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GRADIENT CONFIGURATIONS AND QUADRATIC FUNCTIONS

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PREFACE

When certain quantities of various commodities are purchased at certain prices, the direction represented by the price vector is interpreted as giving the direction of the gradient of a preference function at a point represented by the quantity vector. This interpretation expresses the main principle which is involved in consumer theory, when this theory is founded on the preference hypothesis, or even on the classical, but more obscure, concept of marginal utility.

In the same way, with data of purchases on a variety of occasions, there is provided a configuration formed of a set of directions associated with a set of points. Any function which is increasing and convex being allowed as a preference function, such a configuration admits those preference functions whose gradients have these directions at these points, without distinction between them. But in such a class of preference functions admitted by a given configuration, the quadratic, when admitted, is distinguished first by adequacy and simplicity, and then by a property, which is to be examined elsewhere, which sets it in a central position within the class, and which singles it out as a model in respect to which other preference functions may be considered through their deviation. Thus the investigation of quadratic functions admitted by a given configuration appears with a general importance. A comparison may be made with the role of the linear function in regression analysis.

With directions prescribed, magnitudes only have to be assigned, for gradients to be completely specified. So there has to be considered the existence, and the character, of quadratics with gradients given at certain points; and this is the problem for treatment now. In the fourfold case, the magnitudes of the four gradients may always be chosen, in an essentially unique fashion, in order that quadratics be admitted. Elsewhere it will be shown how analysis involving quadratic preference function arises in a natural fashion within a comprehensive analysis which embraces the totality of consistent preference systems compatible with given data. It is particularly important in theory underlying certain index-number constructions.

In consumer theory, quadratic functions have already made an appearance in the work of R. Frisch, R.G.D. Allen, A. L. Bowley, A. Wold, A. A. Konüs, S. S. Buscheguennce, H. S. Houthakker, H. Schultz, and others. A description of the role of quadratics in this subject, those uses already made, together with further elaborations in which the material presented here has application, will follow in another account.

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1. Quadratics

Let $\varphi = \varphi(x)$ denote a real single-valued, continuously twice differentiable function, defined on the n-dimensional Euclidean space C of vectors x of order n with real elements. The gradient of φ is given by the vector $g = \varphi_x$ of first derivative $\varphi_{\xi_i} = \frac{\partial \varphi}{\partial \xi_i}$ of φ with respect to the elements ξ_i of x, and determines a vector field $g = g(x)$ on C. The matrix $A = \varphi_{xx'}$ of second derivative $\varphi_{\xi_i \xi_j} = \frac{\partial^2 \varphi}{\partial \xi_i \partial \xi_j}$ is symmetric, by the assumed continuity. If A is constant, then φ is a quadratic function, with A as characteristic matrix; and it is called regular if A is regular.

THEOREM 1. *If* g *is the gradient of a quadratic with characteristic matrix* A, *then*

$$g(y) - g(x) = A(y-x) .$$

For $\frac{\partial g_i}{\partial x_j} = A_{ij}$, so that $dg_i = \sum_j A_{ij} dx_j$, whence, by integration, the formula is obtained.

THEOREM 2. *If* φ *is a quadratic with gradient* g *and characteristic matrix* A, *then*

$$\varphi(y) - \varphi(x) = (y-x)' g(x) + \frac{1}{2}(y-x)' A(y-x) .$$

This follows by integration of the expression in Theorem 1.

THEOREM 3. *If* φ *is a quadratic with gradient* g *then*

$$\varphi(y) - \varphi(x) = \frac{1}{2}(y-x)'(g(x) + g(y)) .$$

Thus, by Theorems 1 and 2,

$$\begin{aligned} \varphi(y) - \varphi(x) &= (y-x)' g(x) + \frac{1}{2}(y-x)' A(y-x) \\ &= (y-x)' g(x) + \frac{1}{2}(y-x)'(g(y)-g(x)) \\ &= \frac{1}{2}(y-x)'(g(x) + g(y)) \end{aligned}$$

THEOREM 4. *If* φ *is a regular quadratic, with gradient* g *and characteristic matrix* A , *then* $g(x) = 0$ *at a unique point* $x = c$, *and*

$$g(x) = A(x-c) .$$

For, if $B = A^{-1}$, then, by Theorem 1,

$$x-y = B(g(x) - g(y)) ,$$

so that if

$$c(x) = x - Bg(x) ,$$

then

$$c(x) = x(y) .$$

A unique point $c = c(x)$ is thus defined, which is, moreover, such that

$$\begin{aligned} g(c) &= g(x) + A(c-x) \\ &= g(x) - g(x) = 0 . \end{aligned}$$

Then

$$g(x) = A(x-c) .$$

The unique point at which the gradient of a regular quadratic vanishes defines its *centre*; and the value which a regular quadratic takes at its centre defines its *initial value.* Thus,

$$g(c) = 0 , \varphi(c) = M ,$$

for the centre c and initial value M of a regular quadratic φ with gradient g . A regular quadratic will be called a *central quadratic* if its centre is at the origin, and a *principal quadratic* if its initial value is zero.

THEOREM 5. *If* φ *is a regular quadratic with centre* c , *initial value* M *and characteristic matrix* A , *then*

$$\varphi(x) = M + \frac{1}{2}(x-c)'A(x-c) .$$

This follows by combination of Theorems 2 and 4, and gives a normal form for a regular quadratic.

COROLLARY

$$\varphi(x) = M + \frac{1}{2}(x-c)'g(x) .$$

Singular quadratics, which are not of present importance, have a more complicated description. Thus, for a singular quadratic, the range and null-space of the characteristic matrix form a complementary pair of subspaces, defining the <u>characteristic section</u> and <u>axis</u> of the quadratic; and they are a pair of orthogonal complements, since the characteristic matrix is symmetric.

Any manifold obtained by translation of the characteristic section or axis defines a <u>normal section</u> or <u>axis</u> of the quadratic.

Since $g(x) - g(y) = A(x-y)$, any two gradients differ by a vector in the characteristic section; moreover, gradients are equal at points which differ by a vector of the axis. Accordingly, the gradients determine a particular section, which may be called the <u>gradient section</u>; and the axes are the loci on which the gradient is constant.

The gradient section is either identical with, or different from, being a parallel translation of, the characteristic section. In the former case, the quadratic will be called <u>cylindrical</u>.

