-5}$$

$$\ell = \min(\ell^d, \ell^s) \tag{4-6}$$

$$y = \alpha_7 \ell + \alpha_8 z_4 + u_5 \tag{4-7}$$

Equations (4-1) and (4-2) express the dependence of consumption demand and supply on total income. Equation (4-3) is the usual min condition. Equation (4-4) states, somewhat unrealistically, that labor demand is exogenous, suggesting that it depends only on central plan targets established perhaps as a function of previous years' results. Equation (4-5) states that labor supply depends on the actual volume of consumption and (4-6) is again the usual min condition. Equation (4-7) is in effect the production function.

Our present purpose here is not to belabor how appropriate or inappropriate the particular model may be for describing centrally planned economies, but to examine some of its properties. It is a convenient vehicle by which we can suggest the rather general conjecture that all multimarket disequilibrium models will tend to have either multiple solutions or no solution (in other words, to be inconsistent). The latter problem makes econometric estimation pointless; the former, however, is not necessarily a problem from the econometric point of view[8] but does indicate that the theoretical model is incomplete in some sense.

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[8] See [7] for a treatment of estimation in the case of multiple solutions to structural equation systems.

Both possibilities are demonstrated with the aid of Figure 1. In the first quadrant we represent the consumption demand and supply relations with y on the horizontal axis and consumption quantities on the vertical. All exogenous variables and error terms are assumed to be given for the time period in question and hence the position of all functional relations is momentarily given. Quadrant 2 contains the labor demand and supply functions with labor quantities measured to the left of the origin. Two alternative labor demands are drawn, ℓ_1^d and ℓ_2^d ; we concentrate on the case in which ℓ_1^d is the labor demand. The actual amount of labor determined in quadrant 2 is transferred with the aid of a 45° line in quadrant 3 to the bottom part of the vertical axis and the production function in quadrant 4 determines y . It is clear that y_0, c_0, ℓ_0 represent one solution to the system and y_1, c_1, ℓ_1 represent another. It is easy to verify that the first solution corresponds to excess demand and the second to excess supply in the market for consumer goods; both solutions involve excess demand for labor. In general, to determine whether a set of values is a solution to the system we merely have to verify whether a path such as $y_0A_0c_0B_0\ell_0D_0$ returns to y_0 or whether the path $y_1A_1c_1B_1\ell_1D_1$ returns to y_1 .

What is unsatisfactory about having more than one solution is that so far there is nothing in the model that describes how nature picks among the two possible solutions. Thus, if the exogenous variables and error terms have such values as to place the various functions in the places they occupy in Figure 1, will we actually observe y_0, c_0 and ℓ_0 or y_1, c_1 and ℓ_1 ? One answer to this problem is to specify appropriate dynamics for the model which would make one of the solutions unstable and hence unlikely to be observed. Another answer consists of adding additional relations to close the model and eliminate the indeterminacy. One might wish to define the additional variable I (investment) and assume

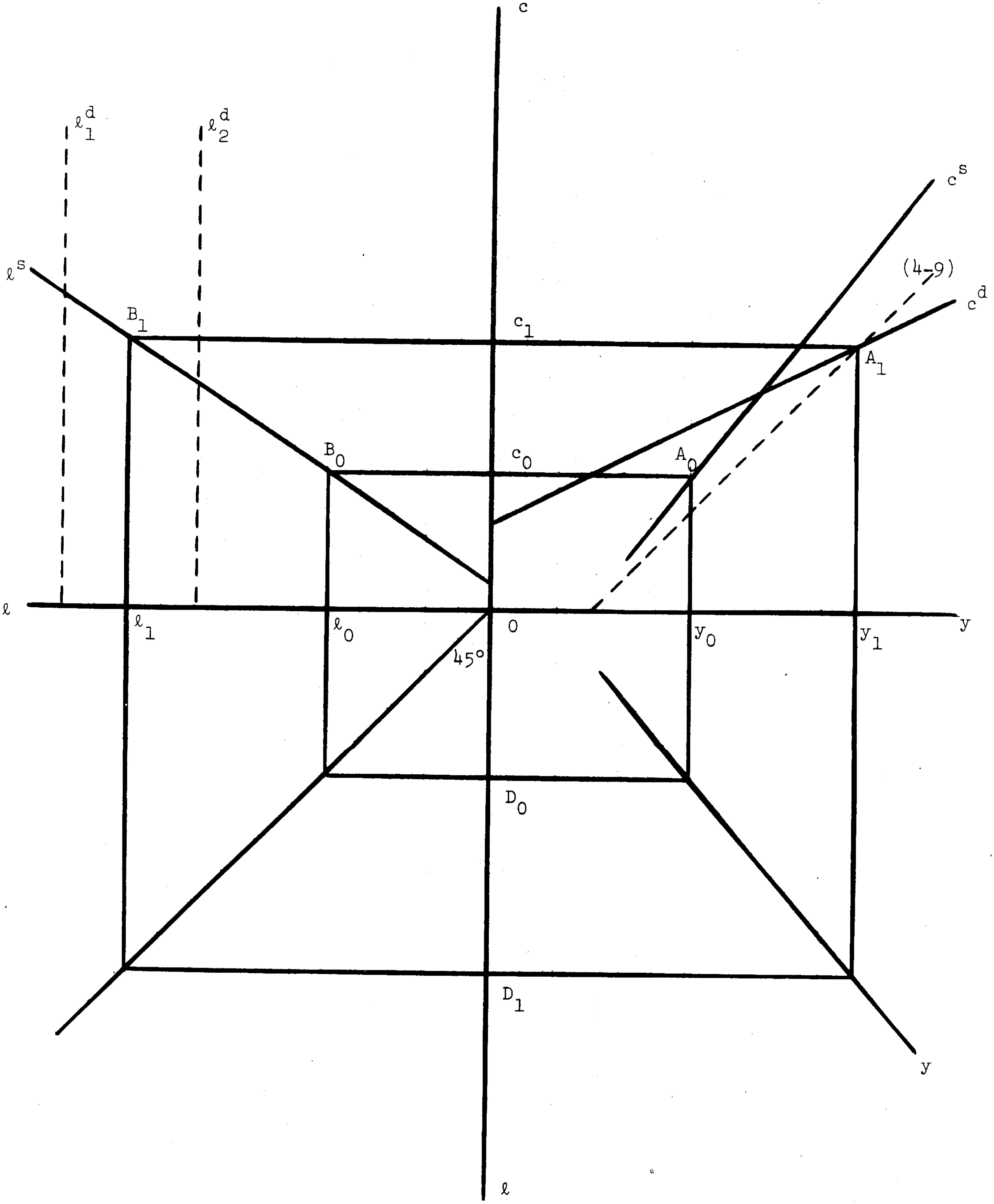

FIGURE 1

$$I = \alpha_9 z_5 + u_6 \qquad (4\text{-}8)$$

i.e., that investment is given exogenously. We also have the obvious identity

$$y = I + c \qquad (4\text{-}9)$$

But now the model may have no solution at all; in fact it is overdetermined with probability 1. Equation (4-9) is a 45° line in quadrant 1 of Figure 1 with intercept equal to $-I$; unless the value of I from (4-8) is such as to make (4-9) be represented exactly by the dashed line through A_1 in quadrant 1, y_1, c_1, ℓ_1 cannot be a solution; unless the dashed line goes through A_0, y_0, c_0, ℓ_0 cannot be a solution. In general, therefore, this enlarged model will have no solution. A more detailed analysis of a similar model appears in the Appendix.

It is noteworthy that one need not have the rather complicated structure of this model to get into difficulties. We shall now state an extremely simple two-market model with no attempt to depict any particular economic circumstances, although the reader may prefer to think of Q as denoting goods and L as denoting labor. We have the following relations, where we reintroduce the time subscript t :

$$D_t^Q = \alpha_1 L_t + \alpha_2 z_{1t} + u_{1t} \qquad (4\text{-}10)$$

$$S_t^Q = \alpha_3 L_t + \alpha_4 z_{2t} + u_{2t} \qquad (4\text{-}11)$$

$$Q_t = \min(D_t^Q, S_t^Q) \qquad (4\text{-}12)$$

$$D_t^L = \alpha_5 Q_t + \alpha_6 z_{3t} + u_{3t} \qquad (4\text{-}13)$$

$$S_t^L = \alpha_7 Q_t + \alpha_8 z_{4t} + u_{4t} \qquad (4\text{-}14)$$

$$L_t = \min(D_t^L, S_t^L) \qquad (4\text{-}15)$$

Equations (4-10) through (4-12) may be thought to represent the market for goods where both the (notional or effective) demand and supply depend on the actual amount of labor traded (and other exogenous variables). Equations (4-13) through (4-15) may be thought to represent the labor market and both the demand and supply for labor depend on the actual amount of goods traded. The reader may verify with a demonstration similar to that employed in Figure 1 that cases with multiple solutions or with no solution are possible, depending on the slopes and intercepts of the structural equations.

The problem bears a superficial resemblance to the case of simultaneous equations with limited dependent variables discussed by Amemiya [1], but appears to be more difficult. One way of formulating it in a completely general way is to assume that there are n interrelated markets. Let y_i $(i=1,3,5,\ldots,2n-1)$ be the demand and y_i $(i=2,4,\ldots,2n)$ the supply in the ith market. Define $z_i = \min(y_i, y_{i+1})$, $(i=1,3,\ldots,2n-1)$. Denote by y the 2n-vector of the y_i, by z the n-vector of the z_i and by b a 2n-vector of constants. Then we can write a general system of the type of (4-10) to (4-15) as

$$By + Cz = b \qquad (4\text{-}16)$$

where B is a $2n \times 2n$ and C a $2n \times n$ matrix of coefficients. The question then is, under what restriction on B, C and b does (4-16) have no solution, exactly one solution or more than one solution. The vector z is a subvector of y and there are 2^n possible such subvectors. Indexing these subvectors by j and collecting terms on the elements of y in y and z we can rewrite (4-16) as

$$B_j y = b \qquad (j=1,\ldots,2^n) \qquad (4\text{-}17)$$

where B_j contains as its kth column either the kth column of B (if y_k does not appear in z) or the sum of the kth column of B and the column in C corresponding to y_k (if y_k does appear in z). Each of these 2^n systems may have (a) no solution if either (i) $\text{rank}(B_j) < \text{rank}([B_j \;\; b])$ or (ii) if the resulting solution has the property for any odd value of k that y_k is included in z and $y_k > y_{k+1}$ or y_{k+1} is included in z and $y_k < y_{k+1}$; (b) exactly one solution if $\text{rank}(B_j) = \text{rank}([B_j \;\; b]) = 2n$ and the condition in (a) (ii) does not occur; (c) an infinity of solutions if $\text{rank}(B_j) = \text{rank}([B_j \;\; b]) < 2n$ and (a) (ii) does not occur. The system as a whole has a unique solution if and only if (b) occurs for exactly one value of j and (a) occurs for all others. The reader may easily construct examples violating this condition.

5. The Likelihood Function in Many Market Models

There are two major cases to be examined. The first one is the case in which none of the relations defining the unobservable quantities (such as D^Q_t, S^Q_t, D^L_t, S^L_t) contains any actual observed quantity (such as Q_t or L_t) . If this is the case (and, of course, this is <u>not</u> true for the model given by (4-10) through (4-15) unless $\alpha_1 = \alpha_3 = \alpha_5 = \alpha_7 = 0$ a priori), then the derivation of the pdf for the observable quantities is not different in principle from that in the single market case. For we can then derive from the point pdf of the error terms the joint pdf $g(D^Q_t,S^Q_t,D^L_t,S^L_t)$. We can write

$$\begin{aligned} h(Q_t,L_t) = {} & h(Q_t,L_t \mid D^Q_t<S^Q_t, D^L_t<S^L_t) \Pr\{D^Q_t<S^Q_t, D^L_t<S^L_t\} + \\ & + h(Q_t,L_t \mid D^Q_t<S^Q_t, D^L_t\geqq S^L_t) \Pr\{D^Q_t<S^Q_t, D^L_t\geqq S^L_t\} + \\ & + h(Q_t,L_t \mid D^Q_t\geqq S^Q_t, D^L_t<S^L_t) \Pr\{D^Q_t\geqq S^Q_t, D^L_t<S^L_t\} + \\ & + h(Q_t,L_t \mid D^Q_t\geqq S^Q_t, D^L_t\geqq S^L_t) \Pr\{D^Q_t\geqq S^Q_t, D^L_t\geqq S^L_t\} \end{aligned} \qquad (5\text{-}1)$$

which is analogous to (2-4) and where a typical term is, for example,

$$h(Q_t,L_t|D_t^Q<S_t^Q,D_t^L<S_t^L) = \int_{Q_t}^{\infty}\int_{L_t}^{\infty} g(Q_t,S_t^Q,L_t,S_t^L|D_t^Q<S_t^Q,D_t^L<S_t^L)dS_t^Q dS_t^L =$$

$$= \frac{1}{\Pr\{D_t^Q<S_t^Q,D_t^L<S_t^L\}} \int_{Q_t}^{\infty}\int_{L_t}^{\infty} g(Q_t,S_t^Q,L_t,S_t^L)dS_t^Q dS_t^L \qquad (5\text{-}2)$$

Hence $h(Q_t,L_t)$ becomes

$$h(Q_t,L_t) = \int_{Q_t}^{\infty}\int_{L_t}^{\infty} g(Q_t,S_t^Q,L_t,S_t^L)dS_t^Q dS_t^L + \int_{Q_t}^{\infty}\int_{L_t}^{\infty} g(Q_t,S_t^Q,D_t^L,L_t)dS_t^Q dD_t^L$$

$$+ \int_{Q_t}^{\infty}\int_{L_t}^{\infty} g(D_t^Q,Q_t,L_t,S_t^L)dD_t^Q dS_t^L + \int_{Q_t}^{\infty}\int_{L_t}^{\infty} g(D_t^Q,Q_t,D_t^L,L_t)dD_t^Q dD_t^L \qquad (5\text{-}3)$$

and the likelihood function is just the product of terms such as (5-3).[9]

The second major case is the generally more difficult one when the functional relations or structural equations contain the actual values of some variables produced by invoking the min conditions. In this case the conditional densities appearing in (5-1) are not known and hence that expression is not useful. We shall proceed somewhat differently and the motivation may be given in terms of the manner in which nature generates for us observable quantities Q_t , L_t .

Assume that a set of values for all the exogenous variables and for the four error terms is given. Then a pair (Q_t,L_t) is a solution to (4-10) to (4-15) if the value of L_t , when substituted in (4-10) and (4-11) yields a Q_t from (4-12) which, when substituted in (4-13) and (4-14) yields the same value of L_t employed in the first place.

From Equations (4-10), (4-11) and (4-12) we can obtain $f(Q|L)$; that is

[9]The reader will note that the absence of a price adjustment equation will again allow unboundedness of the likelihood function as in Section 3.

the density of the random variable Q conditional on the values of L. If $g(u_{1t}, u_{2t})$ is the joint pdf of u_{1t}, u_{2t}, then $g(D_t^Q - \alpha_1 L_t - \alpha_2 z_{1t}, S_t^Q - \alpha_3 L_t - \alpha_4 z_{2t})$ is the joint density of D_t^Q, S_t^Q conditional on L_t and from the single market model

$$f(Q_t\ L_t) = \int_{Q_t}^{\infty} g(Q_t - \alpha_1 L_t - \alpha_2 z_{1t}, S_t^Q - \alpha_3 L_t - \alpha_4 z_{2t}) dS_t^Q + \int_{Q_t}^{\infty} g(D_t^Q - \alpha_1 L_t - \alpha_2 z_{1t}, Q_t - \alpha_3 L_t - \alpha_4 z_{2t}) dD_t^Q \qquad (4\text{-}16)$$

To obtain $h(Q_t, L_t)$ we further require the marginal pdf of L_t, $g(L_t)$, for then

$$h(Q_t, L_t) = f(Q_t | L_t) g(L_t) \qquad (4\text{-}17)$$

To obtain $g(L_t)$ we note that (4-13), (4-14), (4-15) can be used to obtain $\phi(L_t|Q_t)$ analogously with $f(Q_t|L_t)$. The restriction that the observed values of Q_t, L_t be a solution to the system (4-10) to (4-15) is sufficiently strong to allow us to derive the required marginal density from conditional densities alone; a procedure not normally possible. We now supress the subscript t and subdivide the real line into intervals of finite width; Q_i and L_j will denote the value of Q and L in their ith and jth intervals respectively. For any value L_j the probabilities of the events $Q_i|L_j$ are given by $f(Q_i|L_j)dQ_i$ where dQ_i is the width of the ith interval. For each possible outcome Q_i there is a range of possible outcomes $L_k|Q_i$ with probabilities $\phi(L_k|Q_i)dL_k$. In Figure 2 we measure L on the horizontal axis, both to the right and to the left of the origin; we measure Q on the vertical axis. In the right hand panel we depict $f(Q|L_1)$. For the value L_1 of L $f(Q|L_1)$ gives the densities of the various Q-outcomes. For two of the infinitely many such outcomes, Q_1 and Q_2, we show in the left hand panel $\phi(L|Q_1)$

and $\phi(L|Q_2)$. Each of these Q-values gives rise to infinitely many possible L-values. Of all these L-values, however, only one is observable due to the restriction that L-values solve (4-10) to (4-15), namely the value L_1 . The relative probabilities of observing various L-values, say L_1 , are given by the sum of products of probability elements shown by the shaded areas, i.e. by $P_1P_3 + P_2P_4$ + etc. More formally the mechanism establishes a transition matrix from L-values to L-values (through intermediate Q-values) of which only the diagonal elements can even be realized. The diagonal elements of such a transition matrix are given by $\sum_i \phi(L_j|Q_i)f(Q_i|L_j)dQ_i dL_j$ and in order for these to be probabilities they have to be normalized by the sum of diagonal elements. Hence

$$g(L_j)dL_j = \frac{\sum_i \phi(L_j|Q_i)f(Q_i|L_j)dQ_i dL_j}{\sum_j\sum_i \phi(L_j|Q_i)f(Q_i|L_j)dQ_i dL_j} \qquad (4\text{-}18)$$

Let the intervals dL_j , $dQ_i \to 0$ we obtain as the limit

$$g(L) = \frac{\int \phi(L|Q)f(Q|L)dQ}{\int\int \phi(L|Q)f(Q|L)dQdL} \qquad (4\text{-}19)$$

Combining (4-17) with (4-18) gives the density function $h(Q_t, L_t)$ from which the likelihood function is obtainable in the usual fashion.

6. Tests of Hypotheses

For purposes of discussing tests of hypotheses we return to the single-market models discussed in Section 2.[10] A market in which observations on prices and quantities are available may be modeled either as an equilibrium market or as a disequilibrium market. It may be of considerable practical significance to be able to say which specification is likely to be the correct one. Thus there arises the question of testing the null hypothesis

[10] For a more detailed discussion the reader is referred to [12].

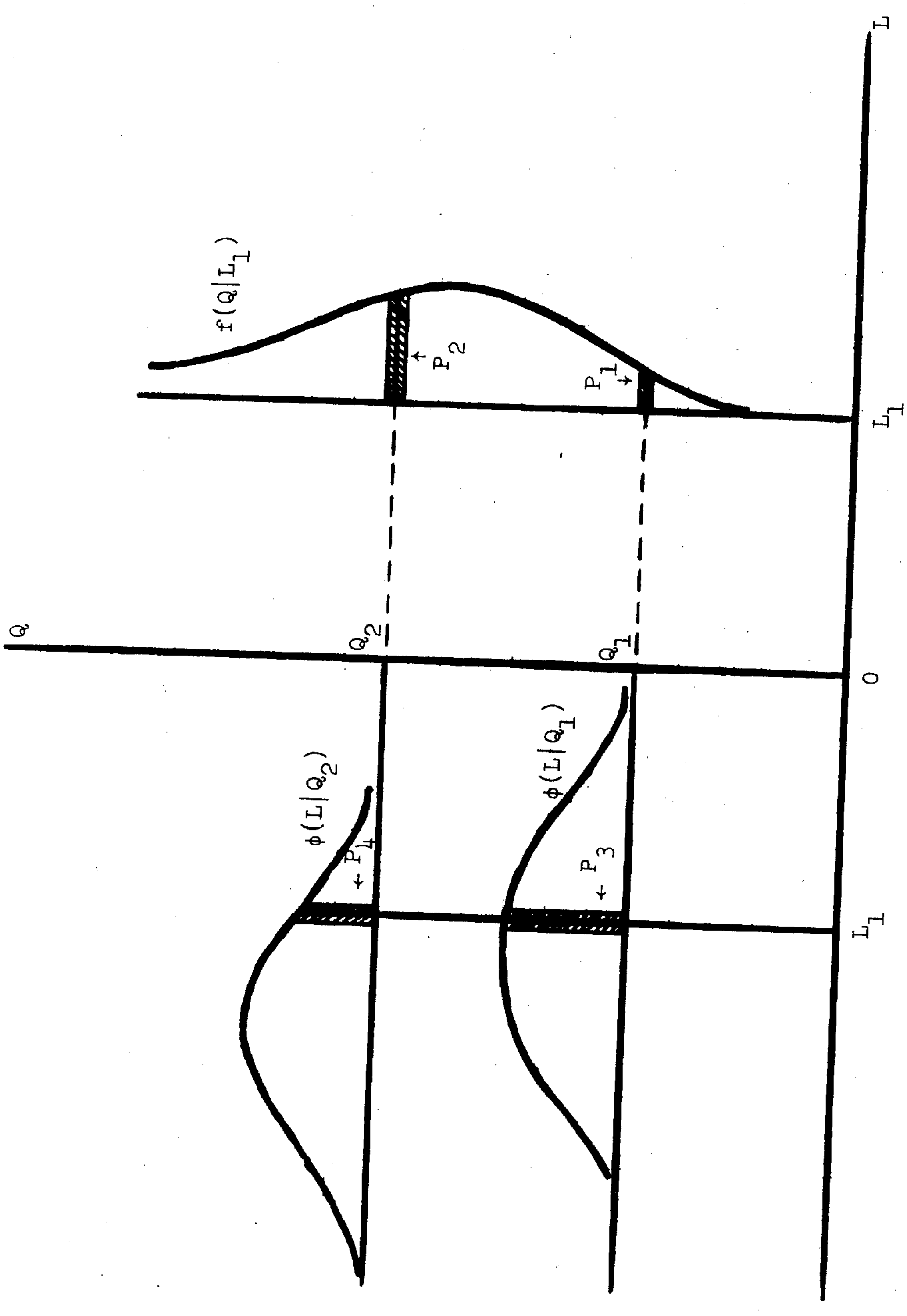

FIGURE 2

of equilibrium as against the alternative that the market is in disequilibrium.[11] It is obvious that this question is an interesting one only if there is no prior knowledge available that $D_t < S_t$ or $D_t > S_t$.

Preliminary investigations suggest that there may not be a uniformly most suitable procedure for testing the null hypothesis. In fact, the best procedure may be dependent on the particular manner in which the equilibrium and disequilibrium models have been formulated. It has been shown in [12], for example, that in the Suits model discussed in Section 2 there are at least two different ways in which "equilibrium" could be understood to exist. In an ordinary demand-supply situation a price adjustment equation may or may not have been specified and if it has, it may have been specified in any number of different ways. Below we mention three test procedures that appear to be promising in a fairly broad set of circumstances.

(1) If a price adjustment equation is specified in the form

$$p_t = p_{t-1} + \gamma(D_t - S_t) + u_{3t} \qquad (5\text{-}1)$$

it is easy to show by solving the appropriate difference equation that if $\gamma \to \infty$, $p_t \to p_e$, the equilibrium price. On this basis it has been suggested by Fair and Jaffee [5] that one might test the hypothesis of equilibrium by estimating the disequilibrium model and testing whether $1/\gamma$ is significantly different from zero. If it is, γ is small (statistically speaking) and the hypothesis of equilibrium is rejected. Clearly the procedure is feasible only if the model does incorporate an appropriate price adjustment equation.

[11] It is also possible to consider cases in which the market is in equilibrium some of the time and in disequilibrium some of the time; however, models of this type have not been sufficiently investigated yet. The most straightforward statistical model permitting this phenomenon is a mixture model in which the density function of the random variables q_t and p_t is a convex combination of the two separate density functions corresponding to the equilibrium and disequilibrium models respectively; these being weighted by the unknown parameters λ_1, λ_2 $(\lambda_1+\lambda_2=1)$.

Failing this other techniques have to be investigated.

(2) Likelihood ratio tests. It seems natural to maximize the likelihood function for the equilibrium model and the disequilibrium model separately and form the ratio λ of the two likelihoods. Superficially, equilibrium would appear to be a special case of disequilibrium and hence in forming the likelihood ratio one would place the maximum likelihood value for the equilibrium model in the numerator. Thus λ would appear to be bounded by 0 and 1 and one might argue that $-2\log\lambda$ has asymptotically χ^2 distribution with degrees of freedom equal to the number of coefficient restrictions implied by the equilibrium model. Unfortunately, things are not nearly as simple. It is shown in [12] that the hypothesis of equilibrium is not necessarily a nested hypothesis[12] in which case the ordinary likelihood ratio is not necessarily bounded above by unity; the theorem about the asymptotic distribution of λ then obviously does not hold. A trivial example of this is provided by the case in which the disequilibrium model is specified as (2-1) to (2-3); the disequilibrium model has exactly as many parameters as the corresponding equilibrium model and hence the density function for either model cannot be a subset of the class of density functions representing the other model. Special and numerically or mathematically very difficult procedures may have to be employed in such cases to solve the problem of hypothesis testing and are based either on deriving the parameters of the asymptotic distribution of the (generalized) likelihood ratio of or estimating a composite model in which both the equilibrium and disequilibrium cases are artificially embedded. Given the difficulty of these procedures, it may be most practical to choose among the alternative

[12] For a discussion of tests in the case of nonnested hypotheses see Cox [3], [4] and Quandt [11].

models on the basis of whether λ is bigger or smaller than 1 (if the two models have the same number of parameters). If the disequilibrium model has more parameters, it may be most practical to form the likelihood ratio in the usual manner and disregard the nonnested nature of the hypotheses, since for large sample values of $\lambda > 1$ will occur only rarely.

(3) It may be preferable to (2) to compute an analogue to the posterior odds based on uninformative prior distributions, namely the ratio of the mean likelihoods over some suitable region of the parameter space. This is clearly an expensive procedure if low levels of accuracy are to be avoided since it is equivalent to a many dimensional numerical quadrature.

Some success with each of these procedures has been achieved but it is clear that a great deal more remains to be done. The same can be said for the fomulation of models of the type discussed in Sections 4 and 5. What is quite clear, however, is that if continued progress is achieved in these areas, we shall be able to obtain a significant practical payoff, namely to estimate macro-disequilibrium models and test them against their equilibrium counterparts.

Appendix

We rewrite in linear form the essential parts of Portes' simplified system given in his Equations (24) [10]. The system is

$$y = \alpha_1 \ell + \alpha_2 \tag{A-1}$$

$$c^s = \alpha_3 y + \alpha_4 \tag{A-2}$$

$$c^d = c^o \tag{A-3}$$

$$c = \min(c^d, c^s) \tag{A-4}$$

$$\ell = \begin{cases} \ell^o & \text{if } c^s \geqq c^d \\ \alpha_5 c + \alpha_6 & \text{if } c^s < c^d \end{cases} \tag{A-5}$$

where we have suppressed for convenience error terms and exogenous variables and lumped them together in the coefficients $\alpha_2, \alpha_4, \alpha_6$ and in c^o and ℓ^o. Consider the two basic possibilities:

(1) $c^d \leqq c^s$. Then

$$\ell = \ell^o \tag{A-6}$$

$$c = c^o \tag{A-7}$$

$$y = \alpha_1 \ell^o + \alpha_2 \tag{A-8}$$

describes the subsystem in effect. This system would fail to have a solution if the value of y from (A-8), call it y^o, were such that $\alpha_3 y^o + \alpha_4 < c^o$. Since α_1 and α_3 are clearly positive, a solution will exist if ℓ^o is sufficiently large; moreover in that case the solution is clearly unique. We posit this to be the case.

(2) If $c^d > c^s$ the subsystem is

$$y = \alpha_1 \ell + \alpha_2$$

$$c = \alpha_3 y + \alpha_4$$

$$\ell = \alpha_5 c + \alpha_6$$

or

$$\begin{bmatrix} 1 & -\alpha_1 & 0 \\ -\alpha_3 & 0 & 1 \\ 0 & 1 & -\alpha_5 \end{bmatrix} \begin{bmatrix} y \\ \ell \\ c \end{bmatrix} = \begin{bmatrix} \alpha_2 \\ \alpha_4 \\ \alpha_6 \end{bmatrix}$$

No solution to this subsystem will exist and hence the solution to the overall system will be unique if $\alpha_1\alpha_3\alpha_5 = 1$ and the rank of the augmented matrix

$$\begin{bmatrix} 1 & -\alpha_1 & 0 & \alpha_2 \\ -\alpha_3 & 0 & 1 & \alpha_4 \\ 0 & 1 & -\alpha_5 & \alpha_6 \end{bmatrix} \qquad \text{(A-9)}$$

is equal to three. Exactly one solution to the subsystem and hence two solutions to the overall system exist if $\alpha_1\alpha_3\alpha_5 \neq 1$ and an infinity of solutions will exist if $\alpha_1\alpha_3\alpha_5 = 1$ and the rank of (A-9) is also two.

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