A MODEL OF MONETARY EXCHANGE

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1. Introduction

It is generally thought that the monetary system plays an important role in the coordination of economic activity in a decentralized economy. However, the theory of general equilibrium, which is designed to analyse the coordination of economic activity, has no meaningful place for a monetary system. Therefore, some authors argue that the usefulness of general equilibrium theory is rather limited. Yet, most economists seem willing to accept general equilibrium theory as an approximate description even of a monetary economy. The present paper uses an explicit model of monetary exchange to analyse what role the medium of exchange actually plays and what precisely are the limitations of general equilibrium theory in the context of a monetary economy.

I study the use of a medium of exchange in an economy with many agents who meet pairwise at random times. If exchanges are voluntary, such an economy cannot in general reach the same allocation as an economy with multilateral trading. To attain at least a degree of multilateral exchange, agents must engage in indirect trading. In the model presented here, indirect exchange is limited to a single commodity, which is traded back and forth and serves as the medium of exchange. This medium of exchange has no intrinsic commodity value of its own.

The specification of expectations is the key to the theory of indirect exchange. An agent's willingness to engage in indirect exchange depends on his expectations about future exchange opportunities. He will not buy a good for which he has no use unless he expects to resell it at a profit. The value of fiat money, which has no intrinsic commodity value of its own, derives solely from the expectation of its value in future exchanges. If fiat money was expected to be worthless in future exchanges, it would be worthless in current exchanges.

The model of this paper contains at least two equilibria in which expectations about future exchange opportunities are self-fulfilling. In one equilibrium, the value of the medium of exchange is always zero and the economy is restricted to the exchanges that are possible under direct barter. In the other equilibrium, the medium of exchange has positive value and supports a certain amount of multilateral exchange.

Once money has been "invented", it seems likely that the direct barter equilibrium can never be reached by a disequilibrium adjustment process. On the other hand, the economy may start in the direct barter equilibrium because agents have never become aware of the scope for indirect exchange. If at some time, they learn about the potential for trade, and begin to use a medium of exchange, the shift to the monetary exchange equilibrium represents a singular and irreversible historical event.

In this paper, the frequency of meetings between agents is treated as an exogenous parameter of the economy. This assumption is artificial and used only as a simple and tractable way to model the frictions which I believe to be essential to the understanding of a monetary economy. In reality, it is likely that an agent with an empty refrigerator will exert himself to meet the grocer. However, I am confident that the results of this paper hold for more general specifications of the trading process.

If the frequency of meetings is large, there is little friction in the trading process, and the cost of arbitraging between different trading partners is small. In the limit as the frequency of meetings grows out of bounds, the allocation of resources in a monetary exchange equilibrium approaches the allocation of resources in frictionless multilateral exchange at competitive equilibrium prices.

Specifically, consumption and production flows approach their competitive equilibrium values. But at the same time, commodity inventories and real money balances, which are designed to smooth the frictions in the trading process, go to zero, because they are no longer needed. If frictions are assumed to be negligible, the theory of general competitive equilibrium approximates the allocation of production and consumption flows in the monetary economy. However, as a theory of the frictionless economy, general equilibrium theory is inherently incapable of analysing how the monetary system itself works, i.e. how the holding of inventories and the use of money as a medium of exchange enable the economy to deal with a given amount of friction in the trading process.

The limitation of general equilibrium theory is illustrated succinctly by the concept of the speculative demand for money. The classical argument that there can be no speculative demand for money in a long run rational expectations equilibrium is valid in the limit as the frequency of meetings becomes infinite. Therefore, the classical quantity equation with a technological velocity of circulation becomes valid as the frequency of meetings becomes infinite.

However, when the frequency of meetings is finite, there exists a speculative demand for money, which invalidates the classical quantity equation. This speculative demand for money is due to the randomness of prices in different meetings. In a given exchange, an agent may hold back on the purchase of a commodity and keep his money until the next meeting because he hopes to get a better price then. Thus, the underlying friction of the trading process is responsible both for the transactions and for the speculative motive for holding money. For all practical purposes it is impossible to distinguish the two motivations.

The next section develops the basic model, first of individual behaviour, then of economic interaction. The first part of section 3 discusses the simultaneous existence of direct barter and monetary equilibria. The rest of section 3 is devoted to a tentative analysis of the motives for holding money. In particular, I consider the role of the speculative motive in any given trading instant, and derive its implications for the pricing of multiple media of exchange. Furthermore, I discuss the importance of consumption and accumulation behaviour for the analysis of long run monetary equilibrium. Section 4 analyses the relationship between monetary exchange and general competitive equilibrium by considering monetary exchange equilibria at high trading frequencies. All proofs are collected in the appendix.

2. The Basic Model

2.1. Individual Behaviour

A representative economic agent solves an intertemporal maximization problem. At each instant, he produces a constant commodity vector $a \in \mathbb{R}_+^{n+1}$, and he consumes a commodity vector $c(t) \in \mathbb{R}_+^{n+1}$. He manages a vector of inventories $y(t) \in \mathbb{R}_+^{n+1}$, so as to maximize the expected present value of utility of consumption over an infinite time horizon. His inventories must never become negative.

The first good (indexed by "0") is money, which is neither produced nor consumed so that $a_o = c_o(t) = 0$.

The evolution of the agent's inventories is governed by two regimes: For almost all t, it is given by a simple accumulation equation:

$$\dot{y}(t) = a - c(t).$$

In addition, there is a random sequence of instants $\{T_i\}_{i=0}^{\infty}$, at which the agent can trade the excess demand vector z_i, so that the inventory dynamics at T_i are given as:

$$y(T_i+) = y(T_i-) + z_i.$$

For any time interval of length I, the number of trading instants in I satisfies a Poisson distribution with mean μI, where μ is an exogenously given parameter.

At any trading instant T_i, there is a price vector p^i and two rationing vectors b^i, s^i, such that the value of the excess demand z_i at the price p^i is zero, and furthermore, the amounts purchased (sold) do not exceed b^i (s^i). The agent assumes that the triple (p^i, b^i, s^i) is drawn from a stationary probability distribution $G(\cdot)$, which is independent and identical for different trading instants. Moreover, he assumes that the distribution $G(\cdot)$ is independent of his own inventories prior to trading.[1]

I shall consider distributions $G(\cdot)$ on the space $\mathbb{R}_+ \times \Sigma^n \times \mathbb{R}_+^{2(n+1)}$: All prices are expressed in terms of an exogenous unit of account; the price of money may be any nonnegative real number; the n-vector of real commodity prices lies in the n-dimensional simplex. The rationing constraints may be given by any pair of nonnegative (n+1)-vectors. I shall take the set of distributions on $\mathbb{R}_+ \times \Sigma^n \times \mathbb{R}^{2(n+1)}$ to be endowed with the topology of weak convergence of probability measures.

In a previous paper [13], I have shown that the agent's problem can be formulated as a dynamic program: Let $V(y)$ be the value of the optimal policy starting from initial inventories y in the absence of a trading instant. Then, one can write:

$$(1) \qquad V(y) = \max_{c(\cdot)} \int_0^\infty e^{-(\rho+\mu)t}(u(c(t)) + \mu W(y(t)))\, dt$$

$$\text{s.t.} \quad \dot{y}(t) = a - c(t)$$
$$y(t) \geq 0$$
$$y(0) = y,$$

where the function $W(\cdot)$ is defined as:

$$(2) \qquad W(y) = \int \max_{z} V(y+z)\, dG(p,b,s)$$

$$\text{s.t.} \quad p'z = 0$$
$$z \leq b$$
$$-z \leq s$$
$$y+z \geq 0.$$

The value of the optimal program for initial inventories y is composed of the value of the optimal consumption path that will be followed until the next trading instant and the expected present value of the optimal consumption program after the optimal trade at the next trading instant.

I shall make the following assumptions about the instantaneous utility function:

A.1: The instantaneous utility function $u\colon \mathbb{R}_+^{n+1} \to [\underline{u}, \overline{u}]$ IR is twice continuously differentiable and bounded.

A.2: The set of all commidites can be partitioned into two sets J_1, J_2 such that:

a: For all $j \in J_1$, and all c, $u_j(c) > 0$;

b: For all c, the matrix (u_{jk}), j, $k \in J_1$ is strictly negative definite;

c: If $\{c^k\}$ converges to $\bar{c}$, where $\bar{c}_j = 0$ for some $j \in J_1$, then

$\lim_{k\to\infty} u_j(c^k) = \infty$

d: The set J_1 contains at least two commodities;

e: For all $j \in J_2$, and all c, $u_j(c) = 0$;

f: $0 \in J_2$.

There may be commodities in which the agent is not interested at all. But for those in which he is interested, his utility is strictly increasing, strictly concave and satisfies an Inada condition at the origin.

Assumptions A.1 and A.2 imply that the valuation functions V, W exist, are continuously differentiable, concave and bounded. Furthermore, they have the same monotonicity properties as u, and V is strictly concave with respect to goods in J_1.

An optimal policy is given as a pair of functions $c(y; G)$, $z(p,b,s; y; G)$. The consumption function c determines the individual's consumption when he does not trade as a function of his inventories and his expectations about future trading conditions. The excess demand function z determines the desired exchange at a trading instant as a function of current trading conditions, inventories and expectations about future trading conditions.

Optimal policies may not be well defined when prices or price expectations are at the boundary of their domain. Therefore, I shall restrict expectations to the set of distribution functions that assign positive weight to the interior of the price set:

$$\Gamma = \left\{ G \mid G(\{p \mid p > 0,\ p_o < \infty\} \times \mathbb{R}^{2(n+1)}) > 0 \right\} .$$

Then the following propostion is a straightforward generalization of Propostions 2.2 and 2.3 in Hellwig [13] and is stated without proof:

Propostion 2.1: Let $G \varepsilon \Gamma$. Under Assumptions A.1 and A.2, the consumption function $c(y; G)$ is single-valued and jointly continuous in y and G. Let $x(y, s) = (\min(y_i, s_i))$; then, for $p > 0$, $p'x(y,s) > 0$, the excess demand function $z(p, b, s; y; G)$ is single-valued and jointly continuous in p, y and G. For any $G \varepsilon \Gamma$, let $(p, b, s, y)^k$ converge to $(\bar{p}, \bar{b}, \bar{s}, \bar{y})$, where $\bar{p}_i = 0$, $\bar{b}_i = \bar{s}_i = \infty$ and $p'x(y, s) > 0$; then $\lim_{k\to\infty} \| z(p^k, b^k, s^k, y^k; G) \| = \infty$.

Thus, consumption and excess demand behaviour are well defined and continuous whenever both, prices and price expectations are in the interior of their domains and moreover, in the given trading instant, the intersection of the budget set with the nonnegative orthant has a nonempty interior. Furthermore, as the current price vector approaches the boundary of its domain, the excess demand for at least one commodity grows out of bounds.

Under Assumption A.2.f, money is not consumed. It is acquired and held, only because it can be resold. Therefore, optimal behaviour satisfies the neoclassical homogeneity postulate: The excess demand for goods and for "real" money balances as well as consumption behaviour are homogeneous of degree zero in money balances and all money prices.[2] To make this statement precise, I first introduce some notation:

Define the function $f: \mathbb{R}_+^{n+2} \to \mathbb{R}_+^{n+1}$:

$$(3) \qquad f(m, x) = (mx_o, x(0)),$$

where m is a positive scalar, x_o the element of x with index zero and $x(0)$ the vector obtained from x by omitting the element with index zero. Further, for $m > 0$ and any set $X \subset IR_+^{n+1}$, let $f^{-1}(m, X)$ denote the inverse image set of X under the function $f(m, \cdot)$.

Then, one obtains a class of transformations $T_m: \Gamma \to \Gamma$ on distribution functions by the condition that for any Borel set $P \times B \times S$,

$$(4) \qquad (T_m G)(P \times B \times S) = G(f^{-1}(m,P) \times f^{-1}(\tfrac{1}{m}, B) \times f^{-1}(\tfrac{1}{m}, S)).$$

To motivate the use of the transformations T_m, note that money prices $q_1,\ldots q_n$ are given in terms of accounting prices as $p_1/p_o,\ldots p_n/p_o$. Thus, multiplying money prices by a constant is equivalent to dividing the accounting price of money p_o by the same constant. To perform the classical homogeneity experiment, one has to divide p_o in every transaction - expected as well as current - by the same factor, by which one multiplies money holdings and the quantity constraints on monetary transactions. Then, it is easy to prove:

Proposition 2.2: Let $G \in \Gamma$. Under Assumptions A.1, A.2, one has for all positive scalars m:

$$c(f(m,y); T_{1/m}G) \equiv c(y; G)$$

$$z(f(1/m, p), f(m,b), f(m,s); f(m,y); T_{1/m}G) \equiv f(m, z(p,b,s; y; G)).$$

2.2 The Structure of the Economy

a: Preferences and Production Vectors

The economy consists of n types of agents (as many types as commodities).

There is a continuum of unit mass of each type. Each type is characterized by a quintuple (u, ρ, a, μ, G) of preferences, a production vector, a trading frequency and expectations.

To keep the notation down, I assume that all agents have the same time preference and the same trading frequency. This has no effect on the results of this paper. Production vectors and preferences are given by the following assumptions:

B.1: The k-th type's production vector is given by:

$$a_o^k = 0$$

$$a^k(0) = a\ e_k,$$

where e_k is the k-th unit vector.

B.2: The instantaneous utility of the k-th type satisfies:

$$J_1^k = \left\{ k,\ k+1 \text{ (modulo n)} \right\},$$

i.e. $u_1^k = 0$ for all $i \neq k$, k+1 (modulo n).