A cylindrical quadratic not only has a constant gradient, but also has a constant value on every normal axis. For if x, y belong to the same axis, $x - y$ will be orthogonal to the gradient $g(x), g(y)$ which belong to the section, so that

$$\varphi(x) - \varphi(y) = \frac{1}{2}(x-y)'(g(x)+g(y)) = 0 .$$

So the quadratic will be specified when it is given on any normal section. The value at any point outside the section being equal to the value at the foot of the perpendicular from the point to that section. By constraining a cylindrical singular quadratic to any of its normal sections, the function obtained is a regular quadratic on that manifold, with a unique centre at which the gradient vanishes. Thus the gradient of a cylindrical quadratic vanishes at points of an axis, defining the <u>central axis</u>. For, if

$g(c) = 0$, then

$$g(x) = A(x-c) ,$$

so that $g(x) = 0$ if x, c lie on an axis through c , and moreover, $g(x)$ always lies in the fundamental section, so the quadratic is cylindrical. Conversely, if the quadratic is cylindrical, so that $g(x)$ lies in the range of A , so there always exists a vector $x - c$ such that

$$g(x) = A(x-c) ,$$

then $g(c) = 0$. Thus a necessary and sufficient condition that a quadratic be cylindrical, having its gradient always in the characteristic section, is that its gradient vanishes at any one point, and therefore on the normal axis through that point. For a non-cylindrical singular quadratic, therefore, the gradient never vanishes; and for such, the gradients lie in a normal section different from the characteristic section.

2. Linear fields

A set $\alpha = \{\alpha_r\}$ of $k + 1$ numbers α_r $(r = 0, \ldots, k)$ will be called a distribution if

$$\alpha_o + \ldots + \alpha_k = 1 .$$

The combination of a set $X = \{x_r\}$ of elements of C by a set of numbers α , is denoted by

$$x_o\alpha_o + \ldots + x_k\alpha_k = X\alpha .$$

A linear manifold in C is a set $\mathfrak{X}$ of elements of C which contains every combination of its elements by a distribution. Thus,

$$X \subset \mathfrak{X} \text{ , and } \alpha \text{ a distribution, implies } X\alpha \in \mathfrak{X} .$$

The linear closure of any set X of elements of C is formed by the set $[X] = \{X\alpha\}$ of combinations $X\alpha$ of those elements by all possible distributions α . The linear closure of any set of elements is a linear manifold, which is said to join those elements or to be spanned by them.

A <u>base</u> for a given linear manifold is a minimal set of spanning elements. A set of elements form a <u>simplex</u> if they are a base for the linear manifold in which they join; for which the condition is that none lies in the join of the others, or, if they are $k + 1$ in number, then that they span a manifold of dimension k. Thus $\{x_r\}$ is a simplex of dimension k, provided

$$\alpha_o + \cdots + \alpha_k = 0 \text{ and } x_o\alpha_o + \cdots + x_k\alpha_k = 0 \text{ implies } \alpha_o, \ldots, \alpha_k = 0 .$$

For otherwise, if the hypothesis holds, and not the conclusion, say $\alpha_o \neq 0$, then there is obtained the relation

$$x_o = x_1\beta_1 + \cdots + x_k\beta_k ,$$

where $\beta_r = \frac{-\alpha_r}{\alpha_o}$, so that $\beta_1 + \cdots + \beta_k = 1$, this showing x_o to be in the join of $x_1, \ldots, x_k$.

Now let $X = \{x_o, \ldots, x_k\}$ be a simplex spanning a manifold $\mathfrak{X}$ of dimension k. Take the matrix $X_o = \{x_r - x_o\}$, of order $n \times k$ and rank k, with the vectors $x_r - x_o$ for its columns; and form the matrix

$$I_o = X_o(X_o{}'X_o)^{-1}X_o{}' ,$$

this being symmetric and idempotent, of order n and rank k. It has the property

$$I_oX_o = X_o ,$$

that is

$$I_o(x_r - x_o) = x_r - x_o ,$$

from which it follows that

$$I_o(y-z) = y-z$$

for every $y,z \in X$, that is

$$I_oy - y = I_oz - z ;$$

or, if

$$J_o = I - I_o ,$$

then

$$J_oy = J_oz .$$

Thus there is a uniquely defined symmetric idempotent $J_{\mathfrak{X}} = I - I_{\mathfrak{X}}$, and a uniquely defined element $\bar{x} \in \mathfrak{X}$ such that

$$J_{\mathfrak{X}} x = \bar{x}$$

for every $x \in \mathfrak{X}$. The idempotent $I_{\mathfrak{X}} = I_o$, first constructed by choice of an arbitrary vertex x_o of the simplex, is independent of this choice, and defines the perpendicular projector onto the manifold. The element $\bar{x} \in \mathfrak{X}$ is such that

$$\bar{x} \perp x - \bar{x} ,$$

for every $x \in \mathfrak{X}$, or equivalently, such that

$$\bar{x} \perp y - z ,$$

for every $y, z \in \mathfrak{X}$. Thus it is perpendicular to every displacement on the manifold, and is the unique vector on the manifold with this property. From this it follows that it is the unique element on the manifold of minimum distance from the origin. It will be called the absolute element on the manifold, with the property that, for all $x \in X$,

$$|x| \geq |\bar{x}| , \text{ and } |x| = |\bar{x}| \text{ if and only if } x = \bar{x} .$$

Now let $g = g(x)$ be any vector field defined on a linear manifold $\mathfrak{X}$ in C . Then it is a linear field if

$$g(x_o \alpha_o + \dots + x_k \alpha_k) = g(x_o)\alpha_o + \dots + g(x_k)\alpha_k ,$$

provided $\alpha_o + \dots + \alpha_k = 1$. Thus, with the notation

$$g\{x_r\} = \{g(x_r)\} ,$$

that is,

$$g(X) = G , \text{ where } X = \{x_r\}, \ G = \{g_r\}, \ g_r = g(x_r) ,$$

the condition is

$$g(X\alpha) = g(X)\alpha$$

for every set $X \in \mathfrak{X}$, and every distribution α . The linear manifold $\mathfrak{X}$ defines the domain; and the values $g(x)$, also forming a linear manifold $\mathfrak{G}$, define the range of the field. A linear field g is defined to be regular if X and $g(X)$ always span manifolds of the same dimension; and otherwise to be singular.

A set $G = \{g_r\}$ of vectors g_r associated with points x_r of a set $X = \{x_r\}$ defines a vector configuration $\{X,G\}$, with base set X and an object set G, spanning its base and object manifolds $\mathfrak{X}$, $\mathfrak{G}$, the dimensions of which define its base and object dimensions. It is called a simplicial configuration if its base set X is a simplex; and a regular configuration if corresponding sets of base and object elements join in linear manifolds of equal dimension, in particular, the base and object dimensions are equal. Thus, for a regular simplicial configuration, also the object set is a simplex. A configuration whose base dimension is n is called complete. A vector configuration is said to be linearly consistent if it can be embedded in a linear vector field; that is, if there exists a linear field g whose domain contains its base set, and is such that

$$g(X\alpha) = G\alpha ,$$

for every distribution α, or equivalently,

$$x_o\alpha_o + \cdots + x_k\alpha_k = 0 \text{ implies } g_o\alpha_o + \cdots + g_k\alpha_k = 0 \text{ , provided } \alpha_o + \cdots + \alpha_k = 0 .$$

If a configuration can be embedded in a field, the field and the configuration are said to admit each other.

THEOREM 1. Any simplicial configuration is linearly consistent.

