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EXPENDITURE CONFIGURATIONS

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PREFACE

Much economic theory is founded on the supposition of given preferences. Yet, in practice, these preferences are seldom given. The problematic character of how preferences can be empirically known is very conspicuous in the literature. There has been a gap between formal theory that has been an intricate preoccupation of economists for fifty, or perhaps a hundred years, and the possibility of obtaining a corresponding empirical content.

At the other extreme of this theoretical activity, which can only be considered a kind of pure mathematics, there is found the attitude that the absence of elaborate mathematics is the sign of empiricism. So long as observed numerical data are handled, and anything at all plausible is done with them, then that is considered being properly practical. But the process to be applied to the data is not a matter of choice, but has to be implicit in the structure of what is asked from what is given. The mathematical task is properly to formalize, and then to explore this structure, to make the latent logical possibilities manifest.

The data have their meaning only in their conceptual setting, and only this should dictate what is done with them. But for some questions that arise quite naturally in economics, it is not at all obvious what is to be done. Often such questions are abandoned as belonging to an abstract doctrine that has no empirical significance. This has been the fate of various parts of consumer theory. The present effort is to take the classical theory of the consumer through as far as possible to its logical conclusion in a system of empirical analysis. It is of no moment nowadays if the calculations are somewhat complicated, because, once their intention is clarified, they are a task for an automatic computer.

An expenditure configuration defines the basic object for analysis which is involved in empirical methods that are logically inseparable from the hypothesis of preferences in consumer theory. It is a configuration formed by a set of hyperplanes in the n-dimensional commodity space, each corresponding to the expenditure-price constraint given on some occasion, together with a set of points in that space, one on each hyperplane, corresponding to the consumption point given on the same occasion. Each consumption point is to be envisaged as a unique and stable equilibrium under the corresponding expenditure-price constraint in respect to some hypothetical preference scale.

The fundamental problem is to characterize the totality of hypothetical preference scales that are admissible on the data presented by such a configuration.

A normal preference scale determines a unique equilibrium on every expenditure-price constraint. Let an admissible equilibrium be one that is determined by an admissible scale. Then the general problem is to determine all the admissible equilibria on an arbitrary constraint. The characterization of the totality of admissible scales is intended to be one that will enable this problem to be solved. It will be shown in a further memorandum how these admissible equilibria form a finitely generated convex region on the constraint hyperplane, defined by a system of linear inequalities. In an analogous fashion, the cost-of-living measurement can be turned into a classical linear programming problem. Every admissible scale determines a corresponding admissible value for the ratio which defines cost-of-living. The admissible values describe an interval corresponding to the totality of admissible scales. Though there is no finite construction for the totality of scales, this interval can be computed exactly, by finite operations. This provides the most exhaustive solution for the cost-of-living measurement problem, by a combinatorial type of analysis. Also, it gives the proper framework for approaching the traditional index-number type of solution.

This memorandum is concerned with certain algebraical aspects of the problem of suitable characterization. (It is associated with material in Research Memoranda Nos. 7, 11, 13, 18 and 20 of this series.) The well-known Houthakker axiom, as applied finitely to such a directly observable expenditure configuration, rather than to the theoretical concept of an expenditure system, which is a kind of infinite configuration, appears in a new fundamental role. It provides necessary and sufficient conditions for the consistency of certain systems of simultaneous linear inequalities. The coefficients in these inequalities are essentially the traditional index-numbers of the Paasche-Laspeyre type.

The material in this and related memoranda is provisional, and is eventually to be absorbed into a single exposition, The Analysis of Expenditures, dealing with that subject in the framework of the hypothesis of preferences. The historical perspective will be left to this account, together with detailed references and acknowledgements. Planned to follow this memorandum, there are further expositions on theory for the construction of a cost-of-living index, on the practice of construction illustrated by numerical data, on the characterization problem, and on application of this characterization to equilibrium analysis and cost-of-living measurements. The material falls broadly into two parts. One is analysis, depending on limiting operations, being concerned, by means of relation-theory, differential calculus, convex sets, and so forth, with preference scales and expenditure systems, and gives the formal basis for the subject. The other is algebra, being concerned with finite operations on finite data, involving systems of linear and quadratic functions, linear equations and inequalities, finite polyhedra and so forth, made formally intelligible on the background of the analysis.