An agent of the k-th type produces commodity k. He consumes commodities k and k+1. Presumably, he would like to sell some of his production of commodity k in return for some units of commodity k+1. However, commodity k+1 is produced by agents of type k+1 who in turn want to buy commodity k+2 rather than commodity k. Thus, the economy has the familiar chain structure of Samuelson's consumption-loan model, in which the principle of the double coincidence of wants is most difficult to meet.

b: The Specification of the Monetary Exchange Process

Agents meet in pairs. At a trading instant, an individual draws a trading partner at random from the rest of the population. The exchange between two agents of types j, k with inventories and expectations (y^j, G^j), (y^k, G^k), is determined by a quintuple (p, b^j, s^j, b^k, s^k) of prices and quantity constraints.

I shall assume that trading leaves the aggregate inventories of all commodities unchanged:

C.1: For any j, k, (y^j, G^j), (y^k, G^k), the quintuple (p, b^j, s^j, b^k, s^k) of trading conditions satisfies:

(5) $$z^j(p, b^j, s^j; y^j; G^j) + z^k(p, b^k, s^k; y^k; G^k) = 0.$$

The chain structure of excess demands under Assumptions B.1 and B.2 implies that there must be indirect exchange, if there is to be any trade at all. Agents must be willing to accept payment in a commodity which they do not want to consume, but which they hope to resell. It will be assumed that all such indirect exchange is limited to a single commodity, money:

C.2: For all k, $s_i^k = 0$ for all $i \neq k$, 0.

An agent of type k is unable to sell any good except commodity k and money. Therefore, he will never acquire inventories of goods other than k, k+1 and money. He derives no utility from other commodities and he cannot resell them; thus he is unwilling to spend any resources on acquiring them.

Assumption C.2 represents the idea that money serves as "the" medium of exchange. It corresponds closely to the view of the "monetary budget constraint" that is prevalent in the literature.[3] Unfortunately, this constraint does not emerge endogenously from the way the economy operates, but has to be imposed by assumption.

However, it should be noted that this assumption can be supported by a set of self-fulfilling expectations. Suppose that agent k is known as the producer of commodity k; other agents do not bother to ask him whether he has anything else to sell. Then, he does not hold any inventories of goods other than k, k+1 and money. In any given trading instant, the constraint C.2 is not regarded as binding, because he has no inventories with which to violate it. Moreover, the odd agent who does ask him whether he has anything else to sell receives a negative answer and will soon learn not to ask.[4]

It remains to specify the determination of prices and the other quantity constraints. Which specification one chooses does not seem to matter very much. But an equilibrium specification is more appealing than <u>any particular</u> disequilibrium specification. Therefore, I shall assume that in any given bilateral meeting prices adjust to clear the "market" between the two agents who meet. Equivalently, with one exception, agents face no quantity constraint in addition to those given by Assumption C.2.

The exception is a prohibition of purchases of commodity k+1 by agents of type k from agents of a type other than k+1. This additional quantity constraint avoids the irrelevant pathology when agent k with $y^k_{k+1} = 0$, $V^k_{k+1} = \infty$, meets agent k-1, and in equilibrium, $p_{k+1} > 0$, $p_k = 0$, $p_o = 0$, so that p_k/p_o is not well defined even though it is the most crucial relative price.

C.3: For all k, $s^k_o = s^k_k = \infty$.

C.4: For all k, all $i \neq k+1$, $b_i^k = \infty$;

$$\text{for all } k, \quad b_{k+1}^k = \begin{cases} 0 & \text{if agent } k \text{ meets an agent of type } j \neq k+1. \\ \infty & \text{if agent } k \text{ meets an agent of type } k+1. \end{cases}$$

For future reference, it is useful to introduce the equilibrium price correspondence P^{jk} for agents of types j, k:

$$(6) \qquad P^{jk}(y^j, y^k; G^j, G^k) = \Big\{p \,|\, z^j(p, b^j, s^j; y^j; G^j) + z^k(p, b^k, s^k; y^k; G^k) = 0;$$
$$b^j, s^j, b^k, s^k \text{ satisfy C.2 -- C.4}\Big\} .$$

Lemma 2.3: Let $G^j, G^k \in \Gamma$. Then the equilibrium price correspondence P^{jk} is nonempty and upper-hemicontinuous with respect to y^j, y^k, G^j, G^k.

The functioning of the monetary exchange process given by Assumptions C.1 - C.4 can now be sketched as follows: An agent of type k holds inventories of commodities k, k+1 and money. When he is not trading, he builds up his inventory of commodity k and runs down his inventory of commodity k+1. When he meets an agent of type k-1, he sells commodity k in return for money at a price that equates demand and supply between these two agents. When he meets an agent of type k+1, he buys commodity k+1 in return for money. When he meets another agent of type k, they may trade commodities k and money, but this sort of exchange is presumably of minor importance. When an agent of type k meets an agent of a type other than k-1, k or k+1, they talk about the weather.

c: The Evolution of the Economy

I shall not consider the time path of inventories for each individual agent. Instead, the evolution of the economy over time is studied in macroscopic terms through the frequency distribution of inventories across agents of a given type.

At any time t, the macroscopic state of the economy is a vector $\mathcal{F}_t = (F_t^k(\cdot))$ of distribution functions on $\mathbb{R}_+^{n+1}$, such that F_t^k is the distribution of inventories across agents of type k at time t. For any measureable set $Y \subset \mathbb{R}_+^{n+1}$, $F_t^k(Y)$ is the proportion of agents of type k with inventories $y^k(t) \varepsilon Y$. The state of the economy evolves over time as the individual agents produce, consume and trade.

The description of the monetary exchange process in C.1 - C.4 is incomplete, because the competitive equilibrium between two agents who meet need not be unique. In this case, the price at which two agents actually trade is assumed to be drawn from an arbitrary, but fixed probability distribution on the set of equilibrium prices. Formally, the trading process is represented by a pair of matrices $\mathcal{Q} = (Q^{jk})$, $\mathcal{K} = (K^{jk})$, such that;

(7) For all j, k, y^j, y^k, $G^j \varepsilon \Gamma$, $G^k \varepsilon \Gamma$,

a: $Q^{jk}(. : y^j, y^k; G^j, G^k)$ is a probability distribution on $\mathbb{R}_+ \times \Sigma^n$.

b: $Q^{jk}(P^{jk}(y^j, y^k; G^j, G^k); y^j, y^k; G^j, G^k) \equiv 1$

c: $K^{jk}(X; y^j, y^k; G^j, G^k) \equiv Q^{jk}(\{P | y^j + z^j(p, y^j, G^j) \varepsilon X\}; y^j, y^k; G^j, G^k)$
for all $X \subset \mathbb{R}_+^{n+1}$.

The mapping Q^{jk} specifies the probability distribution of prices at which agents j and k trade as a function of their inventories and expectations. Similarly, the mapping K^{jk} specifies the probability distribution of agent j's inventories after a trade with agent k as a function of their inventories before the trade and their expectations.

Let the vector of expectations $\mathcal{G} = (G^j(\cdot)) \in \Gamma^n$ and trading matrices $\mathcal{Q}, \mathcal{R}$ satisfying (7) be given and suppose that for all Borel sets X, the function $K^{jk}(X; \cdot, \cdot; G^j, G^k)$ is measurable. Then the evolution of the economy under the consumption functions $c^k(\cdot\, ; G^k)$, $k = 1,\ldots n$, and the trading matrices $\mathcal{Q}, \mathcal{R}$ is determined by the following system of forward equations:

(8) For all k, and all Borel sets $Y \subset \mathbb{R}_+^{n+1}$,

$$F_{t+h}^k(Y) = (1 - \mu h)\, F_t^k(\phi^k(Y))$$

$$+ \mu h \sum_{j=1}^{n} \int_{\mathbb{R}_+^{n+1}} K^{kj}(\phi^k(Y);\, y^k, y^j;\, G^k, G^j)\, F_t^k(dy^k)\, F_t^j(dy^j) + 0(h),$$

where: $\phi^k(Y) = \left\{x \,|\, y(h;\, x) \in Y\right\}$.

and: $\lim_{h \to 0} \dfrac{0(h)}{h} = 0.$

<u>Proposition 2.4</u>: Assume A.1, A.2, B.1, B.2 and let the vector of expectations $\mathcal{G} \in \Gamma^n$ be given. Then, there exist trading matrices $\mathcal{Q}, \mathcal{R}$, satisfying (7), such that the process (8) is well defined and furthermore, there exists a state of the economy that is stationary under the process (8).

The discussion of this paper will be limited to stationary states. I shall neglect the question whether the economy must converge to a stationary state. If the economy is in a stationary state, the population from which an agent draws his trading partners does not change over time. Therefore, the distribution of trading conditions that he encounters is stationary and can be inferred from the actual frequency distribution.

For given expectations $\mathcal{G} \in \Gamma^n$, let $\mathcal{Q}, \mathcal{K}$ be trading matrices satisfying (7) and $\mathcal{F}$ a stationary state for the process (8). Write s^k for agent k's sales constraint according to C.2 and C.3, b^{ko} for his purchasing constraint in trades with agents other than of type k+1 and $b^{k,k+1}$ for his purchasing constraint in trades with agents of type k+1, according to C.4. Then, the experience for an agent of type k in the stationary $\mathcal{F}$ is written as:[5]

$$(9)\quad a:\ E^k(P \times \{b^{ko}\} \times \{s^k\}; \mathcal{Q}, \mathcal{F}, \mathcal{G}) = \frac{1}{n} \sum_{\substack{j=1 \\ j \neq k+1}}^{\mu} \int_{\mathbb{R}_+^{2(n+1)}} Q^{kj}(P;\ y^k, y^j;\ G^k, G^j) F^k(dy^k) F^j(dy^j)$$

$$b:\ E^k(P \times \{b^{k,k+1}\} \times \{s^k\}; \mathcal{Q}, \mathcal{F}, \mathcal{G}) = \frac{1}{n} \int_{\mathbb{R}_+^{2(n+1)}} Q^{k,k+1}(P;\ y^k, y^{k+1};\ G^k, G^{k+1}) F^k(dy^k) F^j(dy^k)$$

for any Borel set P.

The economy will be in a rational expectations equilibrium, if the original expectations $\mathcal{G}$ coincide with the actual experience given by (9):

<u>Definition 2.5</u>: A <u>rational expectations equilibrium</u> is a pair of vectors of distribution functions $\mathcal{F}^*, \mathcal{G}^*$ for inventories and trading conditions and a pair of trading matrices $\mathcal{Q}^*, \mathcal{K}^*$, such that:

a: Given the expectations $\mathcal{G}^*$, the trading matrices $\mathcal{Q}^*, \mathcal{K}^*$ satisfy (7);

b: $\mathcal{F}^*$ is a stationary state under the process (8);

c: for all k, $E^k(\cdot\ ; \mathcal{Q}^*, \mathcal{F}^*, \mathcal{G}^*) = G^{*k}(\cdot)$.

The existence and most important properties of rational expectations equilibria under monetary exchange will be the subject of the following section.

3. Equilibrium under Monetary Exchange

3.1. Rational Expectations Equilibria for Money and Direct Barter

In this section, I propose that monetary exchange works because it is expected to work. Since fiat money has no direct use, agents buy it only because they hope to resell it at a later time in return for real resources. If they expect that the value of money in all future transactions is zero, they are not willing to give up real resources for money, and money cannot function as the medium of exchange. On the other hand, one can find a rational expectations equilibrium in which money has a positive price and serves as the medium of exchange, so that a positive level of trade can be maintained.

Proposition 3.1: Given A.1 - B.2, C.1 - 4, there exists, for any $M > 0$, a rational expectations equilibrium $(\mathcal{F}^*_B, \mathcal{G}^*_B, \mathcal{Q}^*_B, \mathcal{H}^*_B)$, such that:

a: $\sum_{k=1} \int_{\mathbb{R}^{n+1}_+} y^k_o \; F^{*k}_B(dy^k) = M;$

b: for all k, $G^{*k}_B(\{p|p_o = 0\} \times \{b^{ko}, b^{k,k+1}\} \times \{s^k\}) = 1;$

c: for all k, $F^{*k}_B(\{y|y(0) = 0\}) = 1;$

d: for all k, $F^{*k}_B(\{y|c^k(y; G^{*k}_B) = a^k\}) = 1.$[6]

Proposition 3.2: Given A.1 - B.2, C.1 - 4, there exists, for any $M > 0$, a rational expectations equilibrium $(\mathcal{F}^*_M, \mathcal{G}^*_M, \mathcal{Q}^*_M, \mathcal{H}^*_M)$, such that:

a: $\sum_{k=1} \int_{\mathbb{R}^{n+1}_+} y^k_o \; F^{*k}_M(dy^k) = M;$

b: for all k, $G_M^{*k}(\{p|p_o > 0\} \times \{b^{ko}, b^{k,k+1}\} \times \{s^k\}) = 1.$

The no-trade equilibrium of Proposition 3.1 represents a direct-barter economy, in which money has not been "invented" and agents are unaware of the possibility of indirect exchange. In this economy, no exchange takes place at all.

The existence of this no-trade equilibrium presents a technical difficulty for the proof of Proposition 3.2. One needs a mapping whose fixed points are precisely those rational expectations equilibria in which the price of money is positive. The no-trade equilibrium must not be among the fixed points of the mapping.

