PROGRAMMING THE SUPPLY OF STRATEGIC MATERIALS

I. Manganese Model No. 1

II. Computing Manganese Model No. 1

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MANGANESE MODEL NO. 1

Herman F. Karreman

Introduction

The model presented here is the result of an attempt to formulate in an exact way the various possibilities of obtaining manganese. The objective is to determine the cheapest method of providing the economy of the United States, in particular its iron and steel industry, with the amount of manganese which it will need in the future. It should, however, be remembered that the effectiveness of this model has not yet been proven and that some experimentation may make it necessary to introduce minor corrections or changes.

The model is a dynamic one in the sense that it is designed to cover the needs during some future time period, say from 1958 through 1970. The results will therefore be highly dependent on the amounts of manganese needed, on the prices of foreign ores and on the costs of beneficiating the domestic ores in the years to come. The better the ideas are about the magnitudes of these future quantities, prices and costs, the more useful the results will be. It should, however, be stressed that this model is based on a given technology and major changes in the latter (i.e.changes other than mere variation in prices and costs) during the period under consideration will necessitate the construction of another model.

The objective function is in this case a cost function, for which a minimum has to be sought. It is in essence a non-linear function, containing in its simplest form linear and quadratic terms. It might prove to be necessary to replace some of the quadratic terms by linear ones in order to preserve convexity of the objective function which is required for

actually performing the computations. The non-linearities, which do not impair this convexity, will however be retained in order to do as much justice to the economic aspects of the problem as possible. A technique will therefore have to be developed for computing the extreme value of an objective function containing linear and quadratic terms.

The United States economy has been treated in this model as a unit and no distinction has so far been made between private industry and government. Consequently it will not be possible to determine, on the basis of this model, which part of the cost of the program will have to be carried by each of them. Also no provision has been made so far for the possibility that one or both would like to limit the amounts that they are willing to spend on importation of foreign or on beneficiation of domestic ores. These further complications will be introduced in subsequent models.

The model has only a limited number of constraints: the amounts of manganese that can be extracted from the domestic ore bodies are in fact the only important limitations set to the production of manganese from domestic ores. It can, however, be expected that other factors will also restrict the amount of manganese that can be produced in critical times. The production of electric energy that will be available in such times for the production of the alloys or the quantities of beneficiated ores that can be transported in an emergency to the alloy plants, for instance, can also hamper the domestic production of manganese. These other restrictions will be incorporated in later models.

There are other aspects of the manganese problem which have not been taken care of in this first model in order to make it not more complicated than necessary. There is, for instance, definitely a game theoretical

element in the picture resulting from the fact that a neutral country like India supplies almost half the amount of ore imported by the United States and could as a consequence act effectively as a player in a game of strategy. Also, no attention has been paid in this model to the location of plants, particularly of alloy plants, if it turns out that so much more capacity is needed that new plants have to be built. The impact of a large scale beneficiation of domestic ores on other parts of the economy of the United States has not yet been taken into account either.

It will be clear from the preceding that this model is only a first approach, as is also indicated by the title of this paper, to the solving of this sort of problems by the application of more exact techniques. It was definitely intended to keep this model as simple as possible by concentrating on the essence of the problem and neglecting the other aspects of it. Models to be developed later will be more detailed and will take the other constraints as well as the game-theoretical aspects, the location of plants and the impact on other parts of the economy into account too. What model no. 1 achieves is summarized in Section III.

The Model

The amounts of manganese that will be needed by the steel industry[1] in the future will be denoted by ω_i N(et) T(ons); it is assumed that these amounts are given for each future time period i $(i = 1,2,\ldots,n)$.

Manganese can be used by the steel industry in a number of forms, but

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[1] Some manganese will also be needed by the chemical industry and in the manufacture of batteries; these amounts are, however, small in comparison with what the steel industry needs and will therefore be neglected here.

only two of them, viz. ferro- and silicomanganese, are important enough to be included in this model. So far, ferromanganese has predominantly been used in steel making practice and it still covers almost 90 per cent of the current needs. Silicomanganese has become increasingly important throughout the years and covers at the moment almost 10 per cent of the manganese requirements. There are indications that in times of emergency a considerable portion (roughly about 50 per cent) of the ferromanganese could be substituted by silicomanganese. Whether this will actually happen depends to a large extent on the difference in cost of producing ferro- and silicomanganese in those times. And there are good reasons to believe that a substantial portion of the domestic sources, viz. the silicate ores and slags can be converted at a lower cost into silicomanganese than into ferromanganese. Therefore, a distinction has been made in the model between:

1. that part of the manganese requirements for which only ferromanganese can be used; the N.T. of manganese demanded in this form in period i will be denoted by λ_i.
2. that part for which ferro- as well as silicomanganese can be used; these N.T. of manganese will be denoted by μ_i.
3. that part for which only silicomanganese can be used; these N.T. of manganese will be denoted by ν_i.

This leaves us with the equality

$$\omega_i = \lambda_i + \mu_i + \nu_i \quad \text{for every } i \ (i = 1,2,\ldots,n) .$$

The λ's, μ's and ν's are supposed to be given for each time period just like the ω's. What is not known, however, is how much of the μ_i N.T. will have to come as ferromanganese and how much as silicomanganese;

this will depend on the costs to be made to obtain the one or the other. Denoting the parts of the μ_i N.T. that have to come in the form of ferromanganese by $x_{1;i}$ N.T. and in the form of silicomanganese by $x_{2;i}$ N.T we have the equality

(1) $$\mu_i = x_{1;i} + x_{2;i} \quad \text{for every} \quad i \quad (i=1,2,\ldots,n) .$$

The first condition is now that there is sufficient ferromanganese in every time period to meet the requirements. Denoting the amount of manganese to be supplied in the form of ferromanganese in period i by $x_{1,i}$ N.T., we have the inequality

(2) $$x_{1,i} \geq \lambda_i + x_{1;i} \quad \text{for every} \quad i \quad (i=1,2,\ldots,n)$$

and similarly for silicomanganese

(3) $$x_{2,i} \geq x_{2;i} + \nu_i \quad \text{for every} \quad i \quad (i=1,2,\ldots,n) .$$

Ferromanganese can be produced in what is called "blast" furnaces and in electric furnaces. In the past most of the ferromanganese has been produced by the first method, which seems, under normal conditions, also to be the most economical. However, in times of emergency pig iron could very well compete with ferromanganese for blast furnace capacity. On the other hand, other critical materials are made in the same electric furnaces as used for ferromanganese. It is, therefore, quite conceivable that in times of emergency only restricted amounts of blast and electric furnace capacity will be available for the production of ferromanganese. In order to facilitate the inclusion of these restrictions in a more comprehensive model, to emerge from this one, a distinction has been made between the

two ways of producing ferromanganese.[1]

Denoting the N.T. of manganese to be produced as ferromanganese in blast furnaces in period i by $x_{3,i}$

in electric furnaces in period i by $x_{4,i}$

we have the equality

$$(4) \qquad x_{1,i} = x_{3,i} + x_{4,i} \quad \text{for every } i \ (i=1,2,\ldots,n) .$$

The ores from which the ferromanganese is to be manufactured, one way or the other, have to have certain qualifications, which are rather strict: around 45 per cent manganese, around 6 per cent iron and not more than about 6 per cent silicon and aluminum together. A substantial part of the manganese in these high-grade non-silicate ores will be recovered in the process of making ferromanganese, but some of it will be lost. The amount of manganese contained in the high-grade non-silicate ore needed to produce the $x_{3,i}$ N.T. manganese in the form of blast furnace ferromanganese in period i will therefore be $c_{3,i}x_{3,i}$ N.T. , where the $c_{3,i}$'s are all greater than 1. Moreover, there will not be much variation in these $c_{3,i}$'s since the process of making blast furnace ferromanganese is rather standardized. Hence, it will here be assumed that all future $c_{3,i}$'s will be the same for all periods under consideration.