For, provided $\alpha_o + \cdots + \alpha_k = 0$, since $\{x_r\}$ is a simplex, $x_o\alpha_o + \cdots + x_k\alpha_k = 0$ implies that $\alpha_o, \ldots, \alpha_k = 0$, and therefore that $g_o\alpha_o + \cdots + g_k\alpha_k = 0$.

Now the elements of the base and object manifolds $\mathfrak{X}$, $\mathfrak{G}$ of a configuration $\{X,G\}$ are of the form

$$x_\alpha = X\alpha , \quad g_\alpha = G\alpha$$

where α is any distribution. In the simplicial case, for every element

$x \in \mathfrak{X}$ the distribution α such that $x_\alpha = x$ is unique, and so it is possible to define a single-valued vector function g with $\mathfrak{X}$ as domain and $\mathfrak{G}$ as range by taking

$$g(x) = g_\alpha \quad (x = x_\alpha) .$$

This determines a vector field on X, which will be denoted by $\{\mathfrak{X},\mathfrak{G}\}$ and called the linear closure of the simplicial configuration $\{X,G\}$.

THEOREM 2. The linear closure of a simplicial configuration is a linear field containing that configuration, and contained in every linear field containing that configuration.

Since

$$x_{\alpha\sigma+\beta\rho} = x_\alpha\sigma + x_\beta\rho , \quad g_{\alpha\sigma+\beta\rho} = g_\alpha\sigma + g_\beta\rho ,$$

where α,β are any distributions, and $\sigma + \rho = 1$, so that $\alpha\sigma + \beta\rho$ is also a distribution, it appears that the linear closure is a linear field, and it is obviously minimal linear field containing the configuration.

THEOREM 3. Any linear field is the linear closure of a simplicial configuration.

Let X be a simplex spanning the domain $\mathfrak{X}$ of a linear field g, and let $G = g(X)$. Then $\{X,G\}$ is a simplicial configuration; any point x of X is of the form x_α for a unique α; and $g(x_\alpha) = g(X)\alpha = G\alpha = g_\alpha$; so that g is the linear closure of $\{X,G\}$.

THEOREM 4. The gradient of a quadratic function is a linear field, regular or singular according as the quadratic is regular or singular.

By Theorem 1.1, if g is the gradient and A the characteristic matrix of a quadratic, then

$$g(x_r) = g(x) + A(x_r - x)$$

so that

$$\sum_r g(x_r)\alpha_r = g(x) + A(\sum_r x_r\alpha_r - x) = g(\sum_r x_r\alpha_r)$$

provided $\alpha_r = 1$. Moreover, the rank of the vectors $g(x_r) - g(x)$ is always the same as the rank of the vectors $x_r - x$ if and only if the matrix A is regular.

COROLLARY
$$\tfrac{1}{2}(g(x) + g(y)) = g(\tfrac{x+y}{2}) .$$

COROLLARY
$$\varphi(y) - \varphi(x) = (y-x)'g(\tfrac{x+y}{2}) .$$

It appears thus that if the gradient of a quadratic is given at any set of points, then it can be constructed at any point of the linear manifold joining them. Thus the gradient of a quadratic is fully specified when it is known at the vertices of any regular n-dimensional simplex, in other words, at any $n + 1$ points no one of which lies in the linear manifold joining the others.

3. Symmetric fields and configurations

A vector field g will be called symmetric if

$$(x-y)'(g(z) - g(w)) = (z-w)'(g(x) - g(y)) ,$$

and triadic if

$$(x-y)'g(z) + (y-z)'g(x) + (z-x)'g(y) = 0 .$$

THEOREM 1. The conditions for a symmetric and a triadic vector field are equivalent.

By putting $w = x$ in the symmetry condition, it reduces to the triadic condition; for

$$(x-y)'(g(z) - g(x)) = (z-x)'(g(x) - g(y))$$

gives

$$(x-y)'g(z) + (y-z)'g(x) + (z-x)'g(y) = 0 .$$

Thus the first condition implies the second. Conversely, the symmetry condition provides the relations

$$(x-z)'g(w) + (z-w)'g(x) + (w-x)'g(z) = 0$$

$$(x-w)'g(y) + (w-y)'g(x) + (y-x)'g(w) = 0 ,$$

which, by addition, give the symmetry condition

$$(x-y)'(g(z) - g(w)) = (z-w)'(g(x) - g(y)) .$$

Similarly, a configuration $\{x_r, g_r\}$ $(r\text{-}s = 0, \ldots, k)$ is said to be <u>symmetric</u> if

$$(x_p-x_q)'(g_r-g_s) = (x_r-x_s)'(g_p-g_q) ,$$

which condition is again equivalent to the triadic condition

$$(x_r-x_s)'g_t + (x_s-x_t)'g_r + (x_t-x_r)'g_s = 0 .$$

THEOREM 2. <u>A necessary and sufficient condition for the symmetry of the configuration</u> $\{x_r, g_r\}$ <u>is the symmetry of the matrix</u> $X_o'G_o$, <u>where</u> $X_o = \{x_r-x_o\}$, $G_o = \{g_r-g_o\}$.

For the symmetry of this matrix is the condition

$$(x_r-x_o)'(g_s-g_o) = (x_s-x_o)'(g_r-g_o) ,$$

which is necessary for the symmetry of the configuration. Also it is sufficient, since x_o, g_o can be eliminated from this condition for the three pairs from three elements r, s, t to give the triadic condition, equivalent to the symmetry condition.

THEOREM 3. <u>The conditions for the symmetry of a simplicial configuration and its linear closure are equivalent</u>.

Since the closure contains the generating configuration, one condition implies the other; and the converse implication follows from the identity

$$\sum_{p,q,r,s} \alpha_p \beta_q \gamma_r \delta_s \; (x_p-x_q)'(g_r-g_s) = (x_\alpha-x_\beta)(g_\gamma-g_\delta) ,$$

where α, β, γ, δ are any distributions.

THEOREM 4. <u>Any complete symmetric configuration is linearly consistent.</u>

By the symmetry,

$$(x_r-x_s)'(g_\alpha-g_\beta) = (x_\alpha-x_\beta)'(g_r-g_s)$$

from which it follows that $x_\alpha = x_\beta$ implies

$$(x_r - x_s)'(g_\alpha - g_\beta) = 0 \ ,$$

which, by the completeness, implies

$$g_\alpha = g_\beta \ .$$

Thus $x_\alpha = x_\beta$ implies $g_\alpha = g_\beta$, as required.