Curiously, the solution to this problem is based on the homogeneity property of the demand for money.[7] Since agents are interested in real money balances, their demand for nominal money balances rises when the price of money in all current and future transactions declines. As the price of money approaches zero, ceteris paribus, the demand for nominal money balances grows out of bounds, and there is a large disequilibrium in the "money market". The no-trade equilibrium at a zero price of money arises only from a discontinuity in money demand at $p_o = 0$.

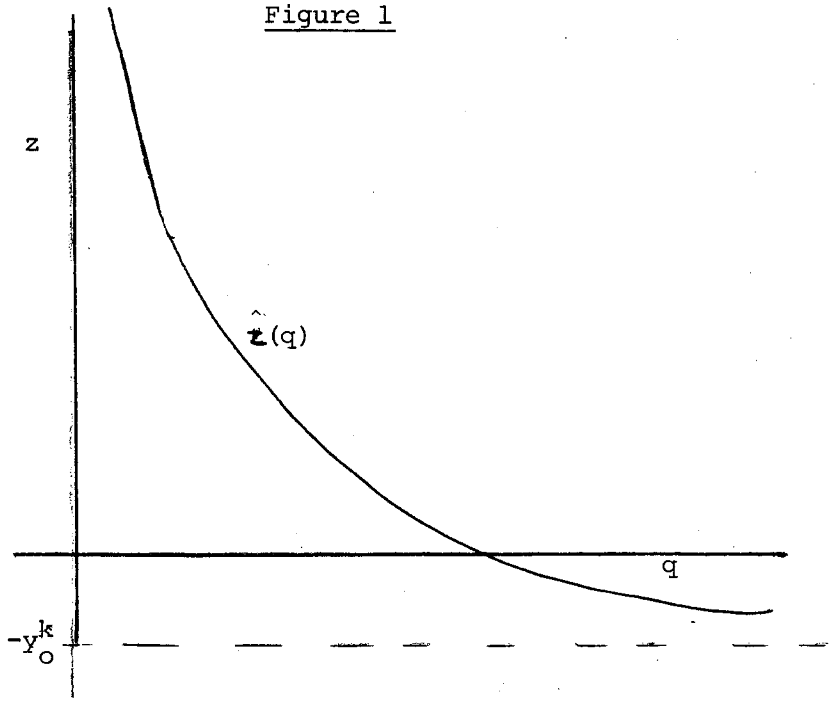
More precisely, for given $G^k \in \Gamma$, $p > 0$, y^k, with $y_k^k > 0$ consider the correspondence:

$$\hat{z}(q) = Z_o^k(f(q, p), b^{ko}, s^k; y^k; T_q G^k),$$

which gives agent k's excess demand for money as a function of the price of money when relative prices and inventories are held constant.

This correspondence is drawn in Figure 1. For $q = 0$, the value of $\hat{z}(q)$ is arbitrary in the interval $[-y_o^k, \infty)$. However, for $q \in (0, \infty)$, $\hat{z}(q)$ is single-valued and continuous, by Proposition 2.1. By Proposition 2.2, it

Figure 1



ranges over the interval $[-y_o^k, \infty)$, with $\lim_{q \to 0} \hat{z}(q) = \infty$, and $\lim_{q \to \infty} \hat{z}(q) = -y_o^k$. Thus, the excess demand for money is discontinuous at $q = 0$.

To apply a fixed-point argument, one must smooth out this discontinuity.[8] Because $\lim_{q \to 0} \hat{z}(q) = \infty$, the smoothing procedure can be chosen so that the point $q = 0$ is automatically eliminated from consideration as an equilibrium.

Figure 1 also suggests that the economy does not arbitrarily "choose" whether to operate under monetary exchange or under direct barter. It seems that, starting from a positive price of money, conventional disequilibrium adjustment processes cannot reach the no-trade equilibrium. As q approaches zero, the excess demand for money becomes large and exerts an upward pressure on the price of money, keeping it away from the point $q = 0$. A move to the no-trade equilibrium at $q = 0$ seems to require a discontinuous break away from monetary exchange and back to direct barter. Under normal circumstances and in the absence of hyperinflation, I do not see any forces that would induce such a break. Although the medium of exchange hangs by its own bootstraps, the bootstraps are both robust and elastic.

Historically, the economy might have started in the no-trade equilibrium of direct barter. Then it would remain there until by some accident agents became aware of the potential for indirect trade and "invented" a medium of exchange. This would represent a singular and irreversible historical event.

The existence of a monetary exchange equilibrium and a direct barter equilibrium, side by side, underlines the importance of expectations for the functioning of bilateral trading processes. Different expectations about future trading possibilities support different levels of exchange, and more than one set of expectations can be self-fulfilling. How much trading actually takes place may depend on the dynamics of extra-economic information processes, through which agents learn about trading opportunities. To the extent that such learning is irreversible, the multiplicity of rational expectations equilibria, in particular, the coexistence of monetary exchange and direct barter equilibria seems to leave room for a weak form of historicity of economic processes.

3.2. On the Demand to Hold Money

A central problem for a more detailed study of monetary exchange is what determines the equilibrium value of money. At an elementary level, this question is, of course, answered by the classical proposition that the price of money is inversely proportional to the quantity of money: A doubling of the nominal quantity of money entails a doubling of all money holdings and all money prices (a halving of the price of money), but leaves all real inventories, consumption and utility unchanged.

Proposition 3.3: Let $(\mathcal{F}^*, \mathcal{G}^*, \mathcal{Q}^*, \mathcal{K}^*)$ be a rational expectations equilibrium for the quantity of money M. Then, for $q > 0, ((T_q F^{*k}), (T_{1/q} G^{*k}), (T_{1/q} Q^{*jk}), (T_q K^{*jk}))$ is a rational expectations equilibrium for the quantity of money qM, where T_q, $T_{1/q}$ are as defined in (4).

The neutrality of money shifts the focus of the analysis to an explanation of the amount of real money balances held in a rational expectations equilibrium. One has to consider the implicit returns which induce agents with positive time preference to hold money even though it bears no explicit interest.

An analysis of this problem must proceed in two steps. In this and the following sections I take agents' consumption behaviours as given and analyse their decisions to trade money for commodity inventories in a given trading instant. In section 3.4, I shall then take the consumption decisions into the analysis.

In a given trading instant, the decision to hold money until the next trading instant is based on two considerations: First, in a meeting with an agent of type k-1, an agent of type k gives up commodity k for money because he hopes to use the money to acquire commodity k+1 and to make an inframarginal gain. Secondly, in a meeting with an agent of type k+1, he may decide to hold on to his money in the hope of obtaining a more favourable price in a later trading instant. Thus, the decision to hold money combines both exchange and speculative motives.

The relationship between the two motives is quite subtle: The speculative motive is subordinated to the exchange motive, but it is not, therefore, insignificant.

In the absence of an exchange motive to hold money, the speculative motive alone cannot support a positive level of real money balances. For suppose that money is held for speculative purposes <u>only</u>. Then, current speculators pay a positive price for money, because they hope for an increase in the value of money. This increase can only be brought about by future speculators who in turn base their behaviour on the hope of an additional increase in the value of money, etc. as in a chain letter. This pattern of speculation is incompatible with a stationary price distribution, contrary to our definition of a rational expectations equilibrium. Thus, the speculative motive for holding money is predicated upont the existence of an additional "real" reason for holding money. In the present model, it cannot operate independently of the exchange motive.[9]

Nevertheless, the speculative motive must not be regarded as negligible. If agents meet only pairwise, the distribution of prices is nondegenerate, and in a monetary exchange equilibrium, there is a significant amount of speculative activity. Consider an agent of type k who has not bought commodity k+1 for a long time, but holds significant money balances. If he meets an agent of type k+1 who has just sold all his inventory of commodity k+1, their combined inventory of commodity k+1 is small and the price of commodity k+1 is very large. As a result, agent k is unwilling to spend <u>all</u> his money to buy commodity k+1. In a given exchange process, this type of meeting has a fixed, nonnegligible probability. The speculative motive is caused by the nondegeneracy of the price distribution and is thus, indirectly, as much a consequence of the bilateral character of trading as the exchange motive itself.

For practical purposes, the close interaction between the exchange motive and the speculative motive to hold money makes it unlikely that the two can be separated, analytically or empirically. But one must be aware that the "demand for money" contains both speculative and exchange motives which are closely intertwined, and neither of which can be neglected.[10]

3.3. Monetary Exchange with Several Media of Exchange.

To underline the importance of the preceding considerations I shall apply them to the analysis of economies with more than one medium of exchange. In particular, I want to focus on the relative prices of different media of exchange. I shall analyse the relative price of two fiat monies and the relative price of a fiat and a commodity money in them.

To study multiple fiat monies, I consider an economy with n goods (indexed "1,2,...n") and m fiat monies (indexed "$o_1, o_2, \ldots o_m$"), in which all fiat monies are equally acceptable as media of exchange. Assumptions C.2 and C.3 are replaced with:

D.2: For all k, $s_i^k = 0$; $i \neq k, o_1, o_2, \ldots o_m$.

D.3: For all k, $s_i^k = \infty$; $i = k, o_1, o_2, \ldots o_m$.

Alternatively to contrast fiat money and commodity money, I consider an economy with n real goods, of which the first is acceptable as a medium of exchange, and fiat money (good 0). The use of the commodity money does not entail any transactions costs. Assumptions C.2 and C.3 are now replaced with:

E.2: For all k, $s_i^k = 0$; $i \neq k, 0, 1$.

E.3: For all k, $s_i^k = \infty$; $i = k, 0, 1$.

Rational expectations equilibria are defined as before, with D.2, D.3 and E.2, E.3 replacing C.2, C.3 in the definition of the trading matrices.

The main results are summarized in the following propositions:

Proposition 3.4: Assume A.1 - B.2, C.1, C.4, D.2, D.3 and let $M_1, \ldots M_m$ be the nominal quantities of the m fiat monies.

a: For any vector $q \in \Sigma^m$, there exists a rational expectations equilibrium with a vector of expectations $\mathcal{G}^{**}$, such that for all k,

$$(10) \qquad G^{**k}(\{p \mid \frac{p_{o_j}}{p_{o_1}} = \frac{q_j}{q_1}, \; j = 1,2,\ldots m\}) = 1.$$

b: For any rational expectations equilibrium with a vector of expectations $\mathcal{G}^{**}$, there exists a vector $q \in \Sigma^m$, such that for all k, (10) is satisfied.

<u>Proposition 3.5</u>: Assume A.1 - B.2, C.1, C.4, E.2, E.3 and let $M > 0$ be the nominal quantity of fiat money. For any rational expectations equilibrium with a vector of expectations $\mathcal{G}^{***}$, one has for all k,

$$(11) \qquad G^{***k}(\{p \mid p_o = 0\}) = 1.$$

The relative price of two fiat monies is arbitrary, but fixed, i.e. the same in every transaction. In contrast, the presence of a commodity money forces the value of fiat money to zero, making it disappear from the economy.

Fiat monies are demanded only for use in future transactions. They are neither produced nor consumed, and no agent ever has a more pressing need for one fiat money than for another. Under D.2 and D.3, an agent who expects the relative price of two fiat monies to be the same in all future transactions will treat them as perfect substitutes with a rate of substitution equal to the expected relative price. If all agents expect the same relative price to prevail in all future transactions, this relative price must also prevail in all current transactions.

If the relative price of two fiat monies in future transactions is random, an agent's marginal rate of substitution between the two fiat monies is strictly larger than the smallest possible future relative price. Therefore, any actual equilibrium price is strictly larger than the smallest possible future price. Then, the distribution of future relative prices and the distribution of current relative prices of the two fiat monies cannot be the same, in contradiction to the definition of a rational expectations equilibrium. It follows that in a rational expectations equilibrium, the relative price of two fiat monies is not random.

Under E.2 and E.3, fiat money is again demanded for future transactions only; but commodity money is demanded for future transactions and for consumption. In particular, agents of type n want to consume commodity 1 and are willing to buy it at a premium over its value as a medium of exchange. Applying the same reasoning as before, one finds that if the price ratio p_1/p_o, i.e. the relative price of commodity money and fiat money, is random, then the equilibrium value of the relative price p_1/p_o in any given meeting must be strictly larger than the smallest possible value of this price in any future transaction. This again contradicts the condition for a rational expectations equilibrium.

On the other hand, the premium over the value as a medium of exchange that agents of type k are willing to pay depends on their inventories and is random, unless $p_o = 0$, and the ratio p_1/p_o is unbounded. Combining the two arguments, one concludes that commodity money crowds out fiat money.

Propositions 3.4.b and 3.5 exemplify the assessment of the speculative motive for holding money in the preceding section. Under D.2 and D.3, there is no real basis for randomness in the relative price of two fiat monies. Therefore, agents do not speculate on this price. But under E.2, E.3, there is a real basis for randomness in the relative price of commodity and fiat money. This generates a speculative motive to hold commodity money, which is strong enough to destroy the fiat money altogether.

If the apparently different fiat monies under D.2, D.3 are traded at constant relative prices, the Hicks-Leontief Aggregation Theorem can be applied to aggregate them into a single fiat money. No matter what the relative prices of fiat monies are, the existence of a rational expectations equilibrium then follows directly from Proposition 3.2. The relative price of two fiat monies is arbitrary.

Under D.2, D.3 different fiat monies should be regarded as different units of the same currency, comparable to nickels and dimes, rather than different currencies. The introduction of an additional fiat money, or a change in the relative prices of fiat monies are purely monetary phenomena. They affect the quantity of the money aggregate, but because of the neutrality of money, they have no effect on the allocation of real resources.