To obtain the high-grade non-silicate ores containing $c_3x_{3,i}$ N.T. of manganese certain costs have to be made, say $\alpha_{3,i}c_3x_{3,i}$, where $\alpha_{3,i}$

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[1] A flowsheet showing the various processes included in this model in their logical sequence can be found in the Appendix.

stands for the cost to be made in period i to obtain 1 N.T. of manganese in high-grade non-silicate ore. These $\alpha_{3,i}c_3x_{3,i}$ will later be defined as the prices paid for the imported ores or the accumulated cost of mining and upgrading domestic ores. Besides the material cost, there will also be all sorts of other costs to be made for the procurement of the $x_{3,i}$ N.T. of manganese in the form of blast furnace ferromanganese. In general, these other costs will be of two sorts, viz. operating costs other than material costs being the costs of reagents, fuel, labor, etc., on the one hand, and capital costs like depreciation and interest of the invested money, on the other.

Denoting the operating costs other than material costs in period i by $\beta_{3,i}$ and the capital cost by $\gamma_{3,i}$ per N.T. of manganese, the following expression gives the total cost of obtaining the manganese in the form of blast furnace ferromanganese in period i :

$$E_{3,i} = \alpha_{3,i}c_3x_{3,i} + \beta_{3,i}x_{3,i} + \gamma_{3,i}x_{3,i} .$$

The costs of obtaining the $x_{4,i}$ N.T. of manganese in the form of electric furnace ferromanganese in period i are accordingly:

$$E_{4,i} = \alpha_{4,i}c_4x_{4,i} + \beta_{4,i}x_{4,i} + \gamma_{4,i}x_{4,i} .$$

Silicomanganese has been obtained in the past mainly from the same high-grade non-silicate ores as used for the production of ferromanganese. Still, it is known that it can be produced from ores which contain on the average only 30 per cent manganese instead of the 45 per cent required for the production of ferromanganese. Moreover, the ores can contain much more silicon and aluminum than the ferrograde ores, the average being

around 20 per cent. The four most important deposits inside the country contain in appreciable quantities only ores with a manganese content of 10 per cent or lower. Moreover, some of these ores and the slags, another major source of domestic supply of manganese, have a rather high silicon content. It might therefore very well be the best to upgrade the domestic manganese, in particular the high-silicon parts of it, to a silico- rather than a ferro-grade ore. In that case silicomanganese could be obtained from high-grade non-silicate ore, the usual way up to the present, as well as from medium-grade silicate ore. It is with this idea in mind that a distinction has been made between the two ways in which silicomanganese can be obtained.

Denoting the N.T. of manganese contained in silicomanganese to be produced from high grade non-silicate ore in period i by $x_{5,i}$, and

" " medium grade silicate " " " " " $x_{6,i}$

we have the equality

$$(5) \qquad x_{2,i} = x_{5,i} + x_{6,i} \quad \text{for every } i \ (i=1,2,\ldots,n) .$$

In conformity with the ferromanganese costs, the costs of obtaining $x_{5,i}$ N.T. of manganese would then be:

$$E_{5,i} = \alpha_{5,i} c_5 x_{5,i} + \beta_5 x_{5,i} + \gamma_{5,i} x_{5,i}$$

and the cost of obtaining the $x_{6,i}$ N.T. of manganese

$$E_{6,i} = \alpha_{6,i} c_6 x_{6,i} + \beta_6 x_{6,i} + \gamma_{6,i} x_{6,i} .$$

In these expressions c_5 and c_6 stand for the N.T. of manganese contained in the high-grade non-silicate and medium-grade silicate ore

needed to produce 1 N.T. of manganese contained in the silicomanganese that is to be produced.

High-grade non-silicate ores will have to be produced in such quantities as to contain sufficient manganese for the production of all the ferro- and part of the silicomanganese. Denoting these amounts of manganese in each period i by $x_{7,i}$ N.T. we have the equality

$$(6) \qquad x_{7,i} = c_3 x_{3,i} + c_4 x_{4,i} + c_5 x_{5,i} \quad \text{for every } i \ (i = 1,2,\ldots,n) .$$

These high-grade non-silicate ores can be obtained in two ways, viz. by importing foreign ores, by far the most important source of supply in the past, and by upgrading domestic ores.

With respect to the foreign ores, a distinction has been made between the major exporting countries according to their geographical location, viz. South Asia (India), South and West Africa (Union of South Africa, Angola, Belgian Congo, Ghana, and French Morocco), and Latin America (Chile, Brazil, Cuba, and Mexico). The N.T. of manganese contained in the imported ores from

South Asia	in	period	i	will	be	denoted	by	$x_{9,i}$
S. and W. Africa	"	"	"	"	"	"	"	$x_{10,i}$
Latin America	"	"	"	"	"	"	"	$x_{11,i}$.

The cost at which these quantities of imported ores can be obtained can be thought to be composed of the following elements:

a) prices fob the exporting countries in period i $\$\pi_i$ per N.T. Mn; these prices include the cost of mining the ore, of transporting it through the country, of export duties eventually levied on it as well as

the cost of loading it in the harbors; these prices will be determined, among other things, by the quantities to be bought by the United States or, to be more explicit,

$\pi_{9,i}$ will be an increasing function of $x_{9,i}$

$\pi_{10,i}$ " " " " " " $x_{10,i}$

$\pi_{11,i}$ " " " " " " $x_{11,i}$

b) transportation costs in period i $\$\rho_i$ per N.T. Mn ;

for Africa and Latin America these rates are compound rates since the exporting countries of these continents have been lumped together in this model; in the past the ocean freight rates have fluctuated tremendously but, since the quantities of manganese are only a small portion of the total quantities moved overseas, it may be assumed here that the ocean freight rates are independent from the amounts of manganese transported to the United States; the cost of transporting the ores from the Atlantic Coast to the Pittsburgh area is also included in the $\$\rho_i$.

c) cost of convoying the ships in period i $\$\sigma_i$ per N.T. Mn ;

in times of emergency it could be necessary to protect the ships that bring the manganese to the United States against hostile actions at sea ; these costs will in general consist of the cost of operating the destroyers, mine sweepers, etc., but not the "values" of ships and crews lost in those actions, in other words the expenses occurred in excess of what it would otherwise cost to run the ships in those times.

d) import duty in period i $\$\tau_i$ per N.T. Mn ;

the import duties levied on the incoming ores might later prove to be a

very useful instrument in the hands of the government to influence the extent to which future manganese requirements are to be met by imported foreign ores.

The cost of the imported ores will then be:

$$E_{9,i} = x_{9,i}(\pi_{9,i} + \rho_{9,i} + \sigma_{9,i} + \tau_{9,i}) ,$$

and there are similarly expressions for $E_{10,i}$ and $E_{11,i}$.

Upgrading domestic ores has barely passed the pilot plant stage. Still there are reasons to believe that some of the newly developed processes will be able to compete with foreign ores in the future. But there are already more beneficiation processes at the moment than could be included in this model, so a selection had to be made. The criteria applied in making this selection were:

1) as many different sorts of domestic supply should be included as possible (it is for this reason that the Chamberlain Region ores, which resemble in composition very much the Cuyuna Range ores, have been left out)

2) if many processes can be applied to the same type of ore, the ones on which we have the best information have been selected (though the Cuyuna Range ores can be treated in many ways, only three processes have been chosen for this reason).

The following processes,[1] all coming up with a high-grade non-silicate ore have been selected on the basis of these criteria:

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[1] A description of these processes is given in the report: "Beneficiation of United States Manganese Reserves," Economics Research Project 1st Technical Report, 15 January 1957.

a) Artillery Peak ore - Flotation process

the N.T. of manganese contained in the high-grade non-silicate ore derived in this way in period i will be denoted by $x_{12,i}$.

b) Cuyuna Range ore - Dean process by $x_{13,i}$.

- Chemico process by $x_{14,i}$.

- Nossen process by $x_{15,i}$.

c) Aroostook County ore - Sylvester-Dean process by $x_{16,i}$.

d) Open-Hearth Slags - " " " " $x_{17,1}$.

The amounts of manganese contained in the ores that have to be mined or produced in the form of open-hearth slags will then be respectively $c_{12}x_{12,i}$, $c_{13}x_{13,i}$, $c_{14}x_{14,i}$, $c_{15}x_{15,i}$, $c_{16}x_{16,i}$, and $c_{17}x_{17,i}$ N.T. where the c's stand in these cases for the N.T. of manganese contained in the raw material needed to produce 1 N.T. of manganese contained in the high-grade non-silicate ores.