Underlying the entire subject is the principle formulated by Samuelson, by which preferences are determined from expenditures. This, together with the axiom of Houthakker for the strict consistency of preferences, constitutes the basic method, and supports the theoretical structure for the empirical analysis of consumer behaviour in terms of preferences. The earlier axiom formulated by Samuelson in conjunction with his principle is contained in the Houthakker axiom. But it does have an importance on its own, in that it is preserved under small disturbances applied to an expenditure system; whereas the axiom of Houthakker can be destroyed. Thus, in empirically estimating parameters in a model expenditure system, one can expect to find only the weaker of these conditions, given by the negativity but not the symmetry of the Slutsky matrix. However, by an operation of symmetrization, it is possible to view a Samuelson system as disturbed from an associated Houthakker system. Hence, according to the approach, the one or the other can be the more basic empirical condition.

I have to express my thanks to Professor Oskar Morgenstern, who has in every way aided this work. Also my thanks are due to Mrs. R. M. Crooks, who has done the typing.

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A simple commodity $\mathcal{C}_o$ available on the market has, on any occasion, a certain price π_o; and it can be obtained by purchase on that occasion, in any amount ξ_o, for an expenditure ϵ_o equal to the cost $\pi_o\xi_o$ of that amount at that price. Thus, in the nature of the market, there is the market equation

$$\epsilon_o = \pi_o\xi_o ,$$

the market price existing as the cost of each unit of the commodity, irrespective of the number of units obtained.

Let there now be considered some n commodities $\mathcal{C}_1, \ldots, \mathcal{C}_n$ forming a composite commodity $\mathcal{C} = \{\mathcal{C}_1, \ldots, \mathcal{C}_n\}$, with prices $\pi_1, \ldots, \pi_n$ on any occasion forming the composite price vector $p = \{\pi_1, \ldots, \pi_n\}$. Amounts $\xi_1, \ldots, \xi_n$ of these commodities for a composite amount given by the vector $x = \{\xi_1, \ldots, \xi_n\}$, defining a composition in those commodities. The purchase of composition x at prices p requires an expenditure e equal to the total cost of the separate amounts at the separate prices. Thus:

$$e = \pi_1\xi_1 + \cdots + \pi_n\xi_n = p'x ,$$

where $p'x$ denotes the scalar product of the price and composition vectors p and x. This is the composite form of the market equation.

2. Purchases

The action of a consumer is to make purchases on the market, represented by expenditure on certain commodities, at certain prices, to obtain the commodities in certain amounts.

Let there be considered a particular consumer, irrespective of the nature of that consumer as an individual, or a population of individuals consuming from the same market; and let there be considered the purchases of that consumer on a variety of occasions, say k in number, in respect to a class of some n simple commodities, forming a composite commodity $\mathcal{C}$. On any occasion $r = 1,\ldots,k$ let p_r be the price vector, and x_r be the composition vector, these defining the purchase (p_r, x_r) of the consumer on the occasion. Then the market equation provides

$$e_r = p_r' x_r$$

as the total expenditure on the occasion r.

Any data on the consumer is to be in the form of purchase data, for a variety of commodities on a variety of occasions. The principle adopted is that any assertions to be made about the consumer must be obtained on the basis of an analysis of such data.