Corollary 3.6: Assume A.1 - B.2, C.1, C.4. For any rational expectations equilibrium under D.2, D.3, there exists an economy with a single fiat money, satisfying C.2, C.3 with the same allocation of consumption, real goods inventories and real money balances.

3.4. Wealth and Welfare Gains From Monetary Exchange

Up to now, I have analysed the holding of money in terms of the choice between money and commodity inventories in a given trading instant. A complete analysis of the level of real money balances in a rational expectations equilibrium must go beyond the choice between non-interest-bearing money and non-interest-bearing commodity inventories and must encompass the decision to accumulate commodity inventories in the first place. Unfortunately, an explicit analysis of the relationship between the instantaneous utility functions and the equilibrium level of real money balances is beyond the scope of this paper.

Instead, I shall merely use an example to illustrate the importance of consumption behaviour for the monetary exchange equilibrium. Returning to the comparison between monetary exchange and direct barter, I shall discuss the proposition of Pesek and Saving [20] that the welfare gain from monetary exchange can be measured by the difference in the level of private wealth between monetary exchange and direct barter. This proposition has previously been criticized by Johnson [16], but Johnson's criticism is seriously incomplete, because it neglects the role of accumulation behaviour.

In a monetary exchange equilibrium, an agent regards his commodity inventories and his real money balances as tangible wealth. In the absence of a single equilibrium price vector, I shall use the agent's own marginal valuation to calculate the shadow value of his inventories as $y'V_y(y;\, G)$. Under direct barter, he holds no inventories of goods or real money balances.

Let $\mathcal{F}_\mu^*$, $\mathcal{G}_\mu^*$ be vectors of inventory distributions and expectations in a monetary exchange equilibrium at the frequency μ. Let $\mathcal{G}_B^*$ be the vector of expectations in the no-trade equilibrium of direct barter. Then the proposition of Pesek and Saving would use the quantity:

$$(13) \qquad \int_{\mathbb{R}_+^{n+1}} y^{k\prime} V_y^k(y^k;\, G_\mu^{*k})\; F_\mu^{*k}(dy^k)$$

as a measure of the average welfare gain from monetary exchange:

$$(14)\quad \int_{\mathbb{R}_+^{n+1}} V^k(y^k;\, G^{*k}_\mu)\; F^{*k}_\mu(dy^k) - V^k(0;\, G^{*k}_B).$$

Proposition 3.7: Assume A.1 - C.4. For all k,

a: $$\int_{\mathbb{R}_+^{n+1}} [V^k(y^k;\, G^{*k}_\mu) - y^{k\prime}V^k_y(y^k;\, G^{*k}_\mu)]\; F^{*k}_\mu(dy^k) - V^k(0;\, G^{*k}_B) > 0;$$

b: $$\lim_{\mu\to\infty} \frac{V^k(0;\, G^{*k}_\mu) - V^k(0;\, G^{*k}_B)}{\mu} = 0$$

c: $$\lim_{\mu\to\infty} \int_{\mathbb{R}_+^{n+1}} y^{k\prime}V^k_y(y^k;\, G^{*k}_\mu)\; F^{*k}_\mu(dy^k) = 0$$

$$\lim_{\mu\to\infty} \int_{\mathbb{R}_+^{n+1}} [V^k(y^k;\, G^{*k}_\mu) - y^{k\prime}V^k_y(y^k;\, G^{*k}_\mu)]\; F^{*k}_\mu(dy^k) = \lim_{\mu\to\infty} V^k(0;\, G^{*k}_\mu).$$

Proposition 3.7a confirms Johnson's assertion that a measure of wealth must always understate the true welfare gains from monetary exchange.

It is instructive to decompose the bias into two parts by adding and subtracting the term $V^k(0;\, G^{*k}_\mu)$ on the left hand side of Proposition 3.7a:

$$(15)\quad \int_{\mathbb{R}_+^{n+1}} [V^k(y^k;\, G^{*k}_\mu) - y^{k\prime}V^k_y(y^k;\, G^{*k}_\mu) - V^k(0;\, G^{*k}_\mu)]\; F^{*k}_\mu(dy^k) + $$
$$+ V^k(0;\, G^{*k}_\mu) - V^k(0;\, G^{*k}_B).$$

Both components of this expression are positive. The integral represents inframarginal gains from inventories within the monetary exchange process. Because the valuation function is strictly concave, this term is strictly positive whenever inventories are positive.

The term $V^k(0; G^{*k}_{\mu}) - V^k(0; G^{*k}_{B})$ is strictly positive because monetary exchange is preferable, even if one has no inventories. Under monetary exchange one can accumulate inventories over time and then trade, but under direct barter this possibility is precluded.

Johnson's criticism of Pesek and Saving rests entirely on the first component of expression(15), i.e. on their failure to account for consumer surplus above the measured level of wealth. Johnson neglects the possibility of inventory accumulation and does not consider the second component of (15). This neglect is justified when the trading frequency is small. Because of the Inada condition on V, the consumer surplus term then is of the first order of smalls while the second component of (15) is of the second order of smalls (Proposition 3.7b) and can be neglected.

In general, this neglect can lead to misleading results. For instance, if the trading frequency is large, commodity inventories and real money balances are small; both the shadow value of inventories and the first component of expression (15) are small. On the other hand, the welfare gains from monetary exchange are large because agents have many trading opportunities which would all remain unused under direct barter. In the limit as the trading frequency goes to infinity, inventories and real money balances go to zero; both, the Pesek-Saving estimate of the welfare gain from monetary exchange and Johnson's measure of their bias go to zero, even as the welfare gain from monetary exchange actually reaches its maximum.

Equilibrium holdings of inventories and real money balances reflect agents' desires to trade at the next opportunity. They may be unwilling to accumulate large inventories either because the chance of another trading instant is so remote that it is not worth while or because additional trading instants after the next one are so likely that they do not need to make the most of the next one. In particular, when the trading frequency is large, an agent of type k is willing to rely on future production of commodity k and additional trading instants to provide for his future consumption of commodity k+1.

This example indicates the importance of consumption and accumulation behaviour for the equilibrium amount of real money balances. Because money serves as the medium of exchange, the equilibrium amount of real money balances will be of the same order of magnitude as the level of commodity inventories held in the economy. A low level of real money balances does not necessarily mean that the trading frequency is small and monetary exchange functions so badly that agents do not care much about it. It may mean instead that the trading frequency is large and the monetary exchange process functions so well that agents do not need to carry large inventories and real money balances in order to profit from monetary exchange.

4. Monetary Exchange and General Competitive Analysis

4.1. Monetary Exchange at High Trading Frequencies

To study the relationship between the monetary exchange process and the theory of Walrasian competitive equilibrium, I consider a sequence of monetary exchange equilibria at successively higher trading frequencies. As the frequency of meetings grows out of bounds, the monetary exchange equilibrium approaches a Walrasian equilibrium allocation. The present section discusses the economic forces behind this convergence result. In the next section, I shall apply this result to assess the usefulness of general equilibrium theory for the analysis of monetary exchange equilibria.

In the frictionless competitive economy, agents have no incentive to hold inventories. Therefore, they decumulate their inventories of commodities and real money balances towards zero. One can then define the set of stationary competitive equilibria as:

$$W^* = \left\{(p^*, c^*, y^*) \;\middle|\; \begin{array}{l} c^{*k} \text{ maximizes } u^k(c^k) \text{ s.t. } p^{*\prime}(a^k - c^k) = 0, \text{ all } k; \\ y^{*k}(0) = 0, \text{ all } k;\ p^*_o = 0;\ \sum_{k=1} c^{*k} = \sum_{k=1} a^k \end{array}\right\}.$$

Proposition 4.1: Given A.1, A.2, B.1, B.2, C.1 - C.3, and the aggregate nominal quantity of money M, let $(\mathcal{F}^*_\mu, \mathcal{G}^*_\mu, \mathcal{Q}^*_\mu, \mathcal{H}^*_\mu)$ be a rational expectations equilibrium for the trading frequency μ, such that for all k

$$G^{*k}(\{p | p_o > 0\}) = 1.$$

For all $\varepsilon, \eta > 0$ there exists μ_o, such that for $\mu \geq \mu_o$, there exists $(p^*, c^*, y^*) \in W^*$ such that:

a: for all k, $\quad F^{*k}_\mu(\{y^k | \; ||y^k(0)|| > \eta\}) < \varepsilon;$

b: for all k, $\quad F^{*k}_\mu(\{y^k | \; ||c^k(y^k) - c^{*k}|| > \eta\}) < \varepsilon$

c: for all k, $\quad G^{*k}_\mu(\{p \;|\; ||p - p^*|| > \eta\}) < \varepsilon.$

The monetary exchange equilibrium of Proposition 3.2 converges in distribution to the set of Walrasian equilibria. In particular, the price distribution any type faces converges to a single Walrasian equilibrium price. As the frequency of meetings becomes large, the cost of waiting for a better price at another meeting becomes small, so that agents are able and willing to arbitrage away any price dispersion between different meetings.

Furthermore, an agent of type k who expects to trade again very soon does not need to purchase a large quantity of commodity k+1. Hence, he need not hold a large inventory of commodity k or of real money balances. In the limit as the trading frequency goes to infinity, both his goods inventories and his real money balances go to zero.

The willingness to arbitrage between different trading partners is the main economic force behind the convergence to Walrasian equilibrium. This factor operates under most bilateral trading processes; it generates similar convergence results for an economy with commodity money (Assumptions C.1, C.4, E.2, E.3) or generalized indirect barter when every commodity can serve as a medium of exchange. The main exception is the rational expectations equilibrium under direct barter, i.e. the no-trade equilibrium of Proposition 3.1 which is independent of the trading frequency and does not converge to a Walrasian equilibrium.

Under monetary exchange, the arbitrage motive is not the only factor which drives the economy to a competitive allocation. Even if expectations are not rational and agents unwilling to arbitrage there is an additional force at work to make the economy converge to a competitive equilibrium relative to revealed preferences.[11]

Suppose that expectations of future prices $\mathcal{G} \in \Gamma^n$ and of the trading frequency $\mu^e < \infty$ are given and give rise to fixed valuation functions $V^k(y^k)$, $W^k(y^k)$ and policy functions $c^k(y^k)$, $z^k(p, y^k)$, $k = 1,2,\ldots n$. Then, one can show:

Proposition 4.2: For given valuation and policy functions V^k, W^k, c^k, z^k, $k = 1,2,\ldots n$ let $\mathcal{F}_\mu$ be a stationary state for the monetary exchange process (8) with the actual trading frequency μ. For all ε, $\eta > 0$, there exists μ_o, such that for all $\mu \geq \mu_o$, there exists a price vector $p \geq 0$, such that for all k,

$$\hat{F}^k_\mu(\{y^k \mid \|z^k(p, y^k)\| > \eta\}) < \varepsilon.$$

If the actual trading frequency is large, exchanges occur almost as soon as new production and consumption flows make them desirable. In a stationary state most of the gains from trade have already been taken. The remaining potential gains from trade depend on the lag of exchanges behind production and consumption activities, which is of the order of $1/\mu$. In the limit as the actual trading frequency goes to infinity, no noticeable gain from an additional trade can be left.

By this argument the stationary state of an bilateral trading process approaches a pairwise optimal allocation for large μ, whether expectations are rational or not. If expectations are not rational, a pairwise optimal allocation need generally not be Pareto optimal, so that it need not be supported by competitive prices.[12]

The monetary exchange process has the property that a pairwise optimal allocation is always Pareto optimal relative to the revealed valuation functions

V^k. The medium of exchange is desirable for every agent in the economy. Therefore, it provides a common yardstick for the measurement of relative values. Every discrepancy in marginal rates of substitution that gives rise to advantageous multilateral trade must also give rise to advantageous bilateral trade with money serving as the medium of exchange. Therefore, a pairwise optimal allocation must be Pareto optimal.

Proposition 4.2 indicates that the choice of the medium of exchange as numeraire is less arbitrary than the classical approach to price theory might suggest. A good can only serve as numeraire if every agent is willing to pay a positive price for it. A good with this property can serve as a medium of exchange to bring about the equivalence of pairwise optimality and Pareto optimality. Conversely, in a complex economy with a single medium of exchange, the medium of exchange is the only commodity which is guaranteed to be desired by every agent in the economy. Thus, the use of money as a unit of account reflects a very real economic function that is served by the medium of exchange.

4.2. Monetary Exchange, Walrasian Equilibrium and the Quantity Equation

Proposition 4.1 illustrates both the strength and the weakness of general equilibrium theory in the analysis of a monetary economy. For high trading frequencies, both the allocation of real consumption flows and the allocation of real money balances and of commodity inventories are close to the Walrasian allocation. Thus, Proposition 4.1 supports the view that general equilibrium theory provides an approximation for the allocation of real resources in a monetary economy at high trading frequencies.

The usefulness of this approximation depends on the questions one wants to ask. On the one hand, the Walrasian approximation for relative commodity prices and for the allocation of consumption flows is acceptable, because it focuses precisely on those problems that a value theorist is interested in.

On the other hand, the Walrasian approximation for commodity inventories and for real money balances merely confirms the fact that as friction disappears from the economy, agents no longer need to hold inventories of goods or money. It contributes nothing to our understanding of monetary exchange at a given level of friction, which is the problem the monetary theorist is interested in.

Thus, it appears that general equilibrium theory provides us with a good analysis of the allocation of resource flows, even in a monetary economy; however, it is not the appropriate tool to analyse the functioning of the monetary mechanism itself. The Walrasian approximation for the allocation of real resource flows is the better, the less need for money there is. Money is designed to deal with circumstances that preclude the strict validity of general equilibrium theory.

It follows that the use of general equilibrium theory to criticize Keynesian ideas that are related to the monetary exchange process is illegitimate. More generally, general equilibrium theory cannot be used to analyse macroeconomic phenomena that arise from the monetary exchange process.

The concept of the speculative demand for money may serve to illustrate this point. For the long run rational expectations equilibrium of the competitive economy, Leontief [17] has shown that the speculative demand for money must be zero. In a long run rational expectations equilibrium, the rate of interest is constant and is expected to remain constant. Then the Keynesian speculative demand for money must be zero, because it is positive only when the rate of interest is expected to rise.

On the other hand, in the monetary exchange equilibrium there always is a speculative motive for holding money. When the frequency of meetings is finite, the distribution of prices encountered by the individual agent is nondegenerate. In a rational expectations equilibrium, the distribution of prices does not change, whereas the individual price may differ from meeting to meeting.

The speculative motive to hold money arises from the desire to search for the most favourable price.[13]

In the limit as the frequency of meetings goes to infinity, the distribution of prices becomes degenerate. Therefore, the speculative motive for holding money disappears. Not only real money balances, but also nominal money balances held for speculative purposes go to zero. All nominal money balances are devoted to the exchange motive. In the limit, the demand to hold money reflects the purely mechanical need to carry out transactions and becomes "rather impervious to direct economic incentives" (Hicks [14, p. 15]). As a result, the classical quantity equation with a technologically determined velocity of circulation becomes valid.

Proposition 4.3: Given A.1 - B.2, C.1 - C.4, and the aggregate nominal quantity of money $M > 0$, let $(\mathfrak{F}^*_\mu, \mathcal{G}^*_\mu, \mathcal{Q}^*_\mu, \mathcal{H}^*_\mu)$ be a rational expectations equilibrium at the trading frequency μ, such that for all k,

$$G^{*k}_\mu(\{p \mid p_o > 0\}) = 1.$$

Then,

$$\lim_{\mu\to\infty} \left| M - \frac{1}{\frac{\mu}{n}} \sum_{K=1}^{n} q_k \int_{\mathbb{R}^{n+1}_+} [a^k_k - c^k_k(y^k; G^{*k})] \, F^{*k}(dy^k) \right| = 0.$$

where for all k,

$$q_k = n \int \frac{p_k}{p_o} \, E^{k-1}(dp \times \{b^{k-1,k}\} \times \{s^{k-1}\} \; ; \mathcal{Q}^*, \mathfrak{F}^*, \mathcal{G}^*).$$

In a stationary state of the economy, the mean accumulation of commodity k by agents of type k must equal the mean acquisition of commodity k by agents of the type k-1. The latter, multiplied by the money price of commodity k must be equal to the mean sales of money from agents of type k-1 to agents of type k. In the absence of a speculative motive, sales of money by agents of type k-1 are equal to their money holdings, so that the quantity equation holds, with the frequency of meetings between agents of types k and k-1 as the velocity of circulation.

For finite frequencies of meetings, the classical quantity equation breaks down because of the nondegeneracy of the price distribution and the presence of speculative behaviour.

Thus, the theory of general competitive equilibrium may be applied to develop the monetary theory of a frictionless economy in which money is invisible. It is inappropriate for the analysis of the monetary exchange process in an economy with frictions, in which money has a significant role to play.

5. Concluding Remarks

The present model of monetary exchange makes the role of the medium of exchange explicit, according to the principle that "money buys goods and goods buy money, but goods do not buy goods". The model has confirmed Clower's [6] suspicion that general equilibrium theory has little to contribute to the analysis of monetary exchange. However, the presence of a monetary system need not impair the approximate validity of general equilibrium theory as a theory of the allocation of resources.

In many respects the model presented here is rather crude and can only be regarded as a first step in the development of a basic framework for the analysis of decentralized monetary exchange. In particular, it is necessary to consider alternative meeting processes and make the frequency of meetings endogenous. In a simple way this could be done by making the frequency of meetings a function of the amount of effort spent in looking for a trading partner. This formulation would present some technical difficulties, but would not change the conceptual basis of the present model.

A more important generalization would take account of the fact that agents do not draw their trading partners at random from the rest of the population. In this context, one must analyse the phenomenon of intermediation, i.e. the appearance of agents whose function it is to serve as trading partners and thereby facilitate exchange. Presumably, such intermediaries would determine prices and evict the auctioneer as well as the auction hall from the theory of decentralized exchange.

Whereas the foregoing desiderata pose considerable conceptual and technical difficulties, it is fairly easy to generalize the specification of preferences and technologies. It seems that necessary conditions for the existence of a monetary exchange equilibrium can be developed along the lines of Hayashi [12] to link the existence of monetary equilibrium to the underlying desire to exchange real commodities.

This paper is based on the proposition that monetary exchange is designed to attenuate the stringency of decentralized bilateral exchange. From this point of view it is necessary to reexamine the propositions of monetary theory that were derived within the general equilibrium framework to see which of them can be extended to an economy with frictions. In the present paper, the neoclassical neutrality postulate was confirmed even for finite frequencies of meetings. On the other hand, the validity of the quantity equation was limited to the frictionless economy with an infinite frequency of meetings. It will be important to consider other propositions, on the optimum quantity of money or the role of monetary policy, to determine their validity at finite frequencies of meetings.

Appendix: Proofs

An optimal solution to problem (1) satisfies the Euler equation:

(A.1) $$\dot{u}_c \leq (\rho + \mu)\, u_c - \mu W_y,$$

with inequality only if the corresponding consumption is zero; furthermore, the transversality condition:

(A.2) $$\lim_{t\to\infty} e^{-(\rho + \mu)t}\, y(t)' V_y(y(t)) = 0.$$

The envelope theorem implies:

(A.3) $$u_c \leq V_y,$$

with inequality only if the corresponding consumption is zero; furthermore:

(A.4) $$V_y = \lim_{t\to\infty} e^{-(\rho + \mu)t}\, V_y(y(t)) + \int_0^\infty e^{-(\rho + \mu)t}\, \mu W_y(y(t))\, dt.$$

Proof of Proposition 2.2: It is sufficient to note that the set of feasible consumption paths for every realization of trading instants and trading conditions is independent of m. Then the optimal consumption program must be independent of m. This is achieved by the policy rule in question.

Proof of Lemma 2.3: Existence of equlibrium:

a: If $j \neq k-1, k, k+1$, choose p_o, p_j, p_k, so that $p_o/p_k = V^k_{y_o}/V^k_{y_k}$ and $p_o/p_j = V^j_{y_o}/V^j_{y_j}$.

b: If $j = k-1$ and $y_o^j > 0$, $y_k^k > 0$, the existence of equilibrium follows from Proposition 2.1 and Theorems 5.2 and 5.4 in Arrow and Hahn [1]. If $j = k-1$ and $y_o^j = 0$, choose p_o, p_j, p_k as in (a). If $j = k-1$ and $y_k^k = 0$, choose $p_o/p_k = \min [V^j_{y_o}/V^j_{y_k}, V^k_{y_o}/V^k_{y_k}]$. The cases $j = k$ and $j = k+1$ are treated analogously.

Upper-hemicontinuity of the equilibrium price correspondence in the one-point compactification of the price space $(\mathbb{R}_+ \cup \{\infty\}) \times \Sigma^n$ follows directly from the continuity of the demand functions when inventories are positive and by inspection of (b) above when inventories are zero. Q.E.D.

Let $\mathbb{P}_m$ be the space of probability distributions on $\mathbb{R}^m_+$ and $\mathbb{P}^n_m$ be the space of n-vectors of probability distributions on $\mathbb{R}^m_+$. The distance between two elements F_1, F_2 of $\mathbb{P}_m$ is given by the Prohorov metric (Billingsley [3], p. 238), $\rho(F_1, F_2)$ which induces the topology of weak convergence of probability measures. The distance between vectors of distributions $\mathfrak{F}_1$ $\mathfrak{F}_2 \in \mathbb{P}^n_m$ is given as $\sum_{i=1}^{n} \rho(F_1^i, F_2^i) = d(\mathfrak{F}_1, \mathfrak{F}_2)$.

To prove Propositions 2.4 and 3.2, I shall need a fixed point theorem for probability distributions. The following lemma combines the basic ideas of the Kakutani and Schauder theorems:

<u>Lemma A.1</u>: For any compact subset $\mathbb{F}_o \subset \mathbb{P}_m$, let $\mathbb{F}$ be the set of distributions F, such that for any compact set $K \subset \mathbb{R}^m_+$, there exists $P \in \mathbb{F}_o$, such that $F(K) \geq P(K)$.

If the correspondence $\phi: \mathbb{F}^n \to \mathbb{F}^n$ is convex valued and closed, it has a fixed point.

Proof: Let $\{r(1), r(2),....\}$ be an enumeration of the rationals in $\mathbb{R}^m_+$. For any q, let $\mathbb{F}(q)$ be the set of distributions in $\mathbb{F}$ whose mass is concentrated at r(1), r(2),....r(q). For any distribution $F \in \mathbb{F}$, let $\psi^o_q(F) \subset \mathbb{F}(q)$ be the distributions in $\mathbb{F}(q)$ that are closest to F:

$$\psi^o_q(F) = \left\{P \in \mathbb{F}(q) \mid \rho(P,F) \leq \rho(Q,F) \text{ for all } Q \in \mathbb{F}(q)\right\} .$$

Let ψ_q: $\mathbb{F}^n \longrightarrow \mathbb{F}(q)^n$ be the vector correspondence with typical element $\psi^o_q(F^i)$. Clearly, ψ_q is upper-hemicontinuous and convex valued. The composition $\psi_q \circ \phi$: $\mathbb{F}(q)^n \longrightarrow \mathbb{F}(q)^n$ satisfies the conditions of Kakutani's fixed point theorem and has a fixed point $\mathfrak{F}^*_q \in \psi_q \circ \phi(\mathfrak{F}^*_q)$.

By Prohorov's Theorem (Billingsley [3], p. 37), the set $\mathbb{F}_o$ is tight. Hence, the set $\mathbb{F}$ is also tight, by construction, and therefore compact, again by Prohorov's Theorem. Hence the sequence $\mathfrak{F}^*_q$ has a limit point $\mathfrak{F}^* \in \mathbb{F}^n$.

Because the rationals are dense in $\mathbb{R}^m_+$, it is easy to show that $\lim_{q\to\infty} d(\psi_q(\mathfrak{F}), \mathfrak{F}) = 0$, uniformly for $\mathfrak{F} \in \mathbb{F}^n$. It follows directly that $\mathfrak{F}^*$ is a fixed point of the correspondence ϕ. Q.E.D.

In the following, a state of the economy is an n-vector $\mathfrak{F} \in \mathbb{P}^n_{n+1}$ of probability distributions on $\mathbb{R}^{n+1}_n$. Expectations are given by a vector $\mathfrak{G} \in \mathbb{P}^n_{3(n+1)}$ of probability distributions on the space of trading conditions $\mathbb{R}^{3(n+1)}_+$. Consider the correspondence λ^{jk}: $\mathbb{R}^{2(n+1)}_+ \times \Gamma^2 \longrightarrow \mathbb{P}_{n+1}$, κ^{jk}: $\mathbb{R}^{2(n+1)}_+ \times \Gamma^2 \longrightarrow \mathbb{P}_{n+1}$, from inventories and expectations into probability distributions on prices and inventories:

$$\lambda^{jk}(y^j, y^k; G^j, G^k) = \left\{Q \in \mathbb{P}_{n+1} \,\middle|\, Q(P^{jk}(y^j, y^k; G^j, G^k)) = 1\right\};$$

$$\kappa^{jk}(y^j, y^k; G^j k\, G^k) = \left\{K \in \mathbb{P}_{n+1} \,\middle|\, \text{(7.c) is satisfied for } Q \in \lambda^{jk}\right\}.$$

Now Lemma 2.3 has the following obvious

Corollary A.2: For all j,k, G^j, $G^k \in \Gamma$, the correspondences λ^{jk}, κ^{jk} from inventories and expectations into probability distributions on prices and inventories are nonempty, upper-hemicontinuous and convex valued.

Proof: It suffices to note that the set of equilibrium prices is closed. Q.E.D.

Lemma A.3:

a: There exist measurable selections of the correspondences λ^{jk}, κ^{jk}.

b: Let Q^{jk} be a measurable selection from the correspondence λ^{jk}. For any Borel set P, the function $Q^{jk}(P; ., .; G^j, G^k)$ is measurable.

c: Let K^{jk} be a measurable selection from the correspondence κ^{jk}. For any Borel set X, the function $K^{jk}(X; ., .; G^j, G^k)$ is measurable.

Proof: a: By the neutrality of the unit of account, one may restrict the correspondence λ^{jk} to distributions on the n+1-dimensional simplex. This restriction to the n+1-dimensional price simplex satisfies the conditions of Lemma 1, p. 55 of Hildenbrand [15]. Hence it has a measurable selection, say Q^{jk}. By continuity of the demand function, the mapping K^{jk} defined by (7.c) and a measurable selection Q^{jk} of λ^{jk} is measurable.

b: It is to be shown that for every closed set A, the set of pairs (y^j, y^k), such that $Q^{jk}(P; y^j, y^k) \in A$ is closed. Let $\mathbb{F}_A = \left\{F \in \mathbb{P}_{n+1} \,\middle|\, F(P) \in A\right\}$. Clearly, $\mathbb{F}_A$ is closed and furthermore,

$$\left\{(y^j, y^k) \,\middle|\, Q^{jk}(P; y^j, y^k) \in A\right\} = \left\{(y^j, y^k) \,\middle|\, Q^{jk}(. \,; y^j, y^k) \in \mathbb{F}_A\right\} .$$

The assertion follows directly from the definition of Q^{jk}. Part c follows immediately. Q.E.D.