The cost of producing the $x_{12,i}$ N.T. of manganese in the form of high-grade non-silicate ore will then be, in conformity with the cost of producing ferro- and silicomanganese:

$$E_{12,i} = \alpha_{12,i}c_{12}x_{12,i} + \beta_{12,i}x_{12,i} + \gamma_{12,i}x_{12,i}$$

and there are similar expressions for the other quantities of manganese to be produced in this form.

It should be remarked here that, contrary to the production of ferro- and silicomanganese, the plants that will upgrade these domestic ores have still to be set up. The relation between the $\gamma_{12,i}$ up to $\gamma_{17,i}$ and

the construction cost of these plants will be discussed later.

Medium-grade silicate ores will have to be produced in such quantities as to suffice the manganese requirements of that part of the silicomanganese that is not to be derived from high-grade non-silicate ores. Denoting the N.T. of manganese in these medium-grade silicate ores in period i by $x_{8,i}$ we have the equality

$$x_{8,i} = c_6 x_{6,i} \quad \text{for every} \quad i \quad (i = 1,2,\ldots,n) \; . \tag{7}$$

The sources from which these medium-grade silicate ores can be obtained and the processes applicable to these sources are the following:

a) Aroostook County ores - Udy process

the N.T. of manganese contained in the medium-grade silicate ores derived in this way in period i will be denoted by $x_{18,i}$.

b) Aroostook ores - Bureau of Mines process by $x_{19,i}$.

c) Open-Hearth Slags - " " " " " $x_{20,i}$.

d) High Manganese "Hot Metal" - Wright process by $x_{21,i}$.

The amounts of manganese contained in the ores that have to be mined or the slags and high manganese pig iron that has to be produced will then be respectively $c_{18}x_{18,i}$, $c_{19}x_{19,i}$, $c_{20}x_{20,i}$, and $c_{21}x_{21,i}$ N.T. where the c's stand again for the N.T. of manganese contained in the raw material needed to produce 1 N.T. of manganese contained in the medium-grade silicate ore.

The cost of producing the $x_{18,i}$ N.T. of manganese in the form of medium-grade silicate ore will then be:

$$E_{18,i} = \alpha_{18,i} c_{18} x_{18,i} + \beta_{18,i} x_{18,i} + \gamma_{18,i} x_{18,i}$$

and there are similar expressions for the other quantities of manganese to be produced in this form.

The relationship between the $\gamma_{18,i}$ up to $\gamma_{21,i}$ and the cost of constructing the new plants will also be defined later.

The low-grade domestic ores and slags that have to be mined or produced will have to contain so much manganese as will be required for the production of the high-grade non-silicate ores and the medium-grade silicate ores. The required amount of manganese to obtain from each of the sources of domestic supply can directly be derived from the preceding.

Denoting the N.T. of manganese to be extracted in period i from the Artillery Peak by $x_{22,i}$ we have the equality

(8) $$x_{22,i} = c_{12} x_{12,i} \quad \text{for every } i \ (i = 1,2,\ldots,n) .$$

Similarly we have for the Cuyuna Range

(9) $$x_{23,i} = c_{13} x_{13,i} + c_{14} x_{14,i} + c_{15} x_{15,i} \quad (i = 1,2,\ldots,n) ;$$

for the Aroostook County

(10) $$x_{24,i} = c_{16} x_{16,i} + c_{18} x_{18,i} + c_{19} x_{19,i} \quad (i = 1,2,\ldots,n) ;$$

for the Open-Hearth Slags

(11) $$x_{25,i} = c_{17} x_{17,i} + c_{20} x_{20,i} \quad (i = 1,2,\ldots,n) ;$$

and for the High-Manganese "Hot Metal"

(12) $$x_{26,i} = c_{21} x_{21,i} \quad (i = 1,2,\ldots,n) .$$

The cost of obtaining the $x_{22,i}$ N.T. of manganese from the Artillery Peak will then be

$$E_{22,i} = \beta_{22,i}x_{22,i} + \gamma_{22,i}x_{22,i}$$

and similarly for the other low-grade ores and slags.

It will be noticed that there are in these expressions no α-terms representing material costs as there were in the previous ones. The reason for this is, of course, that we are at this stage at the origin of the domestic supply of manganese. The same remark as made before applies to the relationship of the γ's and the cost of constructing the mines.

Having indicated the variables which are included in the model, we can now turn to the restrictions imposed on them. These restrictions are in general of two kinds, viz.

a) to make sure that the required amounts of ore will always be available.

b) to keep the ore to be extracted from the ground within the boundaries set by nature. Where this last restriction is actually a physical constraint over which we have no control the other restrictions are more dictated by a wish to be "on the safe side."

The first "safety" restriction affects the variables x_7 and x_8, the quantities of manganese to be obtained in the form of high-grade non-silicate and medium-grade silicate ores. To be sure that there will always be enough manganese in the first form, the inequality

$$x_{7,i} \leq S_{1,i-1} \qquad (i=1,2,\ldots,n)$$

has been included in the model.

This inequality tells us that the amount of manganese that is needed in this period for the production of ferro- or silicomanganese is always equal to or smaller than the amount of manganese kept available in stock at the end of the previous period.

For the latter can also be written

$$S_{1,i-1} = S_{1,0} + \sum_{j=1}^{i-1} x_{9,j} + \sum_{j=1}^{i-1} x_{10,j} + \sum_{j=1}^{i-1} x_{11,j} + \sum_{j=1}^{i-1} x_{12,j} +$$

$$+ \sum_{j=1}^{i-1} x_{13,j} + \sum_{j=1}^{i-1} x_{14,j} + \sum_{j=1}^{i-1} x_{15,j} + \sum_{j=1}^{i-1} x_{16,j} +$$

$$+ \sum_{j=1}^{i-1} x_{17,j} - \sum_{j=1}^{i-1} x_{7,j} \qquad (i = 1,2,\ldots,n).$$

In other words, the amount of manganese in the form of high-grade non-silicate ore in stock at the end of the $(i-1)^{\underline{st}}$ period is equal to the amount of manganese in that form in stock at the end of period 0 (i.e. the beginning of the 1st period) plus what has been added to that stock by importation or upgrading domestic ores minus what has been taken from it for the production of ferro- and silicomanganese from the first up to the $(i-1)^{\underline{st}}$ period.

Combining the last two relationships we have

$$(13) \qquad \sum_{j=1}^{i} x_{7,j} - \left[\sum_{j=1}^{i-1} x_{9,j} + \sum_{j=1}^{i-1} x_{10,j} + \sum_{j=1}^{i-1} x_{11,j} + \sum_{j=1}^{i-1} x_{12,j} + \right.$$

$$\left. + \sum_{j=1}^{i-1} x_{13,j} + \sum_{j=1}^{i-1} x_{14,j} + \sum_{j=1}^{i-1} x_{15,j} + \sum_{j=1}^{i-1} x_{16,j} + \sum_{j=1}^{i-1} x_{17,j}\right] \leq S_{1,0}.$$

In the same way we have for the amount of manganese contained in medium-grade silicate ore

$$(14) \qquad \sum_{j=1}^{i} x_{8,j} - \left[\sum_{j=1}^{i-1} x_{18,j} + \sum_{j=1}^{i-1} x_{19,j} + \sum_{j=1}^{i-1} x_{20,j} + \sum_{j=1}^{i-1} x_{21,j} \right] \leq S_{2,0}$$

where $S_{2,0}$ stands for the amount of manganese contained in medium-grade silicate ore in stock at the end of period 0 (i.e. beginning of the 1st period).

It can be taken for granted that the amount of manganese in the form of high-grade non-silicate ore in stock at the beginning of the first period will be quite substantial (in other words $S_{1,0}$ is large) where that in the form of medium-grade silicate ore in stock can probably be neglected.