3. Expenditure balance

The cost of any composition x at the price p_r of occasion r is $p_r' x$. Hence, the condition that any composition x could have been obtained instead of x_r, on occasion r, for no greater expenditure than that made in x_r, is

$$p_r' x \leq p_r' x_r .$$

It would have been possible on occasion r to exchange x_r for any such composition x, with a saving, or at worst an exact balance of resulting expenditure. All such compositions x under this condition form a region, which may be called the expenditure domain, determined by the price and the expenditure of the occasion. The condition can be written

$$u_r' x \leq 1 ,$$

where

$$u_r = \frac{p_r}{e_r}$$

is the vector of ratios of money prices to total expenditure, defining the balance vector, whose elements define the relative prices for the occasion.

Any composition x is said to be within, on, or over the balance u_r according as $u_r'x \leq , = ,$ or > 1 . Thus the expenditure domain consists of all compositions within the balance. Since there is the relation

$$u_r'x_r = 1 ,$$

which is to define the balance condition, the composition x_r lies on the balance u_r . Any composition x is exchangeable with x_r with a balance of expenditure, or with no greater expenditure, provided it is within, or on the balance u_r .

Thus, from a purchase (p_r, x_r) there is derived the vector pair $[u_r;x_r]$, such that $u_r'x_r = 1$, where $u_r = \frac{p_r}{e_r}$ and $e_r = p_r'x_r$; and this is to define the expenditure figure associated with the purchase, with u_r, x_r for its balance and composition vectors.

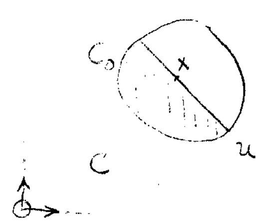
4. Expenditure configurations

Given any expenditure figure $[u;x]$, belonging to purchases made on a certain variety of occasions, the balance and composition vectors are considered restricted to certain balance and composition regions B_o and C_o , which are in the positive orthants B, C of Euclidean vector spaces which are replicas of each other, being of the same dimension n . Thus:

$$u \in B_o , x \in C_o$$

where

$$B_o \subset B , C_o \subset C .$$



The domain and frontier of a balance $u \in B_o$ are defined as the sets of compositions $y \in C_o$ such that $u'y \leq 1$, and $u'y = 1$. Since $u'x = 1$ for an expenditure figure $[u;x]$, the composition in the figure lies on the frontier of its balance.

Geometrically, the frontier of a balance $u \in B_o$ is the intersection with C_o of the hyperplane given by the locus of composition y such that $u'y = 1$. Thus an expenditure figure $E = [u;x]$ appears geometrically as a hyperplane section of C_o, determined with coordinate vector u, together with a point x on it.

Any set of expenditure figures forms an <u>expenditure configuration</u>, with those figures for its elements. Thus a configuration is, geometrically, a set of hyperplanes together with a point on each. Now, for the purchases (p_r,x_r) $(r=1,\ldots,k)$ there are obtained the expenditure figures $E_r = [u_r;x_r]$, forming the configuration $\mathfrak{F} = \{E_r\}$ associated with those purchases. It will be generally assumed that $u_r \neq u_s$ and $x_r \neq x_s$ $(r \neq s)$. The analysis of the purchases is to be an analysis applied to this configuration. The <u>composition</u> and <u>balance sets</u> of the configuration are defined by $X = \{x_r\}$, $U = \{u_r\}$, and with these the configuration may be denoted by $\mathfrak{F} = [U;X]$.

5. Cross-expenditures

Given a set of purchases (p_r,x_r) for occasions $r = 1,\ldots,k$, the expenditure e_{rs} which is required to obtain, at the price p_r of occasion r, the composition x_s found in occasion s, is determined by

$$e_{rs} = p_r'x_s ,$$

and defines the <u>cross-expenditure</u> from occasion r to occasion s. With this definition, the cross-expenditure from one occasion to itself is the same as the expenditure on that occasion:

$$e_{rr} = e_r .$$

If the cross-expenditure from one occasion to the other is compared with the expenditure on the one occasion, there is obtained the <u>relative cross-expenditure</u>

$$f_{rs} = \frac{e_{rs}}{e_r} ,$$

from the one occasion r to the other occasion s. With this definition,

$$f_{rr} = 1 .$$

Accordingly,

$$f_{rs} = \frac{(p_r{}'x_s)}{e_r} = (\frac{p_r}{e_r})'\, x_s$$

$$= u_r{}'x_s .$$

The relative cross-expenditure f_{01} between two occasions $0,1$ is identical with the Laspeyres index with $0,1$ representing the base and current periods, respectively; and, correspondingly, $\frac{1}{f_{10}}$ in the Paasche index.