Lemma A.4: Assume A.1, A.2. For all $x \in \mathbb{R}_+^{n+1}$, the set

$$Y(x) = \left\{ y \in \mathbb{R}_+^n \;\middle|\; \exists\, y_o \in \mathbb{R}_+,\ G \in \Gamma, \text{ such that } c((y_o, y); G) \leqq x \right\}$$

is compact.

Proof: By concavity of V, $V((y_o, y); G) - V(0; G) \geqq (y_o, y)' V_y((y_o, y); G)$. By boundedness of V, $(\bar{u} - \underline{u})/\rho \geqq y_i V_{y_i} \geqq y_i u_{c_i}$, where the latter inequality follows from the envelope condition (A.3). The lemma follows immediately. Q.E.D.

Lemma A.5: Assume A.1 - B.2. Let $\mathbb{F}^n$ be the set of states of the economy that satisfy, for some $\mathcal{G} \in \Gamma^n$, the conditions:

$$\text{i:} \quad \sum_{k=1}^{n} \int y_o^k F^k(dy^k) = N;$$

$$\text{ii:} \quad \sum_{k=1}^{n} \int c^k(y^k; G^k) F^k(dy^k) \leqq \sum_{k=1}^{n} a^k.$$

Then $\mathbb{F}^n$ is compact.

Proof: For given k, (i) and (ii) imply:

$$F^k\left(\left\{y^k \;\middle|\; y_o^k \geqq A\right\}\right) \leqq M/A \quad \text{for all } A < \infty;$$

$$F^k\left(\left\{y^k \;\middle|\; y^k(0) \notin Y\right\}\right) \leqq \min_i \sup_{\substack{y_o \\ y(0) \in \mathbb{R}_+^n - Y}} \frac{a_i^j}{c_i^k((y_o, y(0)); G^k)}$$

for all compact Y. It follows from Lemma A.4 that $\mathbb{F}^n$ is the finite product of tight sets of distributions. Q.E.D.

Proof of Proposition 2.4: Measure the distance between sets of distributions by the Hausdorff distance δ with respect to the Prohorov metric ρ. The forward equations for the evolution of the economy are given by the correspondences

$$\chi_h^*: \mathbb{P}^n_{n+1} \longrightarrow \mathbb{P}^n_{n+1}, \text{ such that:}$$

a: $\chi^*_{h+g} = \chi^*_h \chi^*_g$, and

b: for any compact set $\mathbb{F}^n \subset \mathbb{P}^n_{n+1}$, there exists $A < \infty$, such that for all $\mathcal{F} \in \mathbb{F}^n$

$$\delta((\chi_h^* \mathcal{F})_k, (\chi_h \mathcal{F})_k \leq A\, h^2, \quad k = 1,2,\ldots n$$

where $\chi_h: \mathbb{P}^n_{n+1} \longrightarrow \mathbb{P}^n_{n+1}$ is given by the system:

(A.5) For all k, all Borel sets $Y \subset \mathbb{R}^{n+1}_+$:

$$(\chi_h \mathcal{F})_k(Y) = (1 - \mu h)\, F^k(\phi^k(Y)) + h \sum_{j=1}^{n} \frac{1}{n} \int \kappa^{kj}(\phi^k(Y;\, y^k, y^j;\, G^k,\, G^j) F^k(dy^k)\, F^j(dy^j),$$

where

$$\phi^k(Y) = \left\{ x \,|\, y(h;\, x) \in Y \right\}.$$

The correspondence χ_h is obtained from the forward equations (8) by dropping terms of order higher than h and by admitting any measurable mapping $K^{jk} \in \kappa^{jk}$.[14]

It follows from Corollary A.2 and Lemma A.3 that the correspondence χ_h is nonempty, upper-hemicontinuous and convex valued. Furthermore, the correspondences χ_h and χ_h^* map the compact set of states of the economy $\mathbb{F}^n$, defined in Lemma A.5 into itself. Therefore, χ_h satisfies the conditions of Lemma A.1 and must have a fixed point $\hat{\mathfrak{F}}_h$. By compactness of $\mathbb{F}^n$, the sequence of fixed points $\{\hat{\mathfrak{F}}_{h/q}\}_{q=1}^{\infty}$ has a limit point, say $\hat{\mathfrak{F}}$. It is to be shown that $\hat{\mathfrak{F}}$ is a fixed point of the correspondence χ_h^*.

Consider the distance:

$$\rho((\chi_h^* \hat{\mathfrak{F}}_{h/q})_k, \hat{F}_{h/q}^k) = \rho((\chi_{h/q}^{*q} \hat{\mathfrak{F}}_{h/q})_k, \hat{F}_{h/q}^k)$$

$$\leq \delta((\chi_{h/q}^{*q} \hat{\mathfrak{F}}_{h/q})_k, (\chi_{h/q}^{q} \hat{\mathfrak{F}}_{h/q})_k).$$

Clearly, $\hat{\mathfrak{F}}$ is a fixed point of the correspondence χ_h^*, if the right hand side of this inequality goes to zero as $q \to \infty$. I shall show that the right hand side is uniformly bounded above by the quantity $Ah(e^{\mu h} - 1)/\mu q$. Using the triangle inequality, for any $\mathfrak{F} \in \mathbb{F}^n$,

$$\delta((\chi_{h/q}^{*q} \mathfrak{F})_k, (\chi_{h/q}^{q} \mathfrak{F})_k \leq \delta((\chi_{h/q}^{*q} \mathfrak{F})_k, \chi_{h/q} \chi_{h/q}^{*q-1} \mathfrak{F})_k)$$

$$+ \delta((\chi_{h/q} \chi_{h/q}^{*q-1} \mathfrak{F})_k, (\chi_{h/q} \chi_{h/q}^{q-1} \mathfrak{F})_k).$$

The first term is less than Ah^2/q^2, by part (b) of the definition of χ_h^*. Therefore, the assertion that the left hand side is bounded above by the quantity $Ah(e^{\mu h}-1)/q$ follows directly by induction, if it can be shown that for all $\mathfrak{F}_1, \mathfrak{F}_2 \in \mathbb{F}^n$,

$$\delta((\chi_{h/q} \mathfrak{F}_1)_k, (\chi_{h/q} \mathfrak{F}_2)_k) \leq \rho(F_1^k, F_2^k) + \frac{\mu h}{q} d(\mathfrak{F}_1, \mathfrak{F}_2)/n.$$

But this inequality follows directly by inspection of A.5 if one notes that every element $K^{kj} \varepsilon K^{kj}$ can be represented as the convex combination of mappings that are concentrated on single measurable selections from the equilibrium inventory correspondences. This completes the proof of Proposition 2.4. Q.E.D.

Proof of Proposition 3.1: Let $G_B^K(\{p_o = p_k = 0\} \times \{b^{k,k+1}\} \times \{s^k\}) = 1/n$; $G_B^K(\{p_o = 0\} \times \{b^{ko}\} \times \{s^k\}) = 1 - 1/n$; $k = 1,2,\ldots n$. Then agents expect never to trade and therefore $V_y^k = W_y^k$ for all k. The Euler equation requires $\dot{u}_c \leqq \rho u_c$, with inequality only when the corresponding consumption is zero. Therefore, all commodity inventories are decumulated to zero, proving parts c and d of the proposition. When commodity inventories are zero, commodity prices in a given meeting can be set to satisfy the above construction. Furthermore, from (A.2) and (A.4), $V_{y_o}^k = 0$ for all k, and therefore $p_o = 0$ in every meeting. This proves part b of the proposition. Part a is automatically satisfied. Q.E.D.

To prove Proposition 3.2, I shall use a simpler representation for expectations. Note that in meetings between agents k and k+1, only the relative price of commodity k+1 and money matters. Represent agent k's expectations of the money price of commodity k+1 in a meeting with agent k+1 by the distribution $\bar{G}^{k,k+1} \varepsilon IP_1$, such that, for any measurable set P,

$$\bar{G}^{k,k+1}(P) = n\, G^k(\{p | p_{k+1}/p_o \varepsilon P\} \times \{b^{k,k+1}\} \times \{s^k\}).$$

Similarly, in meetings between agent k and agents other than k+1, only the relative price of commodity k and money matters. Represent agent k's expectation of the money price of commodity k by the distribution $\bar{G}^{ko} \varepsilon IP_1$, such that for any measurable set P,

$$\bar{G}^{ko}(P) = (1 - \tfrac{1}{n})\, G^k(\{p | p_k/p_o \varepsilon P\} \times \{b^{ko}\} \times \{s^k\}).$$

Thus, expectations will be considered as a vector of measures $\bar{\mathcal{G}} \in \mathbb{P}_1^{2n}$. Let $\bar{\Gamma} \in \mathbb{P}_1$ be the set of measures $\bar{G}$, such that $\bar{G}((0, \infty)) > 0$, so that with positive probability, the money price is positive and finite.

In a rational expectations equilibrium, the distribution of prices encountered by agent k in meetings with agents of type k+1 must be the same as the distribution encountered by agent k+1 in meetings with agents of type k. Hence, the analysis will be restricted to the subset $\mathbb{H}^{2n} \subset \bar{\Gamma}^{2n} \subset \mathbb{P}_1^{2n}$, such that for any $\bar{\mathcal{G}} \in \mathbb{H}^{2n}$, there exists $\tilde{\mathcal{G}} \in \mathbb{P}_1^n$, such that:

(A.6) $$\bar{G}^{ko} = \frac{1}{n-1} \bar{G}^{k-1,k} + \frac{n-2}{n-1} \tilde{G}^{k}.$$

<u>Proof of Proposition 3.2</u>: Consider the correspondences $\psi_h: \mathbb{F}^n \times \mathbb{H}^{2n} \longrightarrow \mathbb{F}^n \times \mathbb{H}^{2n}$ given as:

i: $\psi_{h_1}(\mathfrak{F}, \bar{\mathcal{G}}) = \chi_h(\mathfrak{F})$, where χ_h is the correspondence defined in (A.5), with behaviour induced by the expectations $\bar{\mathcal{G}}$.

ii: For any k, the k-th component of $\psi_{h_2}(\mathfrak{F}, \bar{\mathcal{G}})$ is given by the condition that for any measurable $P \subset \mathbb{R}_+$,

$$\psi_{h_2}^{k,k+1}(P; \mathfrak{F}, \bar{\mathcal{G}}) = \int \lambda^{k,k+1}(\{p | p_{k+1}/p_o \in P\}; y^k, y^{k+1}) F^k(dy^k) \; F^{k+1}(dy^{k+1});$$

$$\psi_{h_2}^{ko}(P; \mathfrak{F}, \bar{\mathcal{G}}) = \frac{1}{n-1} \sum_{\substack{j=1 \\ j \neq k+1}}^{n} \int \lambda^{kj}(\{p | p_k/p_o \in P\}; y^k, y^k) F^k(dy^k) \; F^j(dy^j).$$

To extend the correspondences ψ_h to the boundary of the space $\mathbb{F}^n \times \mathbb{H}^{2n}$ I make use of the vectors $\mathfrak{F}_o = (F_o^k)$ where $F_o^k(M/n, 0, 0, \ldots 0) = 1$, $\bar{\mathcal{G}}_o = (\bar{G}^{k,k+1}, \bar{G}^{ko})$, where $\bar{G}^{k,k+1}(\{1\}) = \bar{G}^{ko}(\{1\}) = 1$, $k = 1,2,\ldots n$. $\mathfrak{F}_o$ is a state of the economy, in which money balances are distributed evenly, and not real commodity inventories are held. $\bar{\mathcal{G}}_o$ is a vector of expectations in which the money price of every commodity is one.

One can define a new correspondence ψ^*_h: $\mathbb{F}^n \times \mathbb{H}_c^{2n} \longrightarrow \mathbb{F}^n \times \mathbb{H}_c^{2n}$ of the closure of $\mathbb{F}^n \times \mathbb{H}^{2n}$ into itself as follows:

i: $\psi^*_h(\mathcal{F}, \bar{\mathcal{G}}, A_1, A_2, A_3) = (\mathcal{F}_o, \bar{\mathcal{G}}_o)$ whenever ψ_h is undefined;

ii: $\psi^*_h(\mathcal{F}, \bar{\mathcal{G}}, A_1, A_2, A_3) = \alpha\psi_h(\mathcal{F}, \bar{\mathcal{G}}) + (1 - \alpha)(\mathcal{F}_o, \bar{\mathcal{G}}_o)$,

where $\alpha(\mathcal{F}, \bar{\mathcal{G}}, A_1, A_2, A_3)$ is a continuous function with range [0, 1], such that:

a: $\alpha(\mathcal{F}, \bar{\mathcal{G}}, A_1, A_2, A_3) = 1$ whenever:

$$\max_k \max_{y^{*k}} \int z^k_{k+1}(p, y^{*k})\, d\bar{G}^{k,k+1} + \int z^k_{o}(p, y^{*k})\, d\bar{G}^{ko} \leq A_1,$$

where $c^k(y^{*k}) = a^k$ and $y^{*k}_o \leq A_3$;

b: $\alpha(\mathcal{F}, \bar{\mathcal{G}}, A_1, A_2, A_3) = 0$ whenever:

$$\max_k \max_{y^{*k}} \int z^k_{k+1}(p, y^{*k})\, d\bar{G}^{k,k+1} + z^k_o(p, y^{*k})\, d\bar{G}^{ko} \geq A_2,$$

where $c^k(y^{*k}) = a^k$ and $y^{*k}_o \leq A_3$.

Given any h, A_1, A_2, A_3, the mapping ψ^*_h is upper-hemicontinuous and convex valued whenever the mapping ψ_h is defined. Moreover, ψ^*_h is continuous when ψ_h is undefined: Consider the sequence $\left\{(\mathcal{F}^q, \bar{\mathcal{G}}^q)\right\}_{q=1}^{\infty}$, where expectations converge to the boundary of $\mathbb{H}^{2n}$. If for some k, either the money price of commodity k+1 in meetings with agents of type k+1, p_{k+1}/p_o, or the implied relative price of commodities k+1 and k approaches zero in probability, there exists y^{*k} as specified in (ii.b) above, such that $z^k_{k+1}(p, y^{*k})$ grows out of bounds in probability. This is obvious from the first order conditions (A.1) and (A.3).

Alternatively, suppose that for all k, the money price of commodity k+1, p_{k+1}/p_o grows out of bounds, while all the implied relative commodity prices are bounded away from zero - in probability. By Proposition 2.2, real consumption behaviour is independent of the value of money. By (A.1) and the Inada condition on the instantaneous utility function, $||y^{*k}||$, as defined in (ii.b) above remains bounded away from zero. Excess demand for real money balances at $(0, y^{*k}(0))$ remains bounded away from zero, so that excess demand for nominal money balances at $(0, y^{*k}(0))$ grows out of bounds.

It follows that the correspondence ψ_h^* is upper-hemicontinuous and convex valued over its whole domain $\mathbb{F}^n \times \mathbb{H}_c^{2n}$. Therefore, its range is compact. Moreover, from the preceding paragraph and Proposition 2.1, its range is contained in $\mathbb{F}^n \times \mathbb{H}^{2n}$, i.e. there exists a compact set $\mathbb{F}^n \times \mathbb{H}_o^{2n} \subset \mathbb{F}^n \times \mathbb{H}^{2n}$, such that ψ_h^* maps the set $\mathbb{F}^n \times \mathbb{H}_o^{2n}$ into inself. When restricted to this set, ψ_h^* satisfies the conditions of Lemma A.1 and has a fixed point, $E(h, A_1, A_2, A_3)$.

For given A_1, A_2, A_3, clearly $E(h, A_1, A_2, A_3) \varepsilon\ \mathbb{F}^n \times \mathbb{H}_o^{2n}$ for all h. Then let $\bar{E}(A_1, A_2, A_3) = \lim_{h \to 0} E(h, A_1, A_2, A_3) \varepsilon\ \mathbb{F}^n \times \mathbb{H}_o^{2n}$. By the same argument as in Proposition 2.4, $\bar{E}(A_1, A_2, A_3)$ is the fixed point of the correspondence ψ_h^{**}: $\mathbb{F}^n \times \mathbb{H}_o^{2n} \to \mathbb{F}^n \times \mathbb{H}_o^{2n}$ obtained by substituting the "true" forward equations χ_h^* for χ_h in the definition of ψ_h and ψ_h^* above.

For any A_3 and sufficiently large A_1, A_2, $\bar{E}(A_1, A_2, A_3)$ is a rational expectations equilibrium of the economy. For suppose not, so that for some A_3, and all A_1, A_2 $(\bar{E}(A_1, A_2, A_3), A_1, A_2, A_3) > 0$, and mean excess demand at the stationary point y^{*k} as defined in (ii.a) exceeds A_1 for some k. By Proposition 2.1, the vector of price expectations at $\bar{E}(A_1, A_2, A_3)$ must go to the boundary of $\mathbb{H}^{2n}$ as A_1, A_2 diverge.

Suppose first that the implied relative commodity prices are all bounded away from zero, in probability, while the price of money goes to zero. Taking a subsequence if necessary, let the distribution of implied relative commodity prices converge. By Propositions 2.1 and 2.2 optimal consumption and excess demand functions converge. Then, for $A_4 > 0$, one can find a positive probability that is independent of the price of money, such that an agent of type k is in an η-neighbourhood of a point $(m, y^{*k}(0))$, $m \leq A_4$. In this region, he is starved for commodity k+1, and for any $A_5 > 0$, one can find $\zeta > 0$, such that for $p_o/p_k < \zeta$, his excess demand for money exceeds A_5. This contradicts the assumption that the price of money goes to zero in probability.

Suppose next that for some k, the relative price p_{k+1}/p_k goes to zero in probability as A_1, A_2 diverge. From the first order conditions (A.1) and (A.3), it follows that aggregate consumption of commodity k+1 by agents of type k must grow out of bounds, contrary to condition (ii) of Lemma A.5. The same argument eliminates the possibility that for some k, the relative price p_{k+1}/p_o goes to zero as A_1, A_2 diverge.

Hence the assertion that the vector of expectations at $\bar{E}(A_1, A_2, A_3)$ goes to the boundary of $\mathbb{H}^{2n}$ as A_1, A_2 diverge leads to a contradiction. This proves the assertion that for given A_3 and sufficiently large A_1, A_2, $\bar{E}(A_1, A_2, A_3)$ is the desired rational expectations equilibrium. To complete the proof, it is sufficient to note that condition (a) of the proposition is condition (i) of Lemma A.5 while condition (b) is a direct consequence of the fact that for all A_1, A_2, $\bar{E}(A_1, A_2, A_3) \in \mathbb{F}^n \times \mathbb{H}^{2n} \subset \mathbb{F}^n \times \bar{\Gamma}^{2n}$. Q.E.D.

Proof of Proposition 3.3: This propostion follows trivially from Proposition 2.2 and the definition of a rational expectations equilibrium. Q.E.D.

Proof of Proposition 3.4: I consider only the case m = 2, because the general case follows by induction.

From (A.2) and (A.4) one has:

$$V^k_{y_{o_1}} = \int_o^\infty e^{-(\rho + \mu)t} \mu E \lambda^K(p, y(t)) p_{o_1} dt$$

$$V^k_{y_{o_2}} = \int_o^\infty e^{-(\rho + \mu)t} \mu E \lambda^K(p, y(t)) p_{o_2} dt,$$

where $\lambda^K(p, y(t))$ is a Lagrange multiplier.

If for some fixed, but arbitrary $q \varepsilon [0, 1]$, agent k expects that $p_{o_1} q = p_{o_2}(1-q)$ with probability one, he has:

$$V^k_{y_{o_1}} / V^k_{y_{o_2}} = (1-q)/q.$$

Thus, if all agents expect the degenerate relative price q/(1-q), this expectation is self-fulfilling. By the Leontief-Hicks Aggregation Theorem, an economy in which the relative price of the two fiat monies is fixed at q/(1-q), may be treated like an economy with a single fiat money available in quantity $\bar{M} = M_1 + qM_2/(1-q)$. It follows from Proposition 3.2 that for any $q \varepsilon [0, 1]$, the economy has a rational expectations equilibrium in which the relative price of the two fiat monies is fixed at q/(1-q).

It remains to show that in every rational expectations equilibrium the relative price of two fiat monies in transactions that involve both of them must be nonrandom.[15] Suppose the contrary and define:

$\bar{q}$ = inf q, s.t. $p_{o_1} q = p_{o_2}(1-q)$ for some p supp. G^{**k}, some k.

Define the random variable η, so that $(p_{o_2} + \eta)(1-\bar{q}) = p_{o_1}\bar{q}$. Then, one can write:

$$\frac{V^k_{y_{o_2}}}{V^k_{y_{o_1}}} = \frac{\bar{q}}{1-\bar{q}} + \frac{\int_o^\infty e^{-(\rho+\mu)t} \mu E\lambda^k(p,y)\ p_{o_1}\ \eta\ dt}{\int_o^\infty e^{-(\rho+\mu)t} \mu E\lambda^k(p,y)\ p_{o_1}\ dt} \geq \frac{\bar{q}}{1-\bar{q}}$$

By definition of $\bar{q}$, there exist j, k, such that for every ε > 0, there exist y^j, y^k, such that:

$$\frac{V^k_{y_{o_2}}}{V^k_{y_{o_1}}} \leqq \frac{\bar{q}}{1-\bar{q}} + \varepsilon; \frac{V^j_{y_{o_2}}}{V^j_{y_{o_1}}} \leqq \frac{\bar{q}}{1-\bar{q}} + \varepsilon.$$

As ε → 0, the second term in the expression for $V_{y_{o_2}}/V_{y_{o_1}}$ must vanish. In turn this requires that $V_{y_{o_1}}$ must grow out of bounds. But then it is necessary that y^k_{k+1}, $y^k_{o_1}$, $y^k_{o_2}$ → 0, and similarly for j.

Suppose that j = k-1 and that every neighbourhood of $\bar{q}$ contains a q such that q/(1-q) is the equilibrium relative price for a transaction between agents k-1 and k. Then there must be a sequence of y^k in the support of F^{**k} such that y^k_k, y^k_{k+1}, $y^k_{o_1}$, $y^k_{o_2}$ all converge to zero. But by inspection of the optimal policy for agent k, the support of F^{**k} is bounded away from the vector $\mathbf{y}^k = 0$, so that we have a contradiction. Hence, the relative price of the two fiat monies in transactions between agents j = k-1 and k is bounded away from $\bar{q}/(1-\bar{q})$.

Let $\bar{q}/(1-\bar{q}) + \delta$ be the smallest relative price of fiat money two in transactions between agents k, k+1; k = 1,2,...n. Suppose a relative price $\bar{q}/(1-\bar{q}) + \varepsilon$ is quoted in transactions between agents j and k, $j \neq k-1$. Since agent j holds money only because he eventually wants to buy commodity j+1 from an agent of type j+1, he will for small enough ε offer all his holdings of fiat money one in return for fiat money two. But since this is true for all j, the equilibrium relative price of the two fiat monies is bounded away from $\bar{q}/(1-\bar{q})$, contrary to the definition of $\bar{q}$. This completes the proof of Proposition 3.4. Q.E.D.

Proof of Proposition 3.5: From (A.4) and (A.2), one has, for all k,

$$V^k_{y_o} = \int_o^\infty e^{-(\rho + \mu)t} \mu \, E^K \lambda^k(p, y(t)) \, p_o \, dt,$$

and for $k \neq n$,

$$V^k_{y_1} = \int_o^\infty e^{-(\rho + \mu)t} \mu \, E^K \lambda^k(p, y(t)) \, p_1 \, dt,$$

and:

$$V^n_{y_1} > \int_o^\infty e^{-(\rho + \mu)t} \mu \, E^K \lambda^k(p, y(t)) \, p_1 \, dt.$$

In the same way as before, one shows that if the price ratio p_1/p_o is random, and $\bar{q}/(1-\bar{q})$ is the infimum of the ratio p_1/p_o over all transactions that may occur, then the equilibrium price ratio in any one transaction is bounded above $\bar{q}/(1-\bar{q})$, a contradiction.

However, in transactions involving agents of type n, the price p_1 is random, depending on their need for commodity 1. Then the price ratio p_1/p_o is nonrandom only if $p_o = 0$.

Proof of Corollary 3.6: This follows directly from Proposition 3.4.b, the Hicks-Leontief Agrregation Theorem and Proposition 3.3. Q.E.D.

Proof of Proposition 3.7: Part (a) follows by inspection of (15), from the strict concavity of V^k and the fact that under monetary exchange an agent is free not to trade, thus attaining the same allocation as under direct barter. Part (c) follows from the assertion, to be proved as part of Proposition 4.1 that as $\mu \to \infty$, real inventories and real money balances go to zero. To prove part (b), note that from (1),

$$V^k(0;\ G^{*k}) - V^k(0;\ G^{*k}_B) \leq \int_0^\infty e^{-(\rho + \mu)t}\ \mu[W^k(y(t)) - V^k(0;\ G^{*k}_B))]dt,$$

and that inventory accumulation goes to zero as $\mu \to 0$, from (A.1). Q.E.D.

Proof of Proposition 4.1: First, I show that for all k, the distribution of the price ratio p_k/p_o that agent k encounters must become degenerate in the sense that the ratio $p_k/p_o/E(p_k/p_o)$ converges to one. Suppose not, and consider a strategy of selling commodity k whenever its money price exceeds its mean by x% and buying commodity k whenever its money price falls short of its mean by x%. If there is a positive probability of these two events, there exists for every vector of inventories y^* and every $\varepsilon > 0$ a number n, such that inventories no less than y^* are held after n meetings with a probability no less than $1 - \varepsilon$. But then, there exists a number T, such that inventories exceed y^* after a period T/μ, with probability no less than $1 - \varepsilon$. Letting successively $\mu \to \infty$, $\varepsilon \to 0$, and $y^* \to \infty$, one infers that an optimal strategy must bring utility to its upper bound. This in turn implies that consumption flows are infinite, contrary to the condition that aggregate consumption cannot exceed aggregate production in the long run.

The preceding argument holds for all k. The limit points $\bar{q}_k$ of money prices will not generally be finite. However, it follows from the argument in the proof of Proposition 3.2 that the ratios $\bar{q}_k/\bar{q}_j$ are finite, so that one may introduce the vector q, where $q_k = \bar{q}_k/\bar{q}_j$. By Proposition 2.2, this normalization of prices does not affect real consumption behaviour.

For any μ, the valuation function V^k is given by the solution to the Bellman equation:

$$\text{(A.7)} \quad V^k(y;\ G) = \frac{\mu}{(\rho+\mu)} W^k(y;\ G) + \frac{1}{(\rho+\mu)} \max_c [u^k(c) + (a^k - c^k)\ V_y^k(y;\ G)].$$

Along the same lines as in [13], one can show that as $\mu \to \infty$, the value of trading at the true price distribution and the value of trading at the certain prices $\bar{q}$ converge as $\mu \to \infty$. Therefore, the optimal policy for (A.7) converges to the optimal policy that agent k chooses when he can sell commodity k for real money balances at the fixed price q_k at frequency μ/n and buy commodity k+1 for real money balances at the fixed price q_{k+1} at frequency μ/n.

In [13], it is further shown that the value of the latter problem $V^k(y;\ q)$, in turn converges to the value of the frictionless problem:

$$\text{(A.8)} \quad \text{Max} \int_0^\infty e^{-\rho t} u^k(c(t))\ dt$$

$$\text{s.t.} \quad \dot{y}(t) = a^k - c(t);\ y(0) = y,\ \text{given};$$

$$q'y(t) \geq 0 \text{ for all } t.$$

The optimal policy in the frictionless problem results in inventory decumulation to zero in finite time. Therefore, it follows that real commodity inventories held by any agent in a rational expectations equilibrium must go to zero in probability as the frequency of meetings goes out of bounds. This proves part (a) of the Proposition.

For any vector of inventories y^k in the support of the stationary distribution under random trading at the fixed price q, the optimal excess demand policy requires that in meetings with agent k-1, agent k sells his total inventory of commodity k in return for money, and in meetings with agent k+1, agent k sells his total money balances in return for commodity k+1, both of course at the price q. (Hellwig [13]). If q is an equilibrium price vector and agent k+1's holdings of commodity k+1 go to zero in probability, then real money balances must go to zero, i.e. the price of money p_o goes to zero in probability as $\mu \to \infty$. Otherwise there would be an excess demand for commodity k+1 in a meeting between agents k and k+1. This proves the first half of part (c) of the proposition.

Next, I prove that $q = p^*$ and c^k goes to c^{*k} in probability. Consider agent k in random trading at the frequency μ, in a stationary state $\bar{F}^k_\mu(.; q)$. In a stationary distribution, mean utility does not change, and therefore:

$$\int (c^k - a^k)' V^k_y(y; q)\ \bar{F}^k_\mu(dy; q) = \mu \int [W^k(y; q) - V^k(y; q)]\ \bar{F}^k_\mu(dy; q).$$

Substitute from the Bellman equation for the left hand side and cancel terms to obtain:

$$\frac{1}{\rho} \int u^k(c^k)\ \bar{F}^k_\mu(dy; q) = \int V^k(y)\ \bar{F}^k_\mu(dy; q).$$

From the result in [13], V^k approaches the value of the frictionless problem (A.8) as $\mu \to \infty$. The latter is at least as large as $u^k(c^{*k}(q))/\rho$, so that we have:

$$\lim_{\mu\to\infty} \int u^k(c^k)\ \bar{F}^k_\mu(dy;\ q) \geqq u^k(c^{*k}(q)).$$

Furthermore, from the stationarity of inventories at $\bar{F}^k_\mu$, we have:

$$\int (c^k - a^k)\ \bar{F}^k_\mu(dy;\ q) = \mu \int E\ z^k(q;\ y)\ \bar{F}^k_\mu(dy;\ q).$$

Premultiplying by q, this implies:

$$q' \int (c^k - a^k)\ \bar{F}^k_\mu(dy;\ q) = 0,$$

from the budget constraint on excess demand.

But by definition of $c^{*k}(q)$ and strict concavity of u^k, the constant function $c^k(y;\ q) \equiv c^{*k}(q)$ is the unique maximizer of

$$\int u^k(c^k(y;\ q))\ F(dy)$$

subject to the constraint:

$$q' \int (c^k(y;\ q) - a^k)\ F(dy) = 0,$$

for all distributions F. It follows that c^k must go to c^{*k} in probability. Since furthermore, aggregate consumption in the rational expectations equilibrium is always equal to aggregate production, this completes the proof of the proposition.

Q.E.D.

<u>Proof of Proposition 4.2</u>: At a stationary state, aggregate utility is stationary. This implies:

$$\int [V^k(x) - V^k(y)]\, K^{kj}(dx;\, y,\, y^j)\hat{F}^j_\mu(dy^j)\; \hat{F}^k_\mu(dy) = \frac{1}{\mu}\int (c^k - a^k)\; V^k_y\; \hat{F}^k_\mu(dy).$$

From the Bellman equation for V^k, the right hand side is bounded above by $(\rho + \bar{\mu})(\bar{u} - \underline{u})/\rho\mu$, where $\bar{\mu}$ is the <u>expected</u> trading frequency. Thus, average gains from an additional trade in the stationary state vanish as $1/\mu$. As μ becomes large, the state of the economy becomes approximately pairwise optimal.

By Lemma A.5, the set of stationary states is compact. For any $\varepsilon > 0$, there exists a compact set K, such that $F^k(K) \geq 1 - \varepsilon$. The excess demand function $z^k(p;\, .)$ is continuous on the compact set K. By the definition of weak convergence, it is sufficient to show that any pairwise optimal state of the economy that is stationary under the process (A.5) is supported by a competitive price vector $p > 0$.

To prove this, I assert that for all k, there exists a set Y^k with positive measure under a stationary distribution, such that for $y^k \varepsilon Y^k$, $y^k_k > 0$, $y^k_{k+1} > 0$, $y^k_o > 0$. This will directly imply the existence of a competitive price $p > 0$, because one can set:

a: $p_o = 1$;

b: for almost all $y \varepsilon Y^k$, $p_k = V^k_{y_k}(y)/V^k_{y_o}(y)$;

$$p_{k+1} = V^k_{y_{k+1}}(y)/V^k_{y_o}(y);$$

$k = 1,2,\ldots n.$

To see that under pairwise optimality, the price vector p is well defined by conditions (a) and (b), note that pairwise optimality requires:

i: For all k, almost all x, y ε Y^k, $V^k_{y_k}(y)/V^k_{y_o}(y) = V^k_{y_k}(x)/V^k_{y_o}(x)$, as otherwise agents with inventories x, y could make a profitable trade.

ii: For all k, almost all y ε Y^k, x ε Y^{k+1}, $V^k_{y_{k+1}}(y)/V^k_{y_o}(y) \leqq V^{k+1}_{y_{k+1}}(x)/V^{k+1}_{y_o}(x)$; otherwise the two agents could make a profitable trade.

Finally, stationarity requires:

iii: For almost all y ε Y^k, x ε Y^{k+1}, $V^k_{y_{k+1}}(y)/V^k_{y_o}(y) \geqq V^{k+1}_{y_{k+1}}(x)/V^{k+1}_{y_o}(x)$, for otherwise an agent of type k with inventories y would be unable to buy commodity k+1 and therefore y must be a transient state.

I leave it to the reader to check that the vector p defined in (a) and (b) will also support the inventories of agents not covered by the definition of p.

It remains to show the existence of the sets Y^k, k = 1,2,...n. From (A.1) and the Inada condition, one has for $\mathcal{G}$ ε Γ^n, $c^k_k(0) < a^k_k$, all k. All agents find it advantageous to accumulate at least some inventory of their produce so as to be able to receive some units of the other commodity they consume. Furthermore, an agent of type k will never run down his inventories of commodity k+1 in finite time, and therefore, almost all agents of type k must have positive inventories of commodity k+1.

Furthermore, a set of positive measure of agents of type k must hold money. For suppose not. Then, there exists an index J, such that agents of type J holds a positive amount of money, while almost all agents of type J+1 holds no money. In a stationary state, this implies that agents of type J are not buying commodity J+1 and hence must have zero inventories of commodity J+1. But then they are willing to pay an arbitarily large price to obtain some of commodity J+1 in return for money, implying a contradiction to the statement that they are not buying any of commodity J+1 at all.

Finally, a set of positive measure of agents of type k hold both money and commodity k. For suppose not. Then agent k accumulates inventories of commodity k only if his money balances are zero. This contradicts the continuity of the consumption function. This completes the proof of Proposition 4.2. Q.E.D.

Proof of Proposition 4.3: The stationarity of inventories of commodity k+1 implies:

$$\int [a_{k+1}^{k+1} - c_{k+1}^{k+1}(y^{k+1})] \; F_{\mu}^{*k+1}(dy^{k+1}) = \int c_{k+1}^{k}(y^k) \; F_{\mu}^{*k}(dy^k)$$

Because agent k's inventories of commodity k+1 are stationary, the right hand side is equal to

$$\frac{\mu}{n} \int z_{k+1}^{k}(p;\, y^k) \; Q^{k,k+1}(dp;\, y^k,\, y^{k+1}) \; F_{\mu}^{*k+1}(dy^{k+1}) \; F_{\mu}^{*k}(dy^k).$$

Substituting for z_{k+1}^{k} from agent k's budget constraint, this term is equal to

$$-\frac{\mu}{n} \int \frac{p_o}{p_{k+1}} \; z_o^k(p;\, y^k) \; Q^{k,k+1}(dp;\, y^{k+1},\, y^k) \; F_{\mu}^{*k+1}(dy^{k+1}) \; F_{\mu}^{*k}(dy^k).$$

Multiplying by $\bar{q}_{k+1}$ and adding, one has:

$$\frac{n}{\mu} \sum_{k=1}^{n} \bar{q}_k \int (a_k^k - c_k^k) dF_{\mu}^{*k} = - \sum_{k=1}^{n} \int \bar{q}_{k+1} \frac{p_o}{p_{k+1}} \; z_o^k(p;\, y^k) \; dQ^{k,k+1} \; dF_{\mu}^{*k+1} \; dF_{\mu}^{*k}.$$

As $\mu \to \infty$ the right hand side approaches the quantity:

$$(A.9) \qquad - \sum_{k=1}^{n} \int z_o^k((1, \bar{q}); y^k) \, dF^{*k}_{\mu},$$

from Proposition 4.1. Further, from the argument in the proof of Proposition 4.1, the quantity $z_o^k(1, \bar{q}), y)$ is equal to $- y_o$ for any vector y in the support of a stationary distribution. Hence, the quantity (A.9) approaches the quantity:

$$\sum_{k=1}^{n} \int y_o^k \, dF^{*k}_{\mu} = M. \qquad \text{Q.E.D.}$$

It should be noted that for finite μ, this proof breaks down at two points: First, if there is a speculative demand for money, the quantity (A.9) is strictly less than M. Second, if the distribution of prices is nondegenerate, the covariance between money balances and money prices perturbs the aggregation of excess demands for money. If the equilibrium in every trading instant were unique, it would be easy to show that the bias arising from this must go in the same direction as that from speculative demand, because high excess supplies of money entail high money prices and receive low weight in the aggregation procedure. But it is not clear whether this conclusion can be maintained if equilibrium in a given trading instant is not unique.

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Footnotes

1 This assumption rules out the possibility of strategic consumption behaviour which attempts to affect the trading conditions at the next trading instant. For a detailed discussion of this problem, see Hellwig [13].

2 Trivially, consumption and excess demand are also homogeneous of degree zero in the accounting prices.

3 Clower [6], Barro and Grossman [2].

4 This argument suggests dropping the sales constraint on commodity k+1. This can be done without any change in the results of this paper.

5 Presumably, the agent could also find out the impact of his own inventories on the prices at which he trades. However, the strategic consumption behaviour that this information would induce would cause awkward problems of nonconvexity. I integrate the effect of the agent's own inventories on the prices at which he trades out to avoid these difficulties. For a more detailed discussion of this problem, see Hellwig [13].

6 This proposition was originally introduced in a different context by Hahn [10].

7 This was first seen by Hayashi [12].

8 See e.g. Arrow and Hahn [1], Ch. II, 8.

9 I leave it to the reader to make this argument rigourous to prove that the price of money is zero in a rational expectations equilibrium under multilateral exchange, satisfying C.2 - 4, and, in lieu of C.1, the multilateral market clearing condition:

$$\sum_{k=1}^{n} \int_{\mathbb{R}_+^{n+1}} z^k(p; y^k)\, F^k(dy^k) = 0,$$

so that money is not needed to facilitate exchange.

10 For this view see Pesek and Saving [20], pp. 323 ff. In contrast, Hicks [14], pp. 14 ff. maintains that the exchange motive is "impervious to economic incentives" and "not a demand for money the way that the other is".

11 This observation is due to Feldman [8]. See also Harris [11].

12 This is obvious for a generalized indirect barter process, in which agents have price expectations appropriate to direct barter. For a numerical example see Feldman [8].

13 The reader should note the similarity between the discussion of speculation in Section 3.2 above and the role of search behaviour in the models of Diamond [7] and Butters [5].
It should also be noted that Robertson's [21, p. 25] famous criticism of the General Theory that "the rate of interest is what it is because it is expected to become other than it is; if it is not expected to become other than it is, there is nothing left to tell us why it is what it is," while similar to Leontief's proposition, is directed at the treatment of the speculative demand as an independent phenomenon, apart from other factors.

14 See Hildenbrand [15], p. 53 for the definition of the integral of a correspondence.

15 I neglect the fact that in meetings in which one of the fiat monies is not held by either agent, the relative price of them is arbitrary within certain bounds. This is without economic significance and can be resolved by setting the relative price of the fiat monies equal to one of the two agents' marginal rate of substitution between the two monies. Then the proof can be directly applied to this case.