The second "safety" restriction affects the variables x_{12} up to x_{26}, the quantities of manganese to be produced by beneficiation plants to be constructed and mines to be developed. To be sure that, for instance, the $x_{12,i}$ N.T. of manganese in the form of high-grade silicate ore can always be obtained by the flotation process from mined Artillery Peak ore, the inequality

$$x_{12,i} \leq K_{12,0} + \sum_{j=g_{12}}^{i-1} y_{12,j} \qquad (i = 1,2,\ldots,(n-1))$$

has been included in the model. $K_{12,0}$ stands here for the annual (in case the time periods are years) capacity in terms of N.T. of manganese of the flotation plants at the end of period 0 and $\sum_{j=g_{12}}^{i-1} y_{12,j}$ for the yearly increases in this annual capacity up to $(i-1)^{\underline{st}}$ period taking into account a gestation period of g_{12} at the beginning. Hence, this

inequality tells us that the capacity of the flotation plant at the end of the previous period will always be equal to or larger than what is needed to produce by flotation the required amount of manganese in this period. This inequality can also be written in the form

$$(15) \qquad x_{12,i} - \sum_{j=g_{12}}^{i-1} y_{12,j} \leq K_{12,0} \qquad (i = 1,2,\ldots,(n-1)).$$

It should here be remarked that once the capacity has been increased in period i by $y_{12,i}$ N.T. of manganese per year, it will be possible to produce $y_{12,i}$ N.T. a year more in all subsequent time periods $i+1$, $i+2$, $\ldots, i+\ell$ (ℓ being the "life expectancy" of the capacity increase, usually taken to be about 15 years). After those ℓ time periods we must assume that the capacity increase of the i^{th} period will vanish, so that this inequality will give us the "safety" we want as long as

$$i + \ell \geq n - 1 \quad \text{or} \quad n \leq i + \ell + 1 .$$

If, however, n exceeds this time-span, then there is a chance that some of the capacity increases start to drop out. This will depend on the period in which the capacity had to be increased first. In that case the inequality does not give us anymore the guarantee that the required $x_{12,i}$ N.T. of manganese can be produced.

The following variables have been included in the model for the increases in the capacities of the beneficiation plants and mines:

Flotation plants	(treating	Art. Peak ores)	y_{12}	N.T.	Mn	a	year		
Dean	"	"	Cuyuna	"	y_{13}	"	"	"	"
Chemico	"	"	Cuyuna	"	y_{14}	"	"	"	"
Nossen	"	"	Cuyuna	"	y_{15}	"	"	"	"
Sylvester-Dean	"	"	Arst. ores and slags		y_{16}	"	"	"	"

Udy	plants	(treating Aroostook ores)	y_{18}	N.T. Mn a year
Bureau of Mines	"	(" Arst. ores and slags)	y_{19}	" " " "
Wright	mixers	(" "hot metal")	y_{21}	" " " "
Art. Peak	mines		y_{22}	" " " "
Cuyuna	"		y_{23}	" " " "
Aroostook	"		y_{24}	" " " "
Wright	blast furnaces	(producing "hot metal")	y_{26}	" " " "

It will be noticed that the y's have been numbered as much as possible according to the corresponding x's . Further, that no variables have been introduced for capacity increases of Open Hearth furnaces, which produce the slags as waste products; on the other hand, the model includes a constraint for the maximum amount of slags that can be used as will be shown later. The relationships between the x's and their corresponding y's can be formulated by 11 other inequalities (16) up to (26), which are similar to inequality (15).

The cost of increasing the capacity of the flotation plant in period i by $y_{12,i}$ N.T. of manganese per year will be denoted by $\$ I_{12,i}$, which amount is, of course, a function of $y_{12,i}$.

$$I_{12,i} = \psi_{12,i}\, y_{12,i} \ .$$

By investing this amount in the plant it will be possible to produce during ℓ years on the average $y_{12,i}$ N.T. of manganese per year more, or in total $\ell \cdot y_{12,i}$ N.T. of manganese more. It will now be assumed that the $\$ I_{12,i}$ will be equally distributed over the $\ell \cdot y_{12,i}$ N.T. of manganese, irrespective of the fact whether this capacity will be used or not. Hence, every

N.T. of manganese in the form of high-grade non-silicate ore to be produced by the flotation plant from Artillery Peak ore will be charged with a capital cost equal to the amount of

$$\varphi_{12,i} = \frac{I_{12,i}}{\ell \cdot y_{12,i}} \quad .$$

This $\varphi_{12,i}$ is by definition the same as the $\gamma_{12,i}$ used before in the E_{12} cost expression and the same holds for all other capital costs per N.T. of manganese. But this means that, in order to avoid double counting, we can drop the terms $\gamma_{12,i}x_{12,i}, \ldots, \gamma_{18,i}x_{18,i}, \ldots, \gamma_{22,i}x_{22,i}, \ldots$ in those earlier expressions if we include the amounts $I_{12,i}, \ldots, I_{18,i}, \ldots, I_{22,i}, \ldots$, in the formula which expresses the total cost of the entire program (see below).

This still needs a correction. For at the end of n time periods $(n \leq i + \ell + 1)$ there will probably be unused capacity, the more, the later the capacity has been increased (the higher the value of i). It will now be assumed that the capacity that still exists at the end of the n^{th} period will have a "value" per N.T. of manganese per year which is the same as the "value" of the capacity in the years covered by the program. The consequence of this is that the program has only to be charged with a fraction of the investments $I_{12,i}, \ldots, I_{18,i}, \ldots, I_{22,i}, \ldots$, this fraction being $u_i = \frac{n - i}{\ell}$ where $(n - i)$ stands for the number of years that the capacity increase could effectively be used.

The third restriction affects the variables x_{22}, x_{23}, x_{24} and x_{25}, the quantities of low-grade ores that can be extracted from the

Artillery Peak, Cuyuna Range, and Aroostook deposits as well as the O. H. slags. (It has been assumed that Wright's "Hot Metal" can always be produced in sufficient quantities.) As for the first 3 variables, nature has imposed certain limitations on them and not much can be done to change them (in contrast to the restrictions dealt with so far). True, the estimates of the amounts of manganese in the domestic deposits have been increased repeatedly over the years, but such increases are not limitless. These constraints have been formulated as follows:

$$c_{22} \sum_{j=1}^{i-1} x_{22,j} \leq R_{22,0} + \sum_{j=1}^{i-1} r_{22,j} \quad \text{for every } i \quad (i = 1,2,\ldots,(n-1)) .$$

$R_{22,0}$ stands here for the amount of manganese estimated to be present in the domestic deposits at the end of period 0 i.e. the beginning of the first period; r_{22} stands for the yearly increases in these estimated amounts. c_{22} stands for the N.T.'s of manganese that have to be in the ground in order to obtain 1 N.T. of manganese in the ore extracted from it (c_{22} will be high for what is called "deep mining" and low for "strip mining"). Hence, this inequality assures us that our results will not call for the extraction of a greater quantity of manganese from the ground than it really contains.

To simplify the structure of the model and to facilitate the computations, this inequality will be divided through by c_{22}, which leaves us with:

$$(27) \quad \sum_{j=1}^{i-1} x_{22,j} \leq R'_{22,0} + \sum_{j=1}^{i-1} r'_{22,j} \quad \text{for every } i \quad (i = 1,2,\ldots,(n-1))$$

where $R'_{22,0} = \dfrac{R_{22,0}}{c_{22}}$ and $r'_{22,i} = \dfrac{r_{22,i}}{c_{22}}$.

There are similar expressions (28), (29) and (30) for the amounts of manganese that can be extracted from the Cuyuna and Aroostook deposits as well as can be obtained from the O. H. slags.

Finally, the nature of our problem does not allow us to come up with negative values for the x's. Hence, the model contains also the inequalities

(31) $\qquad x_k \geq 0$ for all k $(k=1,1',2,2',3,4,\ldots,26)$.

Having indicated the variables of the model and the relationships among them, we can now turn to the objective function, which is in this case a cost function that has to be minimized. The cost of this program during one period i is the sum of the cost at the various technical levels during that period, which costs have been defined here before. However, we have to be aware of the fact that some cost elements appear in more than one expression. Thus, for instance, the material costs of producing high-grade non-silicate ore from the Artillery Peak deposit $\alpha_{12,i}c_{12}x_{12,i}$ are the same as $(\beta_{22,i} + \gamma_{22,i})x_{22,i}$, the costs of mining the ore from that deposit. The same holds for the material cost terms in all other expressions. Hence, this α-term should be left out in the total cost expression of that period. Further, it has already been explained that the terms $\gamma_{12,i}x_{12,i}$, etc., standing for the capital cost with which the quantities to be produced $x_{12,i}$, etc., will be charged, can be left out once the investments $I_{12,i}$, etc., are included in the total cost expression.

No attention has, however, been paid to the fact that it costs some money to carry the stockpiles of high-graded non-silicate ore $S_{1,i}$ as

well as that of the medium-grade silicate ore $S_{2,i}$ during period i. These costs have also to be included in the total cost. This calls for the inclusion of an additional cost term in the objective function, viz. $\alpha_s(S_{1,i} + S_{2,i})$ for period i, where α_s stands for the cost of carrying 1 N.T. of manganese in stock one time period further. This α_s has been assumed to be the same for both types of ore in stock and, moreover, to be constant for all time periods.

Finally, all future costs have to be discounted, since \$1 million of expenditures in period i represent less "sacrifice" than \$1 million of expenditures in period $(i - 1)$ and the latter again less "sacrifice" than \$1 million of expenditures in period $(i - 2)$. To take this fact into account the costs of period i $(i = 1,2,\ldots,n)$ have to be multiplied by a factor $d_i = (\frac{1}{1 + r})^i$ where r is the rate of interest which is assumed to be constant over time.

Summing now the cost expressions of period i for all time periods, we end up with the total cost expression of the entire program, which can be written in the following explicit form:

$$E = \sum_{i=1}^{n} d_i[(\beta_{3,i}+\gamma_{3,i})x_{3,i}+(\beta_{4,i}+\gamma_{4,i})x_{4,i}+(\beta_{5,i}+\gamma_{5,i})x_{5,i}+(\beta_{6,i}+\gamma_{6,i})x_{6,i}] +$$

$$+\sum_{i=1}^{n-1} d_i[\alpha_s(S_{1,i}+S_{2,i})+(\pi_{9,i}+\rho_{9,i}+\sigma_{9,i}+\tau_{9,i})x_{9,i} +$$

$$+(\pi_{10,i}+\rho_{10,i}+\sigma_{10,i}+\tau_{10,i})x_{10,i}+(\pi_{11,i}+\rho_{11,i}+\sigma_{11,i}+\tau_{11,i})x_{11,i}] +$$

$$+\sum_{i=2}^{n-1} d_i[\rho_{12,i}x_{12,i}+\rho_{13,i}(x_{13,i}+x_{14,i}+x_{15,i})+\rho_{16,i}(x_{16,i}+x_{18,i}+x_{19,i}) +$$

$$+\beta_{12,i}x_{12,i}+\beta_{13,i}x_{13,i}+\beta_{14,i}x_{14,i}+\beta_{15,i}x_{15,i}+\beta_{16,i}x_{16,i}+\beta_{17,i}x_{17,i}+$$

$$+\beta_{18,i}x_{18,i}+\beta_{19,i}x_{19,i}+\beta_{20,i}x_{20,i}+\beta_{21,i}x_{21,i}+\beta_{22,i}x_{22,i}+$$

$$+\beta_{23,i}x_{23,i}+\beta_{24,i}x_{24,i}+\beta_{26,i}x_{26,i}] +$$

$$+\sum_{i=1}^{n-2} d_i u_i(\psi_{12,i}y_{12,i}+\psi_{13,i}y_{13,i}+\psi_{14,i}y_{14,i}+\psi_{15,i}y_{15,i} +$$

$$+\psi_{16,i}y_{16,i}+\psi_{18,i}y_{18,i}+\psi_{19,i}y_{19,i}+\psi_{21,i}y_{21,i} +$$

$$+\psi_{22,i}y_{22,i}+\psi_{23,i}y_{23,i}+\psi_{24,i}y_{24,i}+\psi_{26,i}y_{26,i}).$$

The first 4 terms under the first summation sign in this expression stand for the cost of producing the alloys, being operation costs other than material costs and capital cost. The first term under the second summation sign stands for the carrying cost of the ore in the stock-piles; the ensuing 3 terms represent the total cost of imported ores. The first 3 terms under the third summation sign stand for the cost of transporting the

upgraded ores to the alloy plants (no domestic transportation costs have been charged against the O. H. slags and the "Hot Metal" since these products are made in the same area as the alloys); the following 10 terms represent the cost of upgrading the domestic ores, followed by 4 other terms standing for the cost of mining the ores as well as producing Wright's "Hot Metal". (Since the slags are a waste-product of the steel industry, no costs are involved in producing these slags and the term $\beta_{25,i}x_{25,i}$ has been left out.) The first 8 terms under the fourth and last summation sign stand for the cost of increasing the capacities of the beneficiation plants; the last 4 terms stand for the cost of increasing the capacities of the mines in the 3 deposits and of the blast furnaces which have to produce Wright's "Hot Metal".

This total cost function has to be minimized. As it stands, it is an expression in 23 x-variables (besides the x's appearing explicitly in the formulae, S_1 and S_2 also include x_7 and x_8); in addition the "constraints" matrix contains 3 more x-variables not appearing in this function i.e. x_1, , x_2, and x_{25}. Moreover, there are 12 y-variables. However, by using equalities of the model it is possible to eliminate the variables x_7, x_8, x_{22}, x_{23}, x_{24}, x_{25}, and x_{26}, so that the number of variables can be reduced to 19 x's and 12 y's, 31 all together. These 31 variables are related to each other by 1 definitional equality, 2 inequalities regarding the amounts of ferro- and silico-manganese to be produced, 2 inequalities regarding the importation and beneficiation of ores, 12 inequalities taking care of the capacities of the plants and mines, and 4 inequalities regarding the availability of low-grade manganese

ore and O. H. slags inside the country. Moreover, there are of course the inequalities which have to guarantee us only non-negative values of the x's . The parameters β, γ, π, and ψ are in general functions of their corresponding variables, making the objective function a non-linear one. However, at this point some sacrifices have perhaps to be made in order to find a minimum for the function as will be explained in Section IV which deals with the computational aspects of the problem.

III Summary

Given the levels of manganese requirements, prices and costs assumed to apply to future time periods, the cheapest method of providing the United States economy with the required quantities of manganese will be found by computing the minimum of the cost function above. The result of the computation will indicate not only the cost of the cheapest method, but also what this method implies in terms of quantities of manganese that have to be obtained from importation of foreign ores, from mining and upgrading domestic ores and . from subtracting from stockpiles. The result will also tell us the capacities of domestic mines and beneficiation plants required to carry out the program as well as the amounts of capital that have to be invested in them.

However, the United States economy is treated in this model as a unit; no distinction has been made between private industry and government. Consequently the model cannot tell us to what extent each of these will contribute to the cost of the program. Later models will deal with this topic.

It should also be remarked that this model does not contain a

restriction on the amount of money that can be spent. In other words it is here assumed that the combined efforts of private industry and government will always succeed in providing all the money required to carry out the program. It will be a matter of later consideration whether this is in correspondence with the facts or whether such a restriction has to be included in the model.

Finally it should be emphasized that the whole result is dependent on the quantities of manganese assumed to be required in the future as well as on the prices and costs assumed to apply to it. A change in any of these parameters will almost certainly lead to another result.

IV Computational Aspects

Before starting the computations, the inequalities in the "constraint" matrix have first to be transformed into equalities, which introduces another 20 "slack" variables into the model. After this transformation the number of variables involved in one time period will be 51, related to each other by 21 equalities. This has to be multiplied by the number of time periods that the model is intended to include, in our case 13. To obtain an idea about the amount of computational work that is here involved, let us assume for a moment that the objective function is a linear one so that the well-known simplex method can be applied to it. We would then have to operate with a square matrix, called "basis", of 13 x 21 = 273 rows and columns, which would surpass the capacity of many of the existing electronic computers and certainly the one here in Princeton. This made us search for other methods of computation; such a method can be found in

Wolfe's paper, "Computing Manganese Model No. 1," which is included in this report.