6. <u>Cross-structure</u>

Let $\mathfrak{F} = \{E_r\}$ be a given expenditure configuration with figures $E_r = [u_r;x_r]$ $(r=1,\ldots,k)$. The <u>cross-deviations</u> between the figures are defined by

$$D_{rs} = u_r{}'x_s - 1 \quad (r,s=1,\ldots,k)$$

and the <u>cross-structure</u> of the configuration is defined by the array

$$D_{\mathfrak{F}} = \{D_{rs}\}$$

formed by the cross-deviations. Since $u_r{}'x_r = 1$, it follows that

$$D_{rr} = 0 .$$

If the figures E_r are derived from purchases (p_r, x_r), then

$$D_{rs} = f_{rs} - 1 ;$$

so a cross-deviation is the deviation of a relative cross-expenditure from unity.

Let $U = \{u_r\}$, $X = \{x_r\}$, and $F = \{f_{rs}\}$, so that

$$F = U'X$$

is a matrix of rank at most $\min\{k,n\}$. Therefore, if $k > n$, every sub-determinant of F of order $n + 1$ must be zero. Thus the cross-coefficients of a k-fold configuration with $k > n$ automatically satisfy a system of algebraical identities.

If $D = D_{\mathcal{F}}$ denotes the cross-structure of a configuration of dimension n, then the configuration $\mathcal{F}$ is said to be a realization of the cross-structure D, in dimension n. While every configuration has a uniquely defined cross-structure, any given cross-structure admits a variety of realizations in every sufficiently large dimension. Since a realization in dimension $n < k$ implies certain identities, such as have been indicated, these identities provide a necessary condition for such a realization. If they are not satisfied, then realizations exist only in dimension $n \geq k$. But if at least one realization exists in dimension m, then a variety of realizations exist in every dimension $m \geq n$. The minimum dimension of realization of a given cross-structure may be taken to define its essential dimension.

7. Preferences

Let $\mathcal{F} = \{E_r\}$ be a given expenditure configuration, with figures $E_r = [u_{ri} x_r]$ $(r=1,\ldots,k)$.

Should one object be chosen for purchase on an occasion and not another which cost no more, and the choice is to be considered regulated by preferences operative on that occasion, then, since it is supposed that no expenditure is ever made without compensation, it must be taken that the one object is preferred to the other.

Accordingly, there is formed the preference relation P_r associated with the figure E_r, with the definition

$$xP_r y \equiv x = x_r \wedge y \neq x_r \wedge u_r' y \leq 1 .$$

That is, x has the relation P_r to y if x is equal to x_r, and y is different from x_r and within the balance u_r. The understanding is that if S is a relation giving a preference system which could be operative on the occasion r to which the figure belongs, then it must contain P_r :

$$P_r \subset S .$$

As applied just to the base set $X = \{x_r\}$ of the configuration, the relations P_r are decided by the cross-deviations between the figures, thus:

$$x_r P_r x_s \iff D_{rs} \leq 0 , \ (r \neq s)$$

it being assumed that $x_r \neq x_s$ $(r \neq s)$.