THEOREM 5. The symmetry of a field g necessary and sufficient for the existence of a function φ such that

$$\varphi(x) - \varphi(y) = \frac{1}{2}(x-y)'(g(x) + g(y)) \ .$$

The existence of such a function implies that

$$\begin{aligned} 0 &= \varphi(x) - \varphi(y) + \varphi(y) - \varphi(z) + \varphi(z) - \varphi(x) \\ &= \frac{1}{2}(x-y)'(g(x) + g(y) + \frac{1}{2}y-z)'(g(y) + g(z)) + \frac{1}{2}(z-x)'(g(z) + g(x)) \\ &= \frac{1}{2}\{(x-y)'g(z) + (y-z)'g(x) + (z-x)'g(y)\} \end{aligned}$$

which is the triadic condition, equivalent to the symmetry condition. Conversely, if the symmetry condition holds, then so does the triadic, so that if

$$\Delta(x,y) = \frac{1}{2}(x-y)'(g(x) + g(y)) \ ,$$

then

$$\Delta(x,y) + \Delta(y,z) + \Delta(z,x) = 0 \ ,$$

which, together with

$$\Delta(x,y) + \Delta(y,x) = 0 \ ,$$

is necessary and sufficient for the existence of a function $\varphi(x)$ such that

$$\Delta(x,y) = \varphi(x) - \varphi(y) \ .$$

THEOREM 6. The gradient of a quadratic is a symmetric field.

For if A is the characteristic matrix, then

$$g(x) - g(y) = A(x-y) \ ,$$

so that

$$\begin{aligned} (z-w)'(g(x) - g(y)) &= (z-w)'A(x-y) \\ &= (x-y)'A(z-w) = (x-y)'(g(z)-g(w)) \end{aligned}$$

by the symmetry of A .

THEOREM 7. Any linear vector field which is also a gradient field must be symmetric.

Thus, let g be a vector field, and let

$$z_t = x + (y-x)t \qquad (0 \leq t \leq 1)$$

and

$$g_t = g(z_t) .$$

Then

$$g_t = g(x) + (g(y) - g(x))t$$

if g is a linear field. Now if g is also a gradient field, say the gradient of some function φ , and if

$$\dot{Z}_t = (y-x)$$

so that

$$dz_t = \dot{Z}_t dt ,$$

then, as t ranges between 0, 1 the point z_t ranges in the linear segment L joining x, y and thus

$$\begin{aligned}\varphi(y) - \varphi(x) &= \int_L g_t {}^{\intercal} dZ_t \\ &= \int_0^1 g_t {}^{\intercal} \dot{Z}_t dt \\ &= \int_0^1 \{g(x) + (g(y) - g(x))t\}^{\intercal}(y-x)dt \\ &= \left[\{g(x)t + \tfrac{1}{2}(g(y) - g(x))t^2\}^{\intercal}(y-x)\right]_0^1 \\ &= \tfrac{1}{2}(y-x)^{\intercal}(g(x) + g(y)) ,\end{aligned}$$

whence the symmetry condition follows.

THEOREM 8. If a quadratic φ is admitted by a gradient configuration $\{X,G\}$, then its gradient is completely determined, and its values completely determined but for an arbitrary additive constant, at every point x_α of the base manifold $\mathfrak{X}$, thus,

$$g(x_\alpha) = g_\alpha , \qquad \varphi(x_\alpha) = \varphi_\alpha ,$$

where

$$\varphi_\alpha - \varphi_\beta = \frac{1}{2}(x_\alpha - x_\beta)'(g_\alpha + g_\beta) .$$

The gradient field of a quadratic is a linear field, so if it contains a configuration, it must contain the linear closure, and then be determined at every point of the base manifold of the configuration. Now the rest follows from Theorem 1.3., in view of Theorems 2.2 and 2.3.

While, if a configuration admits a quadratic, then it must be symmetric, and the quadratic be determined to the extent stated in the Theorem, it is not yet plain that if a configuration be symmetric then it must necessarily admit some quadratic, though this will appear eventually to be the case.

A gradient configuration of a given differentiable function is defined to be any vector configuration contained in the gradient field of that function.

Given any gradient configuration, one may ask whether or not it admits a function of a certain class, in particular a quadratic function, a regular quadratic, or a central quadratic, or further, a positive or negative definite quadratic.

If a configuration is a gradient configuration of a quadratic, it will be called a quadratic configuration. Necessary and sufficient conditions are to be found for a configuration to have this property. If one quadratic is admitted, then, in general, so will be an infinity of quadratics; and investigation is to be made of the class of all quadratics admitted by a given configuration. Thus, for the regular quadratics, the locus of their centres is a certain linear manifold, and with each possible centre there is associated a family of possible characteristic matrices of admitted quadratics with that centre.

Should a configuration $\{x_r, g_r\}$ admit a central quadratic, having its centre at the origin, say with characteristic matrix A, then

$$g_r = Ax_r ,$$

so that

$$x_s' g_r = x_s' A x_r = x_r' A x_s = x_r' g_s .$$

The condition $x_s' g_r = x_r' g_s$ will be called the condition of <u>central symmetry</u>. Now:

THEOREM 9. <u>Central symmetry is necessary for a configuration to admit a central quadratic.</u>

Moreover:

THEOREM 10. <u>Symmetry implies the equivalence of the central symmetry condition</u> $x_r' g_s = x_s' g_r$ <u>to the condition</u> $x_r' g_o = x_o' g_r$.

For the symmetry condition gives

$$(x_o - x_r)' g_s + (x_r - x_s)' g_o + (x_s - x_o)' g_r = 0 ,$$

which, together with

$$x_r' g_o = x_o' g_r , \quad x_s' g_o = x_o' g_s ,$$

implies

$$x_r' g_s = x_s' g_r$$

4. <u>Centre manifold</u>

A <u>quadratic centre</u> of a gradient configuration is defined as a vanishing point of the gradient of any quadratic admitted by the configuration.

THEOREM 1. <u>If</u> c <u>is a quadratic centre of the configuration</u> $\{x_r, g_r\}$, <u>then</u>

$$g_r' x_s - g_s' x_r = (g_r - g_s)' c .$$

For then the configuration formed from $\{x_r, g_r\}$ taken together with (o, c) must admit a quadratic, and therefore satisfy the symmetry condition

$$(x_r - c)'(g_s - o) = (x_s - c)'(g_r - o) .$$

These equations, which must be satisfied by any quadratic centre, will be called the centre equations for the configuration. Their solutions form a linear manifold which will be called the centre manifold of the configuration, which is an orthogonal complement of the gradient manifold.

THEOREM 2. The symmetry condition is necessary for the consistency of the centre equations, and implies their equivalence to the equation

$$G_o' x_o - X_o' g_o = G_o' c$$

where $X_o = \{x_r - x_o\}$, $G_o = \{g_r - g_o\}$ and $X_o' G_o$ is symmetric.

It is necessarily; for if the equations are consistent and c is a solution, then

$$(g_r' x_s - g_s' x_r) + (g_s' x_t - g_t' x_s) + (g_t' x_r - g_r' x_t)$$
$$= (g_r - g_s)' c + (g_s - g_t)' c + (g_t - g_r)' c = 0 ,$$

which gives the triadic condition, equivalent to the symmetry. Now, addition of

$$g_r' x_o - g_o' x_r = (g_r - g_o)' c , \quad g_o' x_s - g_s' x_o = (g_o - g_s)' c ,$$

by virtue of the triadic condition, gives

$$g_r' x_s - g_s' x_r = (g_r - g_s)' c .$$

Hence, the centre equations are equivalent to the subsystem

$$g_r' x_o - g_o' x_r = (g_r - g_o)' c .$$

That is,

$$(g_r - g_o)' x_o - (x_r - x_o)' g_o = (g_r - g_o)' c ,$$

which is

$$G_o' x_o - X_o' g_o = G' c .$$

Moreover, this symmetry condition for the configuration is equivalent to the symmetry of the matrix $X_o{}'G_o$, by Theorem 3.2.