It might also be possible that these computational problems can be solved by taking full advantage of the special structure of the model. This structure comes to the fore if one places the variables and equations for, for instance, 4 times-periods in proper order. It becomes then clear that the overall matrix can be decomposed into a few matrices of not more than 21 rows and 30 to 40 columns, which are grouped in a peculiar way. Because of this special grouping it is possible to transform the overall matrix into one with one of the smaller matrices arrayed along the main diagonal and another small matrix along the sub-diagonal just below it. Such a structure would, according to Dantzig[1], greatly alleviate the computational burden, a possibility which will certainly be explored.

Still it could not be easy to find a minimum for the total costs since the objective function is in essence non-linear in the x's and y's. This results from the fact that the parameters β, γ, π and ψ are in general functions of their corresponding variables contrary to the other parameters ρ, σ, etc., which can be taken to be independent from them. Assuming that β, γ, π and ψ are linear functions of their corresponding x- and y-variables, we end up with an objective function containing linear as well as quadratic x- and y-terms.

Two requirements have, however, to be met by any model to make it in principle computable according to our present knowledge about computations. The first requirement is that all the restrictions imposed on the

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[1] See for this his paper: "On the Status of Multistage Linear Programming Problems," RAND paper No. P-1028, February 20, 1957.

variables are cast in linear form; this requirement is met by this model. The second requirement is that the objective function must be convex to compute its minimum (or maximum); this requirement is here not automatically met. For, an increase in the quantities to be produced could, in the actual situation, lead to a decrease in the production costs β , γ and ψ reflecting in that way "economies of scale" that might be present. This could impair the convexity of the objective function and we might therefore be forced to assume that these parameters are independent from their corresponding variables. This is, according to the industry, a not too stringent assumption, since the technology is such that the quantities to be produced would only slightly affect the costs of production. Hence, the non-convex terms in the objective function could, if necessary, be replaced by linear terms in order to safeguard its convexity.

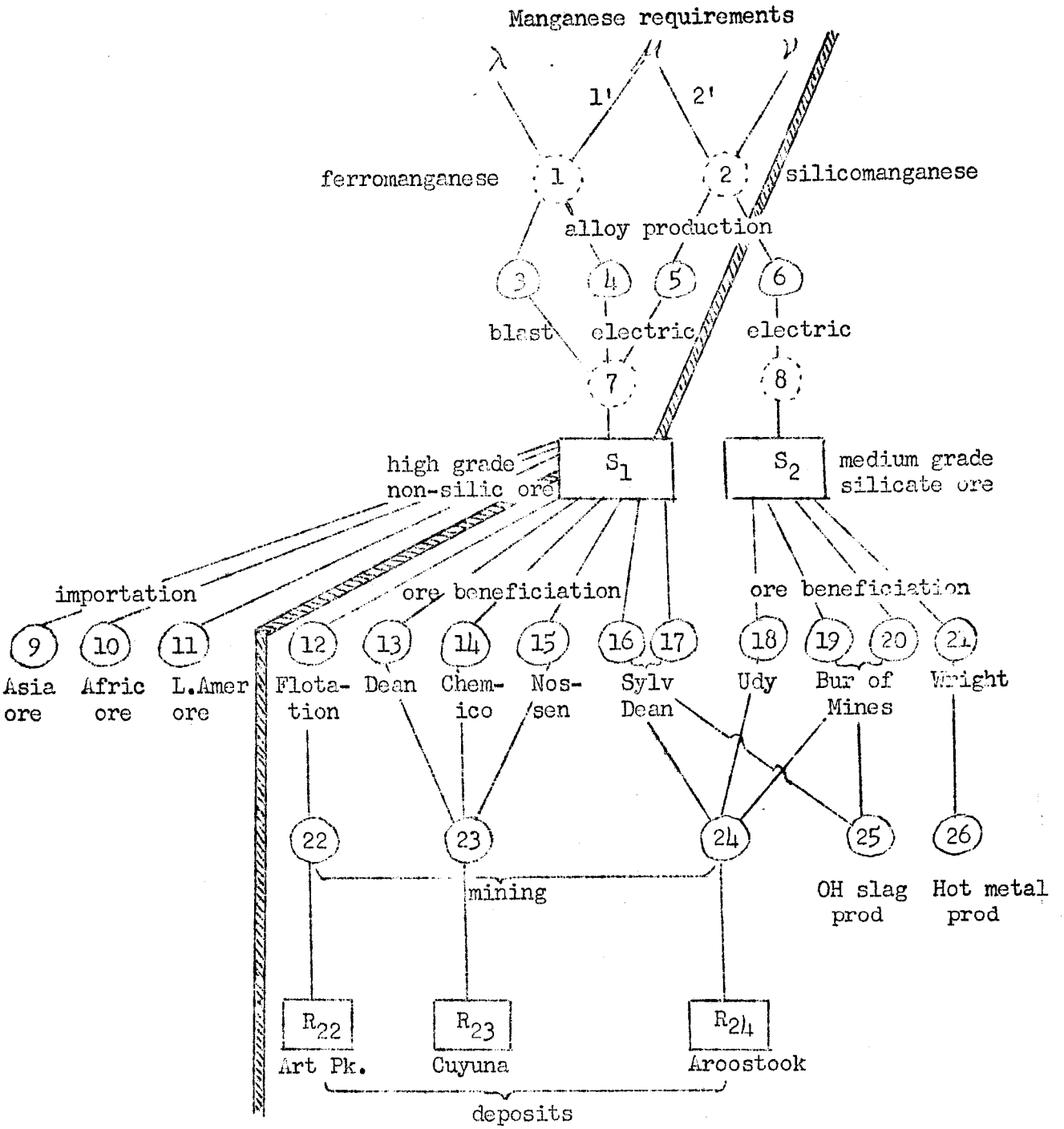
The import prices π have been strongly correlated in the past with the imported quantities. But there is this difference that where an increase in the quantity led before to lower production costs it leads here to higher import prices. (This merely reflects the dominant position of the United States as buyer of foreign ores.) The latter relationship does not impair the convexity of the objective function, on the contrary. Hence there is from a computational viewpoint here less need to replace the non-linear terms by linear ones as there was in the case of domestic production costs. Such a replacement would on the other hand do injustice to the economic aspects of the problem and might, in certain cases, even lead to trivial solutions. Therefore, it has been decided to keep the non-linearities, which do not impair the convexity, in the objective function. It has now become our task to compute the minimum of an objective

function which contains linear as well as quadratic terms.

Finally it should be remarked that all parameters indicated here by Greek letters refer to future needs, costs, and prices; hence, there is much uncertainty about their magnitude. It is therefore our intention to run several computations, each with one particular set of parameters and observe the effect of their changes on the outcome rather than to rely on one such set only. Such a procedure will provide a good basis for decision making under uncertainty.

Economics Research Project
Princeton University
August 29, 1957.

A p p e n d i x



symbolizes storage.

symbolizes activities.

Everything left of the heavy line refers to current practice.

COMPUTING MANGANESE MODEL NO. 1

Philip Wolfe

1. The mathematical model for the allocation of Manganese developed by Karreman [7], despite its conceptual simplicity, presents some formidable difficulties with regard to the available computing facility -- the Institute for Advanced Study computer -- when conventional means of dealing with the computation are considered. These difficulties will be outlined briefly below. They force us to develop a new computational method which, owing to facile use of the data of the problem, is well adapted to the IAS machine. The mathematical development of this procedure, with brief references to the machine characteristics which conditioned it, will be given below.

2. Karreman's formulation [7] of the problem for a period of T years, in each of which n variables are to be chosen to satisfy m constraints, is a programming problem of Tn variables subject to Tm constraints.

The objective function $f(x)$ to be minimized is not rigidly defined; in the original formulation it is quite general, being nonlinear and possibly also non-convex, in which case no conjectured (even) computing method could assure a solution; but it appears that it can, without undue violence, be taken as convex, so we will assume this done throughout. Mathematical techniques exist [6] for the minimization of a quadratic convex function under linear constraints; although they take more computing time than when employed on linear objective functions, they use similar procedures and require only slightly more data-storage; hence their feasibility is about the same as that of the linear programming routine.

If the Manganese problem were to be taken to be a linear problem, it would require storage of $2\,m\,n$ data (using a very compact form) for the constraint matrix, and $T(m+n)$ data consisting of constants: costs and

requirements. These numbers are not large; but the ordinary computational method for this problem — the Simplex Method [4] — requires storage of at least $(Tm+1)^2$ auxiliary data — the inverse of a "basis," a $(Tm+1)$ by $(Tm+1)$ matrix. Thus, at least $(Tm+1)^2 + T(m+n) + 2mn \geq m^2T(T+1)$ words of high-speed storage are needed. At present, the IAS machine has a total capacity of 5,120 words (which will be enlarged to 17,408 this year). Supposing a minimally adequate program (whose limitations will be further considered below) to use 400 words, there are left about 4,700 (17,000) words for storage. The number of constraints m appears to be flexible: we have considered $13 \leq m \leq 20$. For these figures, the size (in number of time periods usable) of the largest problem computable can be given:

storage / m	5,120	17,408
13	4	9
20	2	6

A very crude estimate of computing time needed for the larger problems is of the order of magnitude of one hour.

Experience with large-scale problems, however, suggests that the available storage should be counted as half the above figures because the accumulation of round-off error probably requires "double-precision" coding. A standard code (the RAND code [8] for the Simplex Method) uses 18-decimal floating-point data, whereas single-precision floating data on the IAS machine have 9-decimal precision.

These considerations, as well as the length of time that would be required for the programming and coding of the Simplex Method, have led us to seek other means for our problem. It should be emphasized, however, that a general Simplex Method routine for the IAS machine would be of great value in a variety of other problems in the area of this Project's interest.

Mention should be made of the work of Dantzig, reported briefly in [5], in exploiting the special structure of dynamic problems with a variant of the Simplex Method. It appears [private communication] that these ideas would permit the computation of a fully linear model of the size we are considering by hand in a matter of hours. Unfortunately, this new procedure has not yet been reduced to a machine program, and details on it are not yet available to us. When the procedure becomes fully available, it will certainly supplant the present proposal.

3. Our problem will be approached through an application of the differential procedure of Brown and von Neumann [2] for the calculation of the optimal strategies of a zero-sum two-person game. They show that if H is a skew-symmetric P by P matrix then the differential equations

$$\frac{dz_i}{dt} = \text{Max}\left\{0, \sum_j H_{ij} z_j\right\} , \tag{1}$$

with starting condition

$$z_i(0) \geq 0, \quad \text{some } z_i(0) \text{ positive}$$

have solutions $z_i(t)$ such that for any ϵ there exists T such that for $t \geq T$ the numbers $z_i(t) \Big/ \sum_i z_i(t)$ constitute a mixed strategy for the symmetric matrix game H giving a yield within ϵ of its value (zero).

The connection between this result and programs is given by the theorem of Dantzig [3]: If the m by n linear program

$$\text{Min}\{cx: \ x \geq 0, \ Gx \geq b\} \tag{2}$$

has a solution, then the symmetric matrix game on the next page has an optimal strategy (x,u,s) such that $s > 0$; and for any such optimal strategy, $\frac{x}{s}$ is a solution of the program, and $\frac{u}{s}$ of its dual.

It can further be shown [10] that if the constraint set $\{x: \ x \geq 0, \ Gx \geq b\}$ is bounded and has an interior, then every solution of the game

(3)

	$x_1 \cdots x_n$	$u_1 \cdots u_m$	s
$x_1 \vdots x_n$	0	G^T	$-c^T$
$u_1 \vdots u_m$	$-G$	0	b
s	c	$-b^T$	0

must have $s > 0$. Thus, differential equations for (x,u,s) drawn from the above game will yield solutions for the program. These equations are:

$$(4) \qquad \begin{cases} \dfrac{dx_j}{dt} = \text{Max}\ \{0, \sum_i u_i g_{ij} - sc_j\} \\ \dfrac{du_i}{dt} = \text{Max}\ \{0, sb_i - \sum_j g_{ij} x_j\} \\ \dfrac{ds}{dt} = \text{Max}\ \{0, cx - ub\} \quad . \end{cases}$$

Normalizing, the quantities $\dfrac{x_j(t)}{s(t)}$ are the solutions desired. This is the basis of the iterative procedure we use.

An unsatisfactory feature of these equations is the fact that the x, u, and s never decrease; if a term must be zero in the solution of the program, but is initially positive, it only approaches zero through division by the increasing $s(t)$. As a computational experiment, we shall use differential equations like (4) with added "cutback" terms whose effect will be to reduce unneeded positive variables. Let

$$(5) \qquad \phi[y,z] = \begin{cases} z & \text{if } z > 0 \\ 2^{-k} z & \text{if } z < 0 \text{ and } y > 0 \\ 0 & \text{otherwise} \end{cases}$$

We shall compute with

$$(6)\qquad \begin{cases} \dfrac{dx_j}{dt} = \phi\left[x_j, \sum_i u_i g_{ij} - sc_j\right] \\ \dfrac{du_i}{dt} = \phi\left[u_i, sb_i - \sum_j g_{ij} x_j\right] \\ \dfrac{ds}{dt} = \operatorname{Max}\left\{0, \sum_j c_j x_j - \sum_i u_i b_i\right\} \end{cases} .$$

Experiments can be run with $k = 1,2,\ldots$ ($k = 0$ is known not to work) or with k so large as to reduce (6) to (4).

It seems likely that equations (6), as do (4), should yield convergence to a solution of the program at a rate proportional to t^{-1}; but it has not in fact been proved that (6) provides a solution to the program at all.

We propose also to handle the nonlinearities of the objective function f in (6), by letting

$$(7)\qquad c_j = c_j(x) = \frac{\partial f(x)}{\partial x_j} .$$

Although we have not proved that this device will solve the nonlinear problem, it is essentially the same as that of the method of Uzawa [9] and Arrow-Hurwicz [1] for the case of a strictly convex objective function. In any case, it is the obvious extension of the method for the linear problem. In our problem, each c_j will either be constant or linear in x_j.

Of course, this type of iterative procedure will not yield exact answers for our problem, but only answers valid to the level "ϵ". We see this, however, as fully consonant with the state of the data that will be used.

4. The numerical method for solving (6) is the following: Rewrite the

equations, using difference quotients $\Delta x_j/\Delta t$, $\Delta u_i/\Delta t$, $\Delta s/\Delta t$ in place of the derivatives; select an appropriate value for Δt, and initial values x^o, u^o, s^o; then, for any $n = 0, ., ...,$ calculate the right-hand sides of (6) using x^n, u^n, s^n, and define

$$x^{n+1} = x^n + \Delta x, \quad u^{n+1} = u^n + \Delta u, \quad s^{n+1} = s^n + \Delta s. \tag{8}$$

For computation on the fixed-point IAS machine, it is necessary to adopt a suitable scale in which to express the numerical quantities dealt with. Now the problem we are dealing with is homogeneous, i.e., if all the constants b were multiplied by a constant k, and all the costs c by another constant m, then the solution of the new problem would be proportional to the solution of the old — its activities k times the former's, and its cost km times the former's. Thus, we may assume in advance that the constants have a certain order of magnitude; and we will take them to be of the order of magnitude of 1. This will tend to keep the solution-values x/s and u/s of the order of 1; and since, from any initial solution x,u,s we obtain, on dividing by s, a new initial solution x/s, u/s, 1, we are assured that periodically "normalizing" by division by s will keep all the variables of the order of 1.

Finally, however, the machine representation of the data requires keeping all the data substantially less than one. Thus, we will make the following replacement of all the parameters of the problem

(9)

Replace	b	by	$2^{-10}b$
"	c	"	$2^{-10}c$
"	x	"	$2^{-20}x$
"	u	"	$2^{-20}u$
"	s	"	$2^{-10}s$.

