

An Econometric Definition of the  
Inflation-Unemployment Trade-off

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The main purpose of this paper is to propose a method for ascertaining the most favorable trade-off relationship between inflation and employment implicit in an econometric model of a national economy. Stimulated by and based upon A. W. Phillips' original paper (1958) on the relation between unemployment and the rate of change of money wage rates, numerous studies have appeared to refine, respecify and estimate structural equations explaining the rates of change in the wage rates, the price level, unemployment and related variables. It soon became apparent that these studies, though useful, may not be sufficient for ascertaining the trade-off relationship between unemployment and inflation. If unemployment and inflation are viewed as two of the many endogenous variables which are jointly determined by a system of simultaneous econometric equations, their relationship has to be derived by solving a whole system using alternative values for the policy variables subject to government control. The approach of deriving the unemployment-inflation trade-off by varying the policy variables and solving for these two endogenous variables in an econometric model has been adopted by Andersen and Carlson (1972), Hirsch (1972), de Menil and Enzler (1972), and Hymans (1972), among others.

We would like to suggest in this paper that the most interesting trade-off

relationship between inflation and unemployment consists of a set of points in the inflation-unemployment diagram which cannot be dominated, and that, accordingly, the policy variables should be chosen optimally, rather than arbitrarily, in order to derive this most favorable trade-off relationship. By choosing two alternative paths for the policy variables arbitrarily, one more expansionary than the other, one would expect to obtain two results for unemployment and inflation from an econometric model. However, by a more judicious choice of the policy variables, it may be possible to improve on both of these results. In general, the unemployment-inflation combinations resulting from varying the values of the policy variables in an econometric model would be a scatter and would not all fall on one curve. Thus a unique trade-off relationship may not be obtainable simply by trying out different values for the policy variables. Some form of optimization is required to derive the best possible trade-off relationship.

In section 1 of this paper, we will first point out the various possibilities for unemployment and inflation implicit in a static econometric model consisting of a set of simultaneous equations and propose a method to derive from the model the best trade-off curve. Section 2 generalizes the discussion to the dynamic case. Section 3 applies our approach to derive the best inflation-employment trade-off from the St. Louis Model, and Section 4 applies the same for the Michigan Quarterly Econometric Model. Section 5 contains some concluding remarks.<sup>1</sup>

#### 1. Inflation-Unemployment Possibilities in a Static Model

Given a set of simultaneous equations determining a vector  $y' = (y_1, y_2, \dots, y_p)$  of endogenous variables by a vector  $x' = (x_1, \dots, x_q)$  of policy instruments and a set of exogenous variables not subject to government control which will be treated as fixed, we ask what combinations of unemployment  $y_1$  and inflation  $y_2$  are possible and how one can trace out the best possible combinations.

For our purpose, the set of possible solutions for  $y_1$  and  $y_2$  can conveniently be classified into three categories. The first is a rigid relation, the points all falling on one curve in the  $y_1$ - $y_2$  diagram. The second is a semi-rigid relation, the set of possible points forming an area in the  $y_1$ - $y_2$  plane which has a south-western boundary. The second case is considered most important, and it is the south-western boundary which we would like to ascertain as the best possible trade-off relationship. The third is the least rigid, the set of possible points not being bounded by a south-western boundary. As a special case of the third category, we may have the set of possible points covering the entire  $y_1$ - $y_2$  plane. This is a mathematical possibility, but the model involved would not be economically meaningful because  $y_1$  cannot be negative.

A rigid trade-off curve may be the result of specifying a structural equation explaining  $y_1$  by  $y_2$  and other variables none of which can be influenced, directly or indirectly, by the policy instruments. The set of the other variables in this structural equation may consist entirely of exogenous variables not subject to government control, or of some endogenous variables which are determined completely by uncontrollable exogenous variables. The equation relating  $y_1$ ,  $y_2$  and the other variables so specified might not itself be a structural equation, but the result of combining several structural equations. As an example, let  $y_3$  be the rate of change in the wage rate, and assume a wage-Phillips curve relating  $y_3$  to  $y_1$ ,  $y_2$  and possibly some exogenous variables not subject to government control, and a price-Phillips curve relating  $y_2$  to  $y_1$ ,  $y_3$  and possibly some exogenous variables. Eliminating  $y_3$  from these two structural equations would yield a rigid trade-off relationship between  $y_1$  and  $y_2$ . In terms of the reduced-form equations determining  $y_1$  and  $y_2$  by  $x_1, \dots, x_q$ , the

2 by  $q$  matrix of partial derivatives of  $y_1$  and  $y_2$  with respect to the  $q$   $x$ 's would be of rank 1, so that, starting from a given point, when a small change in  $y_2$  results from whatever changes in the  $x$ 's,  $y_1$  will be changed proportionally.

If any of the other variables in an equation relating  $y_1$  and  $y_2$  (which may be itself a structural equation or, more likely, the result of combining several structural equations) can be influenced directly or indirectly by the policy variables, the relation between  $y_1$  and  $y_2$  will no longer be rigid. The possible combinations of  $y_1$  and  $y_2$  obtained by varying the  $x$ 's will form an area in the  $y_1$ - $y_2$  plane. The 2 by  $q$  matrix of partial derivatives of  $y_1$  and  $y_2$  with respect to the  $q$   $x$ 's via the reduced form will have rank 2 for many values of  $y$  and  $x$ . In general, however, the possible combinations do not take up the entire  $y_1$ - $y_2$  plane as they would in the special case of the third category of our classification. A nonlinear structural equation or set of structural equations explaining the rate of unemployment  $y_1$  may rule out negative values for  $y_1$ . Furthermore, a south-western boundary may exist for the possible combinations of  $y_1$  and  $y_2$  because the values of the  $x$ 's are bounded (such as the tax rates, money supply, and government expenditures taking only non-negative values). It may also exist because in the equation relating  $y_1$  and  $y_2$ , other endogenous variables which can be influenced by government policies are bounded by their own nonlinearities or by the boundedness of the government instruments themselves. Econometric models belonging to category 2, with a south-western boundary for the possible  $y_1$ - $y_2$  combinations, appear to be economically reasonable. Category 1 would rule out the existence of any bad policies which can make both inflation and unemployment worse. Category 3 would imply that one can achieve any desired inflation-employment combination as one pleases.<sup>2</sup>

If a south-western boundary exists for the possible  $y_1$ - $y_2$  combinations, one can obtain points on this boundary by varying the parameters  $k_1$ ,  $k_2$ ,  $a_1$  and  $a_2$  in a quadratic loss function

$$(1) \quad w(y_1, y_2) = k_1(y_1 - a_1)^2 + k_2(y_2 - a_2)^2$$

and minimizing this function subject to the constraint of the econometric model. To see that one point in the boundary will result from such a minimization, let  $k_1=k_2=1$  and  $a_1=a_2=0$ . The points of equal loss will form a circle with center in the origin, and circles closer to the origin will have smaller losses. The minimum occurs when the smallest circle is tangential to the boundary of the possible  $y_1$ - $y_2$  points constrained by the econometric model. Assume that, in the first quadrant of the  $y_1 - y_2$  plane, a south-western boundary exists, which means that the slope of the boundary is negative (or at least non-positive). Since the slope of the circle in the first quadrant is also non-negative, obtaining the smallest circle satisfying the constraint will mean that the  $y_1$ - $y_2$  point is on the boundary -- if it were not, one could use a smaller circle satisfying the constraint and reducing the loss. To obtain another point on the boundary, one could change the ratio of  $k_1$  to  $k_2$ , letting  $k_1=100$  and  $k_2=1$ , say. The points of equal loss would be an ellipse which is elongated vertically. In sacrificing one unit of unemployment, one requires a greater reduction in inflation than before; the slope of the ellipse in the first quadrant of the  $y_1$ - $y_2$  plane is steeper than before. The new minimum will yield a smaller unemployment and a higher inflation rate. We can drop the above assumption that the south-western boundary lies in the first quadrant. If it were in the fourth quadrant, the above analysis would apply by placing the center  $(a_1, a_2)$  of the ellipse below and to the left of the boundary.

Although minimization of (1.1) with  $k_1=k_2=1$  and  $a_1=a_2=0$  will yield a point of the best trade-off curve in the first quadrant if it exists, one cannot anticipate the resulting value for either  $y_1$  or  $y_2$ . To answer the question, what is the lowest inflation rate  $y_2$  for a given unemployment rate  $y_1=5$  (per cent), one may set  $k_1=1000$ ,  $k_2=1$ ,  $a_1=5$  and  $a_2=0$ . The points of equal loss form

a highly vertically elongated ellipse with (5, 0) as center. Minimization yields an unemployment rate close to 5 per cent and the corresponding lowest inflation rate, since it chooses the smallest vertical ellipse centering in (5, 0) which still satisfies the constraint of the econometric model. By the same argument, replacing  $a_2=0$  by  $a_2=2$  in the above minimization would also work provided that the lowest inflation rate for 5 per cent unemployment is above 2 percent.

In this section, we have classified the possible solutions for unemployment and inflation from a static econometric model, defined the best trade-off relationship implicit in the model, and suggested a method for tracing out this relationship.

## 2. Unemployment-Inflation Trade-off Possibilities in the Dynamic Case

It is important to generalize our discussion to the dynamic case because econometric models are dynamic, and economists are interested in the best trade-off relationships between unemployment and inflation through time. In the dynamic setting, we have to consider the 3-dimensional space with unemployment  $y_1$ , inflation  $y_2$  and time  $t$  as the axes. The three categories of trade-off possibilities implicit in an econometric model will be discussed for the very short run, the intermediate run, and the very long-run.

By the very short run, we mean one quarter, if the econometric model is a quarterly model. The dynamic model consists of  $y_t$ ,  $y_{t-1}$  and  $x_t$  as variables, the uncontrollable exogenous variables being considered given as before. Endogenous variables lagged more than one period and lagged policy instruments can be eliminated from any dynamic models by introducing suitable identities, as explained in Chow (1975). For the one quarter immediately ahead, all lagged variables  $y_{t-1}$  are given, and the analysis reduces to the static case. The discussion of section 1 applies entirely to the trade-off possibilities between  $y_{1t}$  and  $y_{2t}$  given all lagged variables.

The intermediate run requires some discussion. When the time interval of interest is from period 1 to period T (where T is not very large), we are concerned with the possible points on the  $y_1$ - $y_2$ -t diagram. The most rigid category 1 would be a surface on this diagram. This means that, for any given t, the possible combinations of  $y_{1t}$  and  $y_{2t}$  will lie on a curve on the  $y_1$ - $y_2$  plane. As an example, there may be a structural equation, or an equation resulting from eliminating other endogenous variables from several structural equations, which relates  $y_{1t}$ ,  $y_{2t}$  and other variables not subject to government control, either directly or indirectly. Again, category 1 is a very special case. By varying the time paths of the policy instruments, one may obtain combinations of  $y_{1t}$ ,  $y_{2t}$  and t which are not confined to a surface. However, not all points in the  $y_1$ ,  $y_2$  and t space are reachable by manipulation of government policies. There may be a set of surfaces serving as the lower boundaries for the possible time paths for  $y_1$  and  $y_2$  in the following sense. For any t, one cannot reduce  $y_{2t}$  without increasing  $y_{1t}$  or  $y_{2s}$  or  $y_{1s}$  for  $s \neq t$ . Thus it may be possible to reduce both  $y_{1t}$  and  $y_{2t}$ , but some  $y_{1s}$  or  $y_{2s}$  in another period s will have to increase -- otherwise, the surface is not a lowest boundary possible. Category 2, where such surfaces exist, is considered more likely than both category 1 and category 3 where such boundary surfaces do not exist.

To obtain a path for  $y_1$  and  $y_2$  on such a boundary surface, we minimize the loss function

$$(2) \quad \sum_{t=1}^T [k_{1t}(y_{1t} - a_{1t})^2 + k_{2t}(y_{2t} - a_{2t})^2]$$

subject to the constraint of the dynamic econometric model. Consider the 2T-dimensional space with  $y_{1t}$ ,  $y_{2t}$  ( $t=1,2,\dots,T$ ) measured along its coordinates.



The points in this space having the same loss are ellipsoids. Contracting one such ellipsoid while keeping the  $y_1$  and  $y_2$  paths attainable by the dynamic econometric model guarantees that the attainable point with minimum loss lies on the boundary surface. To keep the unemployment path close to 5 per cent, say, and to find a best inflation path consistent with the econometric model, one may choose  $a_{1t}=5$ ,  $a_{2t}=0$  (or a small number),  $k_{1t}=1000$ , and  $k_{2t}=1$  for  $t=1,2,\dots,T$ . A numerical method for minimizing a quadratic loss function subject to the constraint of a nonlinear econometric model can be found in Chow (1975, section 12.1), and will not be described here. It will be used for the calculations reported in Sections 3 and 4 below.

Once several optimal paths are obtained, with  $y_{1t}$  aimed at 4, 5, 6 and 7 per cent respectively for instance, one may wish to summarize these paths along the best trade-off boundary in a two-dimensional diagram. One way to do so is to plot the mean inflation rate  $(\sum_t y_{2t}/T)$  against the mean unemployment rate  $(\sum_t y_{1t}/T)$  over the  $T$  periods. This would imply a constant rate of substitution between  $y_{1t}$  and  $y_{1s}$ , and between  $y_{2t}$  and  $y_{2s}$ , contrary to the specification of a quadratic loss function. A second way is to plot  $[\sum_t (y_{2t}-a_{2t})^2/T]^{1/2}$  against  $[\sum_t (y_{1t}-a_{1t})^2/T]^{1/2}$  which would penalize the increase of each variable by the square of its deviation from target. Note that when a two-dimensional diagram is used in the multiperiod case, the index number problem cannot be avoided.

For the very long run, if any one ever cares for such an analysis, we can define the equilibrium  $y_1$  and  $y_2$  combination as the constant values towards which these two variables approach as  $T$  increases in the multiperiod minimization problem specified above, if such constant values exist. Of course,  $y_{1t}$  and  $y_{2t}$  might not approach constant values as  $t$  increases. Since we treat the very long run problem in the same way as the above intermediate run problem by simply increasing the planning horizon  $T$ , we can still use such indices as

$(\sum_t y_{it}/T)$  and  $[\sum_t (y_{it} - a_i)^2/T]^{1/2}$  for  $i=1,2$  even if  $y_{it}$  itself might not approach a limit as  $t$  increases. It is important to note that, as in the case of the intermediate-run problem, optimization is required to obtain the most favorable long-run trade-off. As long as there exist bad policies which would create more inflation in the long run without improving the unemployment situation, the time paths for  $y_{1t}$  and  $y_{2t}$  do not all fall on a rigid surface, and one needs to minimize in order to obtain a path on the lower boundary of all feasible paths.

### 3. Analysis of the St. Louis Model

The St. Louis Model of Andersen and Carlson (1970) is well-known. It has an equation explaining money GNP by the current and lagged values of money supply  $M$  and high-employment Federal expenditures  $E$ . Demand pressure and expected price change will help determine the change in the price level. Since both policy instruments  $M$  and  $E$  affect the economy through the same money GNP variable, the 2 by 2 matrix of partial derivatives of the unemployment rate  $y_1$  and the rate of price change  $y_2$  with respect to these 2 instruments has rank 1. If the analysis is limited to one quarter, the St. Louis model therefore implies a rigid trade-off relation between  $y_1$  and  $y_2$ .

However, in a multiperiod setting, in so far as inflation is affected by expected price change which is determined by past price changes and by the demand pressure which is influenced by the course of real output, alternative paths for the money supply can affect both the expected price change and the demand pressure, thus influencing the paths of inflation and unemployment. There is no reason to expect that arbitrary paths for  $M$  (or  $E$ ) will yield paths for  $y_1$  and  $y_2$  on the lowest boundary for the unemployment-inflation trade-off through time. Using 20 quarters from 1971.1 to 1975.1, and the loss function

$$(3) \quad \sum_{t=1}^{20} k_1 (y_{1t} - a_1)^2 + \sum_{t=1}^{20} k_2 (y_{2t} - a_2)^2$$

with  $k_1=10000$ ,  $k_2= .01$ ,  $a_2= 2$ , and  $a_1= 3.5, 4.5, 5.5, 6.5, 7.0$  and  $8.0$  respectively, and letting  $E_t$  follow its historical path, we have obtained six optimal paths by minimizing (3) with respect to  $M_t$ .  $(\sum y_{2t})/T$  is plotted against  $(\sum y_{1t})/T$  in Figure 1, and  $(\sum y_{2t}^2/T)^{1/2}$  against  $(\sum y_{1t}^2/T)^{1/2}$  in Figure 2. The six optimal points are joined by a solid line. The corresponding points summarizing the paths for  $y_1$  and  $y_2$  resulting from the historical path for  $M_t$  are marked by a cross. The points resulting from using a constant percentage growth for  $M_t$  of 2, 4, and 6 per cent are marked by small circles.

Note that the optimal points dominate the points resulting from the historical path and the smooth growth paths for  $M$ . It would be of interest to examine the dynamic characteristics of the optimizing paths for  $M$ , but space limitation prevents an adequate discussion. Suffice it to say that the optimizing paths for  $M$  exhibit sizable fluctuations. To inhibit large fluctuations, we can add a term  $\sum_{t=1}^{20} k_3 (M_t - a_3)^2$  in the loss function. Minimization of such a function, again using  $a_1= 3.5, 4.5, 5.5, 6.5, 7.0$  and  $8.0$  respectively will yeild a curve above the solid curves in Figure 1 and 2. This curve can also be used to define the best trade-off relationship between  $y_1$  and  $y_2$ , under the assumption that fluctuations in the instrument are also penalized.

#### 4. Analysis of the Michigan Quarterly Econometric Model

Our brief discussion using the Michigan Quarterly Econometric Model by Hymans and Shapiro (1970) follows closely the analysis for the St. Louis model, except that two instruments are used. The instruments are unborrowed reserves UR and non-defense federal expenditures GFO. With more than one instrument, an optimal path along the

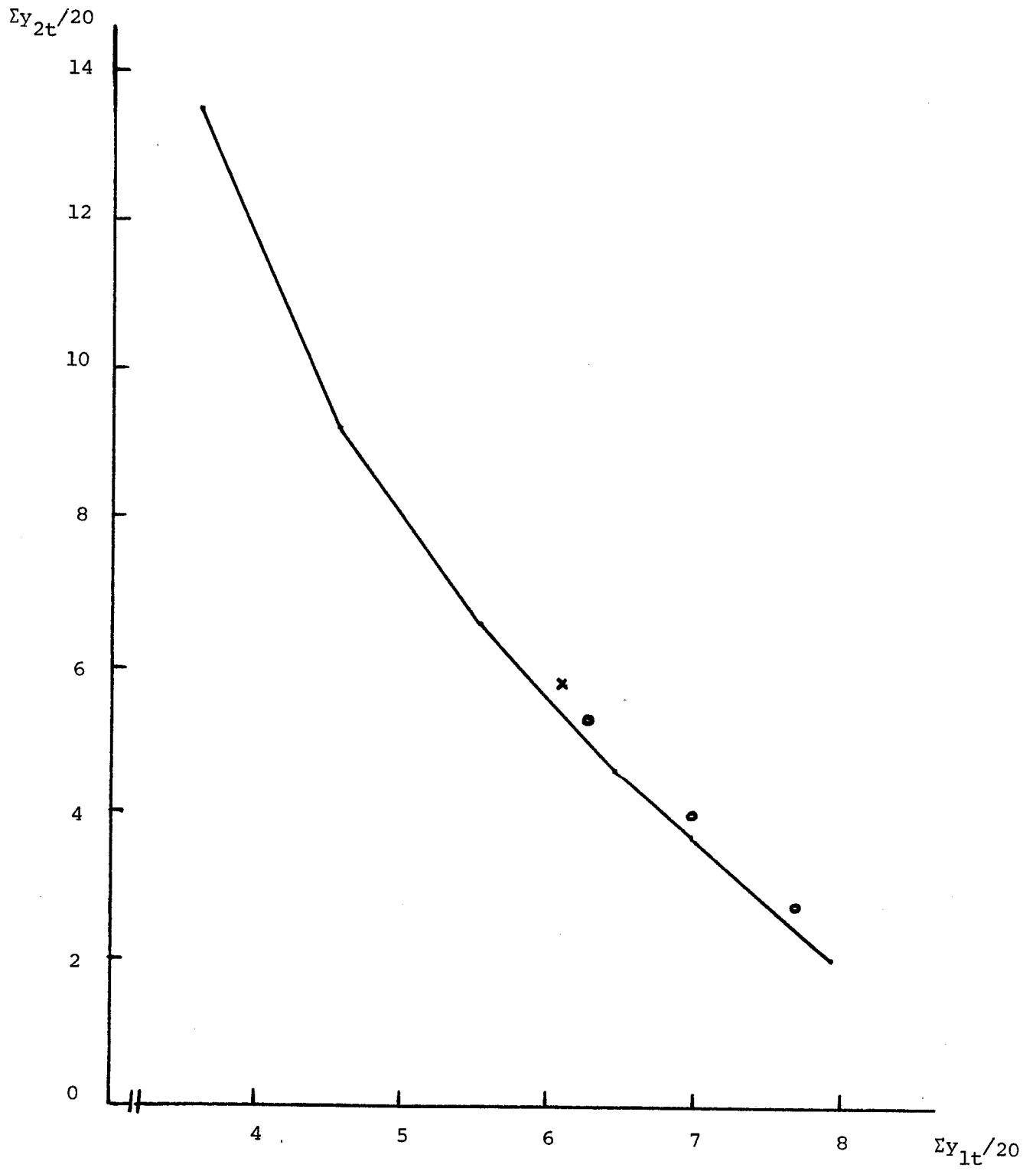


Figure 1

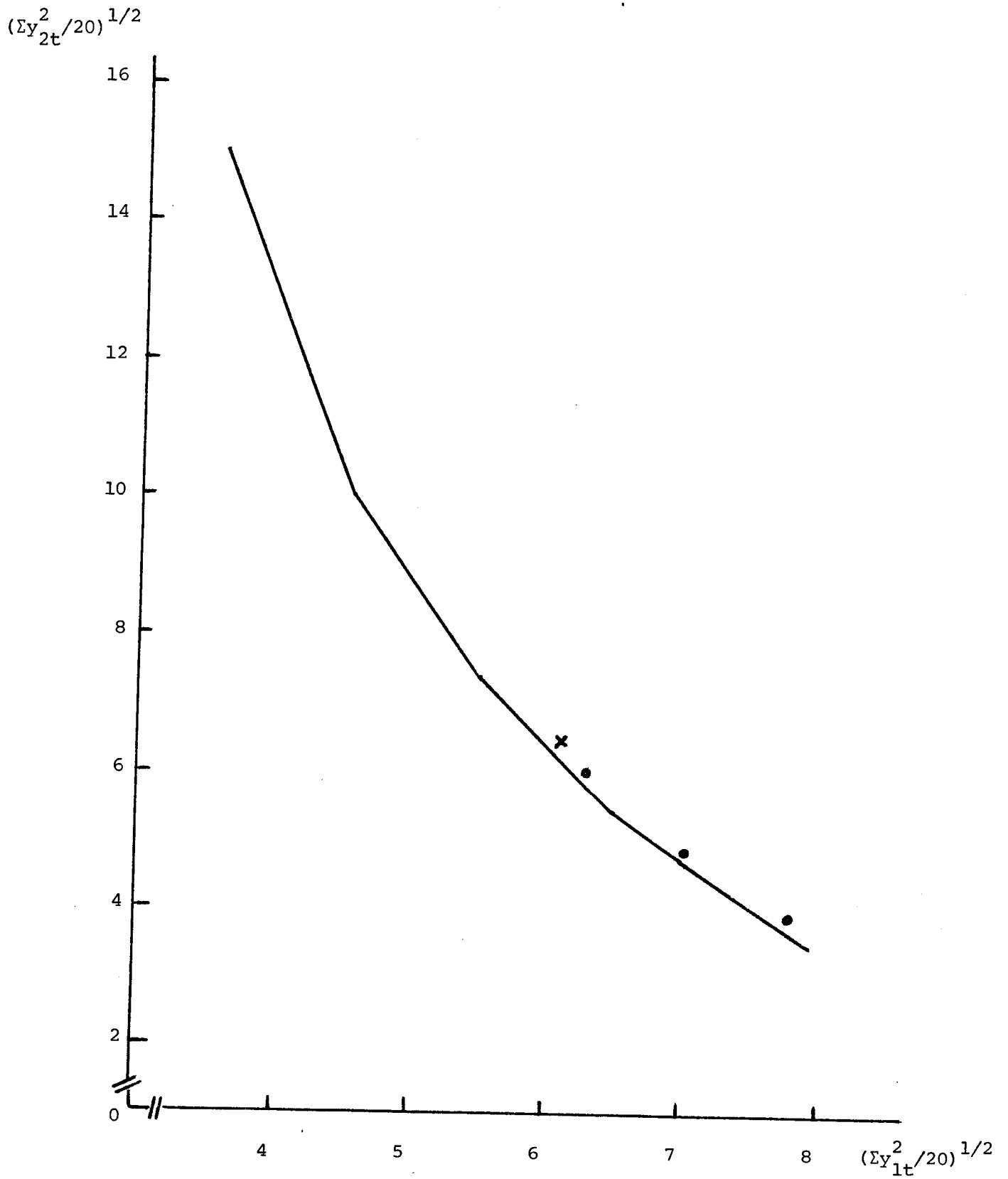


Figure 2

lowest boundary for the dynamic unemployment-inflation trade-off is obtained not only by a suitable dynamic pattern for the one and only control variable (as in the case of the St. Louis model), but also by an optimal combination of the time paths for the instruments. The period for the optimization is from 1971.1 to 1975.1, with  $T=17$ . Due to the structure of the model the level of the GNP deflator  $P$  is included in the loss function rather than the rate of inflation  $y_2$ . The targets for  $P_t$ ,  $a_{2t}$ , grow at an annual rate of 2 per cent with  $a_{20} = P_0$ . Variations in the first instrument  $UR$  are also penalized. The target values for  $UR_t$ ,  $a_{3t}$ , follow the historical path of the instrument. The loss function is

$$(4) \quad \sum_{t=1}^{17} k_1 (y_{1t} - a_{1t})^2 + \sum_{t=1}^{17} k_2 (P_t - a_{2t})^2 + \sum_{t=1}^{17} k_3 (UR_t - a_{3t})^2$$

with  $k_1 = 20000$ ,  $k_2 = 10$ ,  $k_3 = 2$ , and  $a_1 = 3.5$  and  $5.5$  respectively. Figures 3 and 4 have been obtained in the same way as Figures 1 and 2. The two optimal points are joined by a solid line. The corresponding points summarizing the paths for  $y_1$  and  $y_2$  resulting from the historical paths for  $UR_t$  and  $GFO_t$  are marked by a cross. The points indicated by a circle result from letting  $UR_t$  and  $GFO_t$  grow at an annual rate of approximately 4.5 and 16.5 per cent, respectively. These growth rates correspond to smoothed historical paths.

The optimal points clearly dominate the points resulting from the historical paths and smooth growth paths for the instruments. The optimizing paths for  $GFO$  exhibit some fluctuations and  $GFO$  even takes on negative values in some quarters. Total government expenditures, however, are still positive and the economic effects according to the Michigan Quarterly Econometric Model would be the same if we reduce other (defense) government expenditures

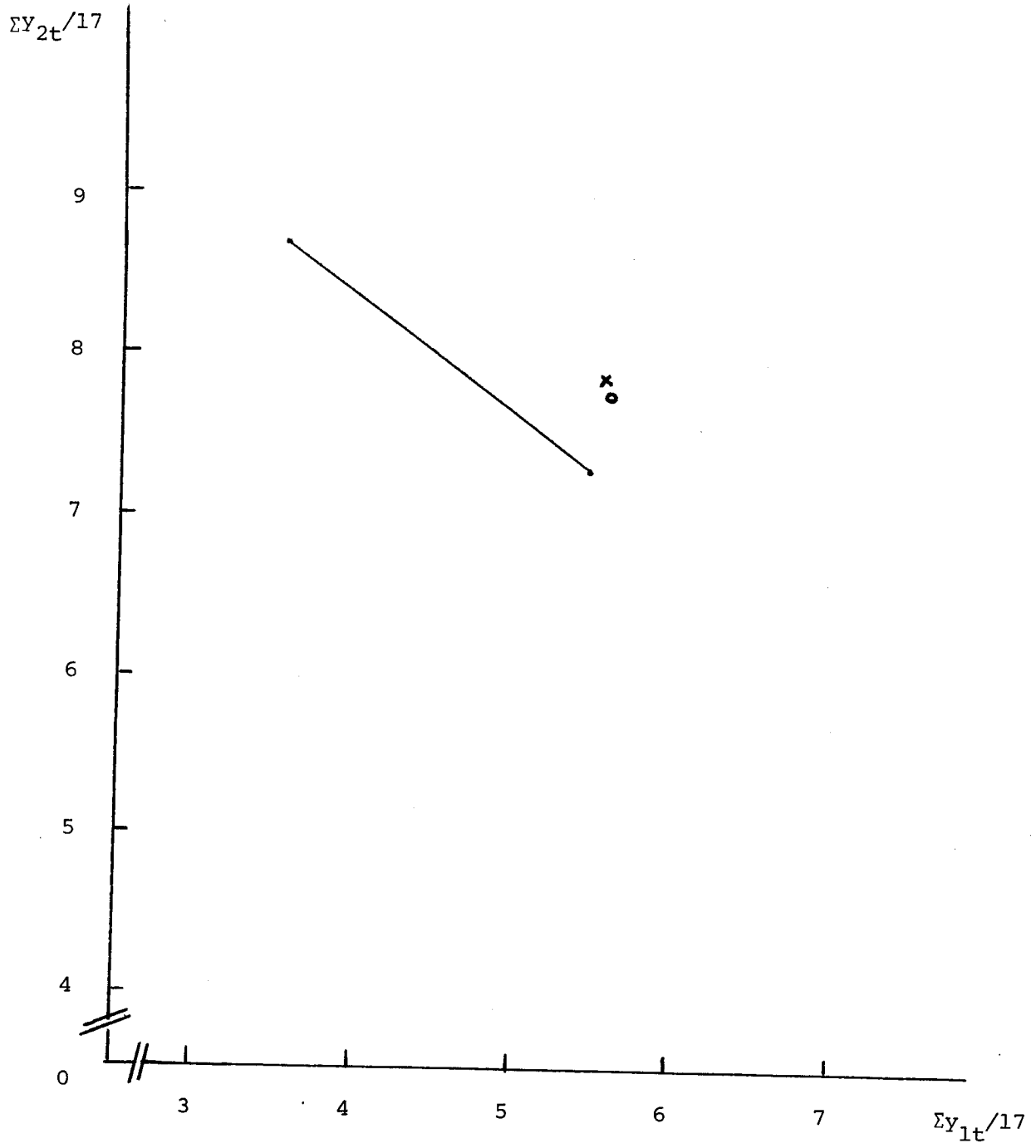


Figure 3

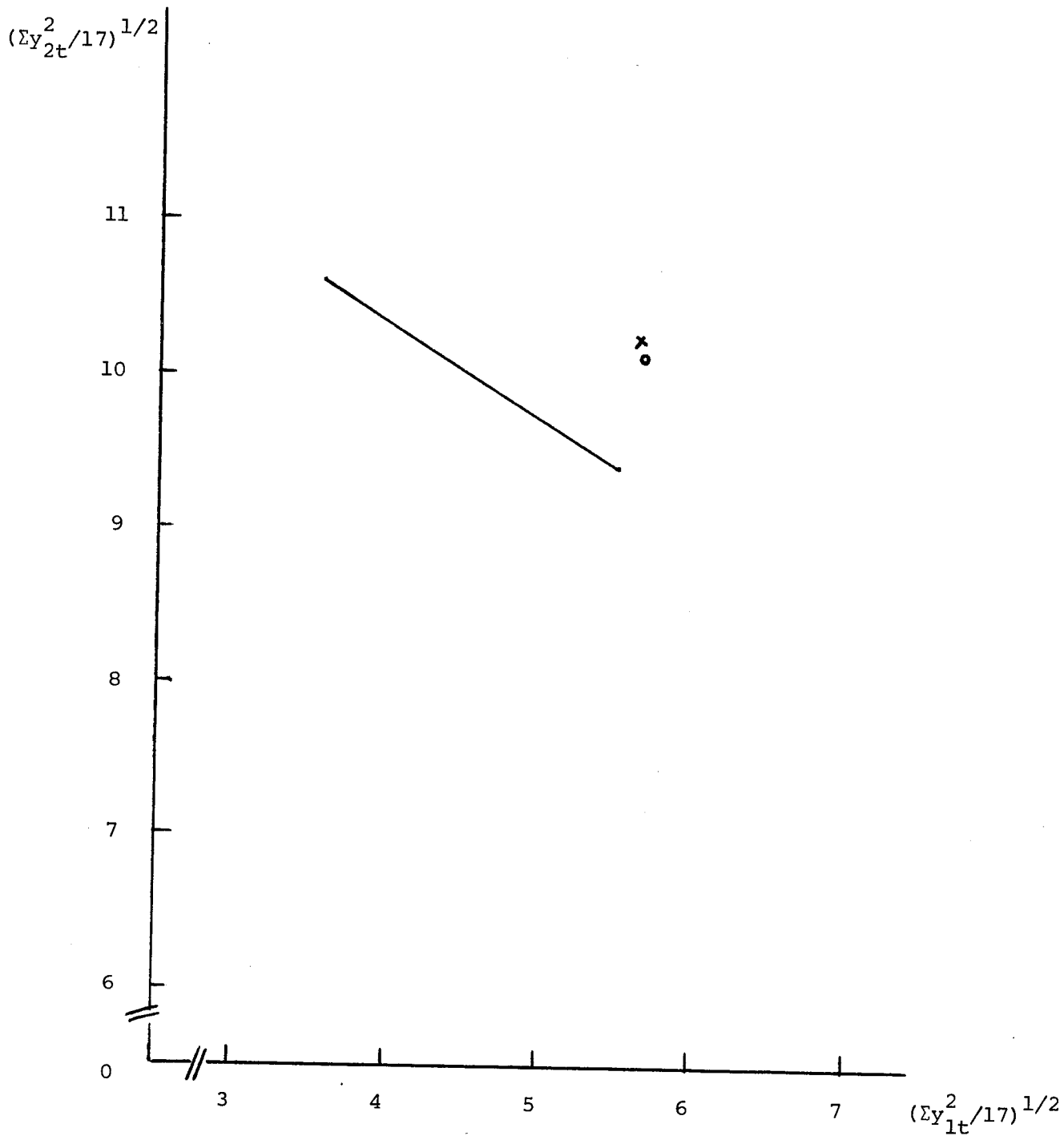


Figure 4



instead of nondefense expenditures.

5. Concluding Remarks

We have proposed a definition of the best unemployment-inflation trade-off, and a method of deriving it numerically from an econometric model. The notion is explained in both a static and a dynamic setting. The St. Louis model and the Michigan Quarterly Econometric Model have been used to derive the proposed trade-off relationships. In both cases, it has been shown that the outcomes of inflation and unemployment resulting from other than optimum values of the policy instruments are dominated by the results obtained by optimization. The examples illustrate clearly the need for optimization in order to ascertain the best possible trade-off. If the optimizing paths of the instruments according to a given econometric model fluctuate too violently for actual implementation, it may be reasonable to define and derive the best trade-off relationships by penalizing and inhibiting the instability in the instruments. The method proposed could also be employed to compare different econometric models in terms of the most favorable trade-off relationships which they imply and of the characteristics of the required time paths of the instruments.

Footnotes

\* We would like to thank Saul Hymans and Harold Shapiro for generously supplying us information concerning the Michigan Quarterly Econometric Model, Keith Carlson for gracious help with the St. Louis Model, Alan S. Blinder for useful comments, and the National Science Foundation for financial support.

<sup>1</sup>In this paper, our main purpose is to propose a definition of the best inflation-unemployment trade-off and a method of deriving the relationship from an econometric model. We are not concerned with the actual shape of the price-Phillips curve, and therefore, would avoid discussion of whether the long-run Phillips curve is nearly vertical.

<sup>2</sup>In Chow (1976), it was found that by manipulating the policy instruments one can reach any desired combinations of the level of employment and the general price index according to the Klein Goldberger model of the U.S. economy.

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