THEOREM 3. <u>The symmetry condition is sufficient for the consistency of the centre equations of a regular simplicial configuration.</u>

For in this case G has $k \leq n$ independent columns; so there exists at least one solution of the equation

$$G_o{}'x_o - X_o{}'g_o = G_o{}'c \ ,$$

to which the centre equations are equivalent, by Theorem 2.

THEOREM 4. <u>The centre equations of a configuration imply the centre equations for any other configuration which is generated by it.</u>

If α, β are any distributions, multiplication of

$$g_r{}'x_s - g_s{}'x_r = (g_r - g_s)'c$$

by $\alpha_r \beta_s$ and summation over r, s gives

$$g_\alpha{}'x_\beta - g_\beta{}'x_\alpha = (g_\alpha - g_\beta)'c \ .$$

COROLLARY. <u>The centre manifold of a simplicial configuration contains the centre manifold of every configuration contained in its linear closure, and is identical with the centre manifold of any such configuration which contains that configuration.</u>

THEOREM 5. <u>The symmetry condition is sufficient for the consistency of the centre equations of a regular complete configuration.</u>

For such a configuration, by Theorem 3.3, is linearly generated by a regular simplicial subconfiguration; and the two configurations have their centre manifold identical, by Theorem 4, Corollary.

A <u>normal</u> <u>configuration</u> is defined to be one for which the base and gradient manifolds are completely inclined, there being no submanifold of which one is perpendicular to the other. Then either one taken with an orthogonal complement of the other provides a complementary pair of manifolds, intersecting in a single point, and joining in the complete space.

THEOREM 6. A necessary and sufficient condition for a configuration $\{x_r, g_r\}$ to be normal is that the matrix $X_o{}'G_o = \{(x_r - x_o)'(g_s - g_o)\}$ be regular.

For it is singular if and only if there exists a distribution α such that for all distributions β

$$(x_\alpha - x_o)'(g_\beta - g_o) = 0 ,$$

and similarly, the other way around.

A configuration will be said to be centred if its centre equations are consistent.

THEOREM 7. For a normal centred configuration, the centre manifold cuts the base manifold in a unique point $\hat{c}$, given by

$$\hat{c} = x_o - X_o(G_o{}'X_o)^{-1}X_o{}'g_o$$

and this is the unique point at which the gradient in the linear closure is perpendicular to the base manifold, that gradient being

$$\hat{g} = g_o - G_o(G_o{}'X_o)^{-1}X_o{}'g_o ,$$

and such that

$$X_o{}'\hat{g}_o = 0 .$$

Evidently $\hat{c}$ belongs to X. Also

$$G_o{}'\hat{c} = G_o{}'x_o - X_o{}'g_o ,$$

so it belongs to Γ, the centre manifold. Now,

$$\hat{c} = x_o(1-\alpha) + (X_o + x_o)\alpha$$

where

$$\alpha = (G_o{}'X_o)^{-1}X_o{}'g_o$$

and so

$$\hat{g} = g_o(1-\alpha) + (G_o + g_o)\alpha$$
$$= g_o - G_o(G_o{}'X_o)^{-1}X_o{}'g_o$$

is the gradient at $\hat{c}$ in the linear closure. Moreover,

$$X_o'g = X_o'g_o - X_o'g_o = 0 .$$

The unique point $\hat{c}$ thus defined on the centre manifold of a normal configuration will be called its principal centre.

The absolute centre of a configuration is defined as the unique point $\bar{c}$ which is on and perpendicular to the centre manifold. It is thus the foot of the perpendicular from the origin to the manifold, and is the unique vector on the manifold of minimum length. Since the centre manifold is orthogonal to the gradient manifold, if c is any centre, then

$$\bar{c} = I_G c .$$

In particular,

$$\bar{c} = I_G \hat{c} .$$

THEOREM 8. The absolute centre is given by

$$\bar{c} = G(G'G)^{-1}(G'x_o - X'g_o) .$$

Evidently, the point $\bar{c}$ thus defined satisfies the centre equation. Moreover, $I_G\bar{c} = \bar{c}$, so it is the absolute centre. It is verified that

$$\bar{c} = I_G(x_o - X_o(G_o'X_o)^{-1}X_o'g_o)$$
$$= I_G\hat{c} ,$$

so that

$$\hat{c} - \bar{c} = J_G\hat{c} .$$

Any point of the centre locus is of the form

$$c = \bar{c} + \bar{k}$$
$$= \hat{c} + \hat{k}$$

where $\bar{k}$, $\hat{k}$ are vectors orthogonal to the gradient manifold, defining the absolute and principal displacements of any centre c.

THEOREM 9. Central symmetry is equivalent to the condition $\bar{c} = 0$.

For $\bar{c} = 0$ is equivalent to the condition $G_o{}'\hat{c} = 0$, which is

$$G_o{}'x_o = X_o{}'g_o \text{ , or equivalently, } g_r{}'x_o = g_o{}'x_r$$

and, with symmetry given, this is equivalent to central symmetry, by Theorem 3:10.

5. Convexity

THEOREM 1. *If* φ *is quadratic, and* $\alpha + \beta = 1$, *then*

$$\varphi(x\alpha + y\beta) = \varphi(x)\alpha + \varphi(y)\beta - (x-y)'(g(x)-g(y)\alpha\beta \text{ .}$$

For

$$\varphi(x) - \varphi(y) = \tfrac{1}{2}(x-y)'(g(x)+g(y)) \text{ ,}$$

by Theorem 1.3. Therefore

$$\begin{aligned}\varphi(x\alpha + y\beta) - \varphi(z) &= \tfrac{1}{2}(x\alpha + y\beta - z)'(g(x\alpha + y\beta) + g(z)) \\ &= \tfrac{1}{2}\{(x-z)\alpha + (y-z)\beta\}\{(g(x)+g(z))\alpha + (g(y)+g(z))\beta\} \\ &= \varphi(x)\alpha + \varphi(y)\beta - \varphi(z) - (x-y)'(g(x)-g(y))\alpha\beta \text{ .}\end{aligned}$$

THEOREM 2. *A necessary and sufficient condition that a quadratic with gradient* g *be convex is that*

$$(y-x)'(g(x)-g(y)) < 0 \qquad (y \neq x) \text{ .}$$

For, by Theorem 1, this condition is necessary and sufficient for

$$\varphi(x\alpha + y\beta) > \varphi(x)\alpha + \varphi(y)\beta$$

whenever $x \neq y$, $\lambda + \mu = 1$ and $\lambda, \mu \geq 0$; and this is the condition for φ to be convex.