A binary relation, as a propositional function of ordered couples of elements, is also a set, of all those ordered couples which have the relation, from the first element to the second. Accordingly, there are operations between relations corresponding to all the basic operations between sets, and the same notation can be used for both. Further to these, there is the operation of adjunction, of two relations Q, R to give a further relation QR , with the definition

$$xQRy \equiv \bigvee_z xQz \wedge zRy .$$

That is, x has the relation QR to y if there exists an element z such that x has the relation Q to z, and z has the relation R to y. Adjunction is associative, that is

$$P(QR) = (PQ)R ;$$

so the expression $R_1 \ldots R_m$ can stand, unambiguously, for the adjunction of a sequence of relations; and

$$R^m = \overset{m}{R \ldots R}$$

can stand for the adjunction of a sequence of m relations all identical with R, defining the m^{th} power of R. Obviously,

$$R^{m+n} = R^m R^n .$$

A sequence of elements $x_o, \ldots, x_m$ is said to form an R-chain, if they have the multiple relation defined by

$$R(x_o, \ldots, x_m) \equiv x_o R x_1 \wedge \cdots \wedge x_{m-1} R x_m .$$

A two-element R-chain is called an R-link, and is formed of any two elements which have the relation R :

$$R(x_o, x_1) \Longleftrightarrow x_o R x_1 .$$

Now $xR^m y$ just means that x,y are connected by an R-chain of m links, or m + 1 elements. There can be formed the relation $\vec{R}$ which all elements have which are connected by an R-chain, of whatever length, thus:

$$\vec{R} = \bigcup_{m=1,2,\ldots} R^m$$

defining the chain extension of R. Since $R^m R^n = R^{m+n}$, it follows that

$$x\vec{R}y \wedge y\vec{R}z \Longrightarrow x\vec{R}z$$

showing that $\vec{R}$ is transitive, which condition also has the statement

$$\vec{R}^2 \subset \vec{R} .$$

So if $T = \vec{R}$, it follows that $T^m \subset T$, and then that $\vec{T} \subset T$. This condition, that a relation contain, and therefore be identical with, its chain extension, is equivalent to the condition that it be transitive.

Now an R-cycle is defined as a cycle of elements in which every link is an R-link; and R is said to be acyclic if no R-cycles exist. Also an element x is called R-reflexive if xRx , that is if it has the relation R to itself; and R is called irreflexive if it has no reflexive elements; equivalently,

$$xRy \Rightarrow x \neq y ,$$

which can be written

$$R \subset \neq .$$

Now any element is $\vec{R}$-reflexive if and only if it is an element of an R-cycle. Hence the acyclicity of R is equivalent to the irreflexivity of $\vec{R}$.

A relation which is irreflexive and transitive is called an order. Thus $\vec{R}$, in any case transitive, is an order if and only if R is acyclic.

Now a preference system is given by an order relation. For it is absurd that any object be preferred to itself, so the relation must be irreflexive. Also, if it is required that any best element in any set be the unique conclusion to all processes of elimination, which reject from arbitrary subsets elements which are inferior to any in those subsets, then it must be transitive.

8. Consistency

Any preferences are considered consistent if they could belong to the same system. Thus a pair of opposite preferences (a,b) and (b,a) is inconsistent. For if they were given to belong to a transitive, irreflexive system, say S , so that it is given that aSb and bSa , then the transitivity would imply aSa , which contradicts the irreflexity. More generally, any preferences forming a relation R , are consistent if there exists an order S containing them, that is,

$$R \subset S .$$

In this case, the order S may be said to <u>obtain</u> the consistency of R as a preference relation. Now, taking chain extensions,

$$\vec{R} \subset \vec{S} .$$

But S is transitive, so that

$$\vec{S} \subset S .$$

Therefore

$$\vec{R} \subset S .$$

But S is irreflexive, so that

$$S \subset \neq ;$$

therefore,

$$\vec{R} \subset \neq ,$$

so that $\vec{R}$ is irreflexive, and, since already transitive by construction, therefore an order; equivalently, R is acyclic.

Thus the consistency of R , that is $R \subset S$ for some order S , implies that $\vec{R}$ is an order. But, conversely, if $\vec{R}$ is an order, since

$$R \subset \vec{R} ,$$

it follows that R is consistent. Thus R being consistent is equivalent to $\vec{R}$ being an order, and this is equivalent to R being acyclic.