This will clearly make the data fit; the equations for the computation now become, with all letters standing for these new variables,

$$\Delta x_j = \Delta t \; \phi \; [x_j, \sum_i u_i g_{ij} - s c_j]$$

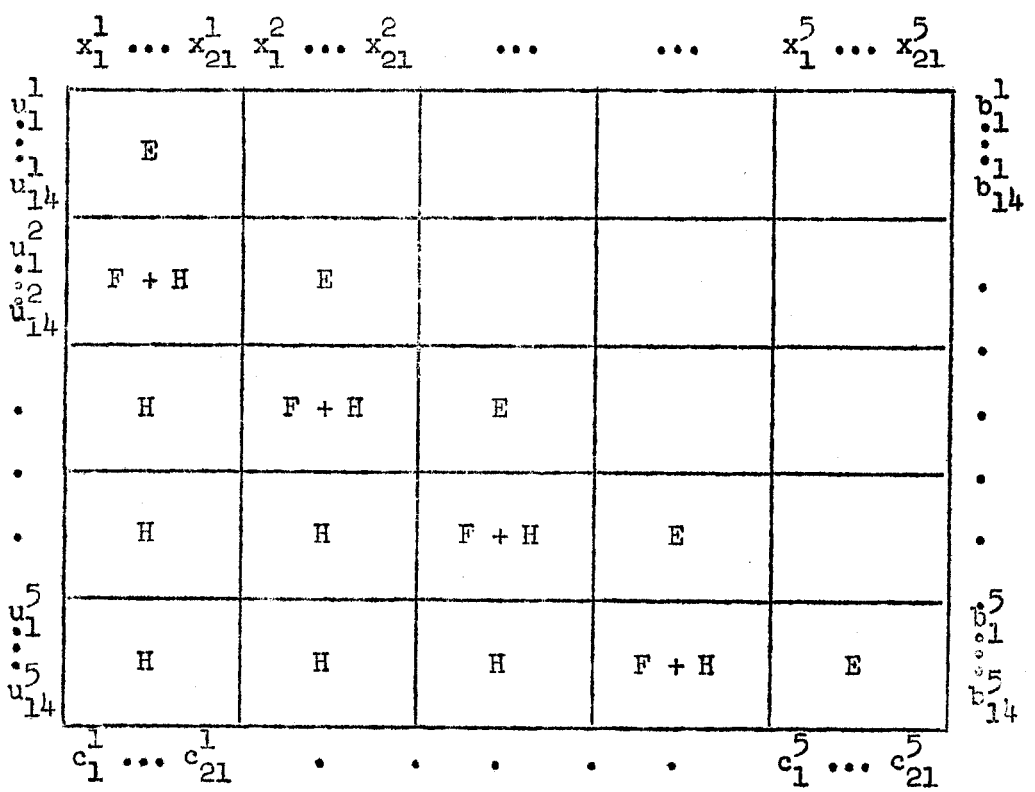
(10)
$$\Delta u_i = \Delta t \; \phi \; [u_i, s b_i - \sum_j g_{ij} x_j]$$

$$\Delta s = 2^{20} \, \Delta t \, \text{Max} \, [0, \sum_j c_j x_j - \sum_i u_i b_i] \quad .$$

Initial values for this system must be taken of the orders $\Delta x = 2^{-20}$, $\Delta u = 2^{-20}$, $\Delta s = 2^{-10}$.

The distance from optimality of the given approximation is measured by s^{-1} times the sum of all the quantities of (10). Our program will use an initial distance, ϵ_o, in conjunction with an initial time-step, Δt_o, proceeding with the iteration until the distance measured by (10) falls below ϵ_o. Then both ϵ and Δt will be reduced by an appropriate factor — 4 seems reasonable — and the process repeated. When finally the distance is less than a preassigned ϵ, we will be through.

5. We shall perform the calculation on a sub-model of Karreman's [7]. Our problem, for T time periods, has 21T variables, 14T constraints. Its matrix can be compactly written in the form (for T = 5)



where E, F, and H are three 14 by 22 matrices given in the appendix. Here x_j^τ denotes the level of activity j in period τ, etc.

If, for convenience, we define the additional quantities $x_j^0 = 0$, $u^{T+1} = 0$, we can rewrite the summations of (10) as follows:

$\sum_i u_i g_{ij}$ becomes (for j in period τ)

$$(11) \qquad \sum_i u_i^\tau e_{ij} + \sum_i u_i^{\tau+1} f_{ij} + \sum_i h_{ij} \sum_{\tau_1=\tau+1}^{T} u_i^{\tau_1} \quad ;$$

and $\sum_j g_{ij} x_j$ (for i in period τ) becomes

$$(12) \qquad \sum_j e_{ij} x_j^\tau + \sum_j f_{ij} x_j^{\tau-1} + \sum_j h_{ij} \sum_{\tau_1=1}^{\tau-1} x_j^{\tau_1} \quad .$$

These forms indicate how much storage will be needed for the problem. Making use of the fact that the matrices E, F, H have very few non-zero entries, the constants of (11) and (12) can be incorporated into a very high-speed program for the formation of these sums, which can be kept entirely in the electrostatic storage unit (WM). Drum storage is needed for the remaining variables and their sums and constants, i.e., for x_j^τ, $\sum_{\tau_1}^{\tau} x_j^{\tau_1}$, c_j, and likewise for u, for $\tau = 0, \ldots, T+1$. A precise count for our problem gives the drum storage requirement of $88(T+1)$ words, permitting calculation with a 20-year model under present conditions (4,096 words, with two words used per datum for certain technical reasons).

In the Appendix we give a more precise statement of the numerical problem and the nature of the program for the IAS machine.

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APPENDIX I: THE MATRICES

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1	1	1																			
2	-1		1	1																	
3		-1			1	1															
4			$-c_3$	$-c_4$	$-c_5$															-1	
5						-1															-1
6										-1											
7											-1										
8												-1									
9													-1								
10														-1							
:																					

Remainder 0

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1																					
2																					
3																					
4							1	1	1	1	1	1	1							1	
5														1							1
6																					
:																					

Remainder 0

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
:										Zeros											
5																					
6															1						
7																1					
8																	1				
9																		1			
10																			1		
11										-1											
12											-1										
13												-1									
14													$-c_{19}$	$-c_{20}$							
:																					

APPENDIX II: NOTE ON PROGRAMMING

The program written for this problem has been designed to minimize the time required for computation. Data-input and output speed has been sacrificed to flexibility; problem data may be individually modified at any point in the calculation.

The bulk of the calculation being the formation of the summations of (10), we illustrate this by sketching the formation of u_1^τ of (12).

In the calculation for time period τ $(= 1,\ldots,T)$, the following quantities from the previous iteration have been brought to WM from the drum $(j = 1,..,21;\ i = 1,..,14)$:

$\{x_j^{\tau-1}\}$, $\{c_j^{\tau-1}\}$, $\{b_i^{\tau-1}\}$, $\{u_i^\tau\}$, $\{x_j^\tau\}$, $\{c_j^\tau\}$, $\{b_i^\tau\}$, $\{u_i^{\tau+1}\}$.

(All data are stored on the drum in the order indicated by the above fragment.)

A new u_1^τ will be formed from the old by adding Δu_1 calculated from (12). Here the expression is quite simple:

$$\Delta u_1^\tau = \Delta t\ \phi[u_1^\tau,\ sb_1^\tau - x_1^\tau - x_2^\tau] \quad .$$

The orders then are:

	1.	MULTIPLY	s and b_1^τ
	2.	SUBTRACT	x_1^τ
	3.	SUBTRACT	x_2^τ
	4.	STORE	result (for future use)
	5.	SHIFT	λ places right (multiplies by $2^{-\lambda} = \Delta t$)
"cutback"	6.	TRANSFER	to 8 if result ≥ 0, otherwise execute 7
	7.	SHIFT	μ places right (multiplies by $2^{-\mu}$)
new $u_1^\tau \geq 0$	8.	ADD	u_1^τ
	9.	TRANSFER	to 11 if result ≥ 0, otherwise execute 10
	10.	STORE	0 for new u_1^τ
	11.	STORE	sum for new u_1^τ .

Only orders 1-3 are specific to the variable (u_1) being acted on; 4-11 appear as above for each of the 35 variables. Such a program, while occupying

a great deal of space in WM, is capable of more efficient operation than a more compact one in which orders 4-11 were formed when needed: Firstly, because modifying orders 4, 8, and 11 would require 3 additions, and secondly because of the possibility of "read-around" due to frequent reference to the storage of the block.

Once the general program has been stored in WM, communication with the machine takes place entirely through punched cards in the read hopper. Single cards bear instructions to load data, to start the computation, to change parameters of the computation (such as λ or μ above), and to print results, which are punched on cards in such a way that they may be used as initiating data.

Department of Mathematics and Economics Research Project
Princeton University
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