COROLLARY (i) *A necessary and sufficient condition that a quadratic* φ *with gradient* g *be convex is that*

$$(y-x)'g(x) > \varphi(y) - \varphi(x) \text{ .}$$

For, by Theorem 1.3

$$\varphi(y) - \varphi(x) - (y-x)'g(x) = \tfrac{1}{2}(y-x)'(g(y)-g(x)) \text{ .}$$

COROLLARY (ii) *A necessary and sufficient condition that a quadratic be* convex is that its characteristic matrix be negative definite.

For, by Theorem 1.1,

$$(y-x)'(g(x)-g(y)) = \frac{1}{2}(y-x)'A(y-x) .$$

The negativity condition for a simplicial configuration $\{X,G\} = \{x_r, g_r\}$, with linear closure $\{X,G\} = \{x_\alpha, g_\alpha\}$, is now defined by the condition $\Delta_{\alpha\beta} < 0$, where

$$\Delta_{\alpha\beta} = (x_\alpha - x_\beta)'(g_\alpha - g_\beta) ,$$

for all distributions α, β.

From the identity

$$\varphi_{\alpha\sigma+\beta\rho} = \varphi_\alpha\sigma + \varphi_\beta\rho + \sigma\rho\Delta_{\alpha\beta} \quad (\sigma+\rho = 1) ,$$

or otherwise obviously:

THEOREM 3. Negativity is necessary for a configuration to admit a convex quadratic.

Since

$$\Delta_{\alpha\beta} = \sum_{r,s} (\alpha_r \beta_r)(\alpha_s - \beta_s)x_r' g_s$$
$$= (\alpha-\beta)'X'G(\alpha-\beta) ,$$

where $1'\alpha = 1 = 1'\beta$, so that $1'(\alpha-\beta) = 0$, the negativity condition requires that $\theta'X'G\theta$ be negative definite under the constraint $1'\theta = 0$. There is also the identity

$$\Delta_{\alpha\beta} = (\alpha-\beta)'\Delta(\alpha-\beta) ,$$

where

$$\Delta = \{\Delta_{rs}\} \text{ and } \Delta_{rs} = (x_r - x_s)'(g_r - g_s) .$$

Accordingly:

THEOREM 4. The negativity of a simplicial configuration $\{X,G\}$ is equivalent to the non-positive definiteness of either of the matrices

$$X'G - 1\{1'(X'G)^{-1}\}1' , \quad \Delta - 1\{1'\Delta^{-1}1\}1' .$$

Now

$$x_\alpha - x_o = \sum_{r=1}^{k} (x_r - x_o)\alpha_r ,$$

since $\sum_{r=0}^{k} \alpha_r = 1$; so that

$$x_\alpha - x_\beta = \sum_{r=1}^{k} (x_r - x_o)(\alpha_r - \beta_r) ;$$

and similarly for gradients. Therefore, with

$$X_o = \{x_r - x_o\} , \; G_o = \{g_r - g_o\} ,$$

these being matrices of order $n \times k$ having the vectors $x_r - x_o$ and $g_r - g_o$ $(r=1, \ldots, k)$ for their columns, and if

$$\alpha = \begin{pmatrix} \alpha_o \\ a_o \end{pmatrix} , \quad \beta = \begin{pmatrix} \beta_o \\ b_o \end{pmatrix}$$

are any distributions, partitioned at their first elements, then

$$\Delta_{\alpha\beta} = (a_o - b_o)' X_o' G_o (a_o - b_o) .$$

Accordingly,

THEOREM 5. <u>The negativity of a simplicial configuration</u>

$$\{X,G\} = \{x_r, g_r\} \; (r=0,1,\ldots k)$$

<u>is equivalent to the negative definiteness of the matrix</u> $X_o'G_o$, <u>where</u>

$$X_o = \{x_r - x_o\} , \; G_o = \{g_r - g_o\} \; (r=1,\ldots k) .$$

Now, if $X_o'G_o$ is negative definite, it must be regular. Therefore:

THEOREM 6. <u>For a simplicial configuration, negativity implies normality.</u>

Consider the matrix

$$K = \{g_r'(x_s - c)\} ,$$

for any vector c . Its symmetry is equivalent to the condition that c lie on the centre manifold of the configuration $\{x_r, g_r\}$, which provides $X_o'G_o$ symmetric, and also the relation

$$G_o'(x_o - c) = X_o'g_o .$$

Now, partitioning, and using this relation,

$$K = \left(\begin{array}{c|c} g_o'(x_o-c) & g_o'(x_s-c) \\ \hline g_r'(x_o-c) & g_r'(x_s-c) \end{array} \right), \quad (r,s=1,\ldots,k) ,$$

$$= \left(\begin{array}{c|c} 2\varphi_o & g_o'X_o + 2\varphi_o \\ \hline X_o'g_o + 2\varphi_o & G_o'X_o + X_o'g_o + g_o'X_o + 2\varphi_o \end{array} \right),$$

where

$$\varphi_o = \frac{1}{2}(x_o-c)'g_o .$$

Therefore, the condition that K be negative definite is that φ_o be negative, and

$$G_o'X_o + X'g_o + g_o'X_o + 2\varphi_o - \frac{1}{2}(X'g_o + 2\varphi_o)\varphi_o^{-1}(g_o'X_o + 2\varphi_o)$$
$$= G_o'X_o - \frac{1}{2}X_o'g_o\varphi_o^{-1}g_o'X_o$$

be negative definite. But

$$|G_o'X_o - \frac{1}{2}X'g_o\varphi_o^{-1}g_o'X_o| = \begin{vmatrix} 2\varphi_o & g_o'x_o \\ X_o'g_o & G_o'X_o \end{vmatrix} = |G_o'X_o| \; (2\varphi_o - g_o'x_o(G_o'X_o)^{-1}x_o'g_o)$$
$$= 2|G_o'X_o| \; (\varphi_o - \hat{\varphi}_o) ,$$

where

$$\hat{\varphi}_o = \frac{1}{2}(x_o-\hat{c})'g_o = \frac{1}{2}g_o'X_o(G_o'X_o)^{-1}X_o'g_o .$$

So, with φ_o negative, this matrix is negative definite if and only if $G_o'X_o$ is negative definite, and $\varphi_o < \hat{\varphi}_o$. But $\hat{\varphi}_o$ is negative if $G_o'X_o$ is negative definite. Therefore:

THEOREM 7. <u>With the matrix</u> $\{g_r'(x_s-c)\}$ <u>symmetric, it is negative definite if and only if the matrix</u> $G_o'X_o = \{(g_r-g_o)'(x_s-x_o)\}$ <u>be negative definite, and</u>

$$(x_o-c)'g_o < g_o'X_o(G_o'X_o)^{-1}X_o'G_o .$$

Now, the last part of this condition is

$$(x_o-c)'g_o < (x_o-\hat{c})'g_o$$

which can be written

$$g_o'(c-\hat{c}) > 0 .$$

However,

$$I_{\mathcal{G}}(c - \hat{c}) = c - \hat{c} ,$$

the centre manifold being perpendicular to the gradient manifold. Also

$$I_{\mathcal{G}} g_o = \bar{g} ,$$

the absolute gradient. Therefore, since $I_{\mathcal{G}}$ is symmetric, the condition becomes

$$(c-\hat{c})'\bar{g} > 0 .$$

Now the condition for K to be negative definite can be given in a form which is symmetrical in regard to the configuration elements, in which the arbitrary distinction of the o-element is removed. Thus:

THEOREM 8. _If_ c _is any centre of the configuration_ $\{x_r, g_r\}$, _then the matrix_ $\{g_r'(x_s-c)\}$ _is symmetric, and it is negative definite if and only if the configuration satisfies the negativity condition, and_ $(\hat{c}-c)'\bar{g} < 0$.

Under this condition, c will be called a _convex centre_ of the configuration.

THEOREM 9. _The centre of any convex quadratic admitted by a configuration is a convex centre of that configuration._

For, if $g_r = A(x_r-c)$, where A is negative definite, then

$$g_\alpha'(x_\alpha-c) = (x_\alpha-c)'A(x_\alpha-c) < 0 .$$

But

$$g_\alpha'(x_\alpha-c) = \sum_{r,s} g_r'(x_s-c)\alpha_r\alpha_s < 0$$

if and only if $\{g_r'(x_s-c)\}$ is negative definite.

What remains to be eventually shown is the converse, that any centre of a configuration is the centre of a quadratic admitted by the configuration, and that, moreover, any convex centre is the centre of such quadratic which is convex.

If the configuration is normal, so that $G_o'X_o$ is regular, the matrix $\{g_r'(x_s-c)\}$ is singular just if $\varphi_o = \hat{\varphi}_o$. Hence

THEOREM 10. *If* $\{x_r, g_r\}$ *is normal, then a necessary and sufficient condition that* $\{x_r'(g_s-c)\}$ *be regular is that* $g_o'(x_o-c)$ *be different from* $g_o'X_o'(G_o'X_o)^{-1}X_og_o$.

6. Initial value

A necessary condition that a configuration $\{x_r, g_r\}$ admit a quadratic is that there exist number φ_r such that

$$\varphi_r - \varphi_s = \frac{1}{2}(x_r - x_s)'(g_r + g_s) ,$$

and such set of numbers defining a *level set*, being uniquely defined but for an additive constant. Then $\{x_r, g_r, \varphi_r\}$ will be taken to define a *quadratic skeleton* based on the configuration, admitting any quadratic with levels φ_r and gradients g_r at the points x_r .

It will appear now that any quadratic admitted by a skeleton has a certain initial value, depending just on its centre.

THEOREM 1. *If* c *is a centre and* $\{\varphi_r\}$ *a level set for a configuration* $\{x_r, g_r\}$, *then*

$$\varphi_r - \frac{1}{2}g_r'(x_r - c) = \varphi_s - \frac{1}{2}g_s'(x_s - c) .$$

This follows directly from the centre equations, together with the equations for the level intervals.

THEOREM 2. *If a configuration* $\{x_r, g_r\}$ *admits a regular quadratic with centre* c *and taking values* $\varphi_r = \varphi(x_r)$ *then its initial value satisfies*

$$M = \varphi_r - \frac{1}{2}(x_r - c)'g_r$$

for all r .

Thus

$$\varphi_r = M + \frac{1}{2}(x_r - c)'A(x_r - c)$$

if A is the characteristic matrix; and

$$g_r = A(x_r - c) ;$$

and the consistency of these conditions on M is given by Theorem 1.

Thus all the concentric regular quadratics on a skeleton $\{x_r, g_r, \varphi_r\}$ have the same initial value, with determination as just described.

THEOREM 3. _If_ $M(c)$ _is the initial value for quadratics with centre_ c _on the skeleton_ $\{x_r, g_r, \varphi_r\}$ _and_ c^* _is any other centre, then_

$$M(c) - M(c^*) = \frac{1}{2}(c-c^*)'\bar{g} .$$

For, immediately,

$$M(c) - M(c^*) = \frac{1}{2}(c-c^*)'g_r .$$

But

$$c - c^* = J_G(c - c^*) ,$$

and

$$J_G g_r = \bar{g} ,$$

where J_G is symmetric; whence the result.

THEOREM 4. _The condition that_ c _be a convex centre, if such exist, is_ that $M(c) < M(\hat{c})$.

For such exist if and only if the negativity condition holds for the configuration, and then a necessary and sufficient condition is that $(c-\hat{c})'\bar{g} > 0$, which is $M(c) < M(\hat{c})$.

7. Characteristic matrix

It will now be shown that, corresponding to every point in the centre manifold of a normal configuration, there exists a regular quadratic admitted by the configuration, with that point on its centre; and the construction will be made of all such quadratics.

If c is a centre of the configuration $\{x_r, g_r\}$, then the matrix

$$K = \{g_r'(x_s-c)\}$$

is symmetric. For a regular quadratic with centre c to be admitted by the configuration, there has to exist a regular symmetric matrix A such that

$$g_r = A(x_r - c) ,$$

or equivalently,

$$x_r - c = Bg_r ,$$

where $B = A^{-1}$. In this case,

$$X_o = BG_o ,$$

so that

$$G_o'X_o = G_o'BG_o ,$$

whence, in the regular simplicial case, in which G_o is of rank k, the matrix $X_o'G_o$ must be regular symmetric and, therefore, the configuration normal symmetric.

Thus there can be formed the matrix

$$B_o = X_o(X_o'G_o)^{-1}X_o' ,$$

symmetric of rank k, such that

$$X_o = B_oG_o .$$

Now

$$B = B_o + S_o ,$$

where S_o is symmetric, and such that $S_oG_o = 0$.

And

$$x_r - Bg_r = x_o - Bg_o = c ,$$

so that

$$x_o - B_og_o - S_og_o = c .$$

But

$$x_o - B_og_o = \hat{c} ,$$

so this relation is

$$\hat{c} - c = S_og_o .$$

Let $\overline{G}_o$ be a base for the orthogonal complement of the range of G_o. Then the symmetric matrices S_o such that $S_oG_o = 0$ are of the form

$$S_o = \bar{G}_o \, \sigma \, \bar{G}_o' .$$

Since $X_o'G_o$ is regular, the partitioned matrix $(X_o\bar{G}_o)$ is square and regular. Accordingly, with $\rho = (X_o'G_o)^{-1}$, the nullity of

$$B = X_o \rho X_o' + \bar{G}_o \, \sigma \, \bar{G}_o'$$

$$= (X_o\bar{G}_o)\begin{pmatrix} \rho & o \\ o & \sigma \end{pmatrix}\begin{pmatrix} X_o' \\ G_o' \end{pmatrix} ,$$

is equal to the nullity of σ ; so B is regular provided σ is regular.

Now, for a unique γ ,

$$\hat{c} - c = \bar{G}_o \gamma ;$$

So, if the admitted quadratic with inverse characteristic matrix B is to have the centre c , there must be the relation

$$\bar{G}_o \, \sigma \, \bar{G}_o' g_o = \bar{G}_o \gamma ,$$

or equivalently,

$$\sigma \, \bar{G}_o' g_o = \gamma ,$$

which may be written

$$\sigma \kappa = \gamma , \text{ where } \kappa = \bar{G}_o' g_o$$

where

$$\kappa' \gamma = g_o' \bar{G}_o \gamma = g_o'(\hat{c}-c) = g_o' J_G(\hat{c}-c) = \bar{g}(\hat{c}-c) ,$$

since

$$\hat{c} - c = J_G(\hat{c}-c), \ \bar{g} = J_G g_o , \text{ and } J_G \text{ is symmetric.}$$

Given κ and γ , such that $\kappa'\gamma \neq 0$, the symmetric matrices σ which satisfy this relation are of the form

$$\sigma = \sigma_o + \sigma_1$$

where

$$\sigma_o = \gamma(\kappa'\gamma)^{-1}\gamma$$

and σ_1 is symmetric, such that

$$\sigma_1 \kappa = 0$$

so it must be of the form $\sigma_1 = \bar{\kappa} \tau \bar{\kappa}'$, where $\bar{\kappa}$ is a base for the orthogonal complement of the ray on κ.

If $\kappa'\gamma = 0$, let

$$\kappa = \begin{pmatrix} \kappa_0 \\ \kappa_1 \end{pmatrix}, \quad \gamma = \begin{pmatrix} \gamma_0 \\ \gamma_1 \end{pmatrix},$$

the partitions being at the leading elements, where, without loss in generality, it can be supposed that $\kappa_0 \neq 0$. Then $\gamma_0 \kappa_0 + \gamma_1' \kappa_1 = 0$, so that $\gamma_1 = -\kappa_0^{-1} \gamma_1' \kappa_1$. Now take

$$\sigma_0 = \begin{pmatrix} 0 & \kappa_0^{-1}\gamma_1' \\ \gamma_1 \kappa_0^{-1} & 0 \end{pmatrix}$$

this again being symmetric, and such that

$$\sigma_0 \kappa = \gamma .$$

Construction has therefore been made of all the regular quadratics admitted by the given configuration, and it appears that these exist, provided $G_0'X_0$ is symmetric. Moreover, a subclass of these has been characterized whose centre is any point c on the centre manifold of the configuration.

THEOREM 1. *Every point on the centre manifold of a normal symmetric configuration is the centre of a regular quadratic admitted by that configuration.*

Now the condition for B to be negative definite is that $\rho = (G_0'X_0)^{-1}$, and σ negative definite. But, given

$$\sigma\kappa = \gamma ,$$

the condition that σ can be chosen negative definite is that $\kappa'\gamma < 0$, or equivalently

$$(\hat{c}-c)'\bar{g} < 0 .$$

But this is precisely the condition that c be a convex centre.

THEOREM 2. *Every point* c *on the centre manifold of a normal negative symmetric configuration such that* $(\hat{c}-c)'\bar{g} < 0$ *is the centre of a convex quadratic admitted by that configuration.*

In other words, every convex centre is the centre of an admitted convex quadratic.

To proceed, again, with another manner of solution: there has to be found all the symmetric matrices A such that

$$G_o = AX_o ,$$

where $G_o'X_o$ is symmetric and regular. Without loss in generality, it can be supposed that

$$G_o = \begin{pmatrix} H_o \\ H_1 \end{pmatrix} , \quad X_o = \begin{pmatrix} Y_o \\ Y_1 \end{pmatrix} ,$$

where Y_o is square and regular, so that

$$Y = \begin{pmatrix} Y_o & 0 \\ Y_1 & 1 \end{pmatrix}$$

is a regular square matrix. Let

$$H = \begin{pmatrix} H_o & U \\ H_1 & V \end{pmatrix} .$$

Then U, V have to be chosen so that

$$A = HY^{-1}$$

is symmetric, an equivalent condition being that $Y'H$ is symmetric.

Now,

$$X_o'G_o = Y_o'H_o + Y_1'H_1 \text{ being symmetric,}$$

$$Y'H = \begin{pmatrix} Y_o'H_o + Y_1'H_1 & Y_o'U + Y_1'V \\ H_1 & V \end{pmatrix}$$

is symmetric if and only if V is symmetric, and

$$H_1' = Y_o'U + Y_1'V .$$

Accordingly, with V an arbitrary symmetric matrix, and

$$U' = (H_1 - VY_1)Y_o^{-1} ,$$

there is obtained a symmetric matrix $A = HY^{-1}$ with the property that

$G_o = AX_o$; and all the symmetric matrices with this property are of this form, in one-to-one correspondence with the symmetric matrices V .

Explicitly,

$$A = \begin{pmatrix} A_o & U \\ U' & V \end{pmatrix},$$

where V is an arbitrary symmetric matrix, U has the value just given, and A_o is determined as a symmetric matrix from the relation

$$Y_o'A_oY_o - Y_1'VY_1 = H_o'Y_o - Y_1'H_1 ,$$

since $H_o'Y_o - Y_1'H_1$ is symmetric, which provides the desired value

$$A_o = (H_o - UY_1)Y_o^{-1} .$$

Now for A to be negative definite, it is necessary and sufficient that

$$V \text{ and } A_o - UV^{-1}U'$$

be negative definite, or equivalently, that V and

$$Y_o'(A_o - UV^{-1}U')Y_o = G_o'X_o - H_1'V^{-1}H_1$$

be negative definite. But this is the condition that

$$Y'H = \begin{pmatrix} G_o'X_o & H_1' \\ H_1 & V \end{pmatrix}$$

be negative definite.

Thus the general solution when $G_o'X_o$ is regular, in which case $k \leq n$, is

$$A = A^* + R$$

where

$$A^* = \begin{pmatrix} Y_o'^{-1}(Y_o'H_o - H_1'Y_1)Y_o^{-1} & Y_o'^{-1}H_1 \\ H_1Y_o^{-1} & 0 \end{pmatrix}$$

and

$$R = \bar{X}_oV\bar{X}_o' , \quad \text{with } \bar{X}_o = \begin{pmatrix} Y_o'^{-1}Y_1 \\ -1 \end{pmatrix}.$$

If $k = n$, there is a unique solution

$$A = G_o X_o^{-1} .$$

If $k > n$, write

$$G_o = (F_o F_1) , \quad X_o = (Z_o Z_1) ,$$

where, in the case of a regular complete configuration, it can be assumed without loss in generality that $F_o' Z_o$ is regular. Then, with $F_o' Z_o$ is symmetric, it follows that

$$A = F_o Z_o^{-1}$$

is symmetric and such that

$$A Z_o = F_o .$$

Moreover, with

$$Z_o' F_1 = F_o' Z_1 ,$$

there follows

$$Z_o'^{-1} F_o' Z_1 = F_1$$

and then

$$A Z_1 = F_1 .$$

Thus there is found a unique regular symmetric matrix A such that $A X_o = G_o$.