THEOREM I. <u>For a relation</u> R <u>to be consistent as a preference relation, it is necessary and sufficient that it be acyclic. In this case</u> $\vec{R}$ <u>is an order, containing</u> R <u>, and contained in every preference system</u> S <u>which contains</u> R <u>and thus obtains its consistency.</u>

Now consider the preference P_r associated with the figure E_r of a configuration $\mathfrak{F} = \{E_r\}$. Any one relation P_r must be consistent. For, from the form of P_r, in which all the links have the same leading element x_r, no P_r-chain of more than one link can be formed, and therefore no P_r-cycles, since a cycle requires at least two links. Thus, for the preferences obtained from a single figure, the consistency condition is vacuous, since automatic. However, when all the preferences from the different figures of are taken together, to form a relation

$$Q_{\mathfrak{F}} = \bigcup_{r=1}^{n} P_r,$$

then inconsistency is possible. The relation $Q_{\mathfrak{F}}$ defines the <u>base preference</u> of the configuration $\mathfrak{F}$, and then

$$P_{\mathfrak{F}} \equiv \overrightarrow{Q}_{\mathfrak{F}}$$

defines the <u>preferences of the configuration</u>. Preferences which are not base preferences are called <u>derived preferences</u>; so that all the preferences are obtained from the base preferences, together with the derived preferences, determined between the extremities of the base preference chains.

Now $\mathfrak{F}$ is defined to be a <u>consistent configuration</u> if its base preferences $Q_{\mathfrak{F}}$ are consistent, that is acyclic; or equivalently, the preference relation $P_{\mathfrak{F}}$ is an order.

Since the superior element in any base preference is a base point, any base preference cycle belongs to a cycle of base points, indexed by a cycle of elements $r, s, t, \ldots, q, r$ taken from $1, \ldots, n$, such that

$$x_r Q_{\mathfrak{F}} x_s,\ x_s Q_{\mathfrak{F}} x_t,\ \ldots,\ x_q Q_{\mathfrak{F}} x_r.$$

But

$$x_r Q_{\mathfrak{F}} x_s \Longleftrightarrow x_r P_r x_s \Longleftrightarrow D_{rs} \leq 0.$$

So this is equivalent to the condition

$$D_{rs} \leq 0 \; , \; D_{st} \leq 0 \; , \ldots , \; D_{pr} \leq 0 \; ,$$

defining the occurrence of a <u>cross-deviation cycle</u>. Acyclicity is now equivalent to the impossibility of a cross-deviation cycle; and this is the <u>Houthakker condition</u>

$$D_{rs} \leq 0 \; , \; D_{st} \leq 0 \; , \ldots , \; D_{pq} \leq 0 \Rightarrow D_{qr} > 0 \; ,$$

usually stated

$$p_r{}'x_s \leq p_r{}'x_r , \; p_s{}'x_t \leq p_s{}'x_s \; , \ldots , \; p_p{}'x_q \leq p_p{}'x_p \implies p_q{}'x_r > p_q{}'x_q$$

in terms of purchase data.

THEOREM II. <u>A necessary and sufficient condition that an expenditure configuration</u> $\mathcal{F}$ <u>with cross-structure</u> $D_{\mathcal{F}} = \{D_{rs}\}$ <u>be consistent is that</u>

$$D_{rs} \leq 0 \; , \; D_{st} \leq 0 \; , \ldots , \; D_{pq} \leq 0 \Rightarrow D_{qr} > 0 \; .$$

When this condition is satisfied, the order relation $P_{\mathcal{F}}$ on the base set $X = \{x_r\}$ of the configuration is determined from the cross-structure $D_{\mathcal{F}}$, with the property

$$D_{rs} \leq 0 \; , \; D_{st} \leq 0 \; , \ldots , \; D_{pq} \leq 0 \Rightarrow x_r P_{\mathcal{F}} x_q \; .$$

It can be defined as the minimal transitive relation with this property.

9. <u>Preference hypothesis</u>

Any order S represents an admissible <u>preference hypothesis</u> applied to the configuration $\mathcal{F}$ if it contains all the base preferences of $\mathcal{F}$, that is

$$P_r \subset S \; ,$$

which is equivalent to

$$P_{\mathcal{F}} \subset S$$

where

$$Q_{\mathcal{F}} = \bigcup_{r=1}^{k} P_r \quad \text{and} \quad P_{\mathcal{F}} = \vec{Q}_{\mathcal{F}} ;$$

and it requires $P_{\mathcal{F}}$ to be an order. Thus if any preference hypothesis is admitted, then $P_{\mathcal{F}}$ represents one such hypothesis since $P_r \subset P_{\mathcal{F}}$; and any other one, say S , is limited to the extent of being a refinement of $P_{\mathcal{F}}$. Thus the consistency of $\mathcal{F}$ is the condition for the admissibility of some preference hypothesis; and then $P_{\mathcal{F}}$ represents a minimal hypothesis, which must be contained in any other.

In principle, the preference hypothesis may be applied to each of the figures of the configuration, separately. For these on their own, there can be no contradiction. For it is always possible to assume distinct orders S_r , operative on each of the different occasions $r = 1,\ldots,k$, such that

$$P_r \subset S_r .$$

The occasions being distinguished as belonging to different times, or different consumers, so also are the hypothetical preference systems operative on those occasions are distinguished. Hence there is no surprise when it turns out impossible to find a single enveloping preference system, which contains all the preferences P_r found in the different occasions. Thus it may turn out impossible to set

$$S_r = S ,$$

where S is to be a preference system operating throughout the occasions. The method then is to admit that the preference system assumed to operate on any occasion, fictitious though it is, must change from occasion to occasion, in such a way that the preferences actually manifested on those occasions cannot be fitted into a single system.

This is the general method for the introduction of preferences into the analysis. It can always be done; but it may be useless. Its object is to search out permanencies underlying a variety. The elements of permanence are to be preferences; and their stability throughout occasions is shown, or at least saved from denial, in the possibility of fitting those which are manifested into a single system, which is to express the desired stability connecting the occasions. The preferences taken separately amount to nothing: it is their joint consistency that has to be decided, before the method has validity. For to have preferences, but different ones on every different occasion, is an empty hypothesis. The hypothesis with content is that preferences regulate choices, and that these preferences have a permanence. Thus, to make choices is not necessarily to have preferences, except in a completely vacuous sense. Preferences are automatically observed whenever a choice is made. But this need not mean anything is revealed. They might be considered revealed when permanence is granted; and the means available to granting this is in their consistency. The essence of preferences is that they should have stability, so that they can be considered perpetuated once observed. Or if they are to change, then it should be with explanation in terms of causal forces for that change. But time alone is always a sufficient cause.

What is here described is the narrowest method, in which stability, or permanence, is interpreted as absolute rigidity. This model must lie at the centre of any method based on preferences. But obviously what is also wanted eventually is a statistical form of analysis, which is not barred by inconsistencies; but which can be universally applied, with a smaller or greater measure of significance; and which can present inconsistencies as disturbances from an underlying consistency. However, the exclusive concern here is with application of the rigid hypothesis of preferences, though this, once completed, gives a basic framework that may be made the starting-point for a more flexible form of analysis.

10. Normal expenditure systems

A complete expenditure configuration $\mathcal{E}$ on regions B_o, C_o in the balance and composition spaces B, C is a configuration whose balance and composition sets are identical with these regions. It is an expenditure system if, for $[u;x]$, $[v;y] \in \mathcal{E}$,

$$x \neq y \Rightarrow u \neq v ,$$

which is defined to be responsive, if, further

$$u \neq v \Rightarrow x \neq y ;$$

and to be uniformly responsive, with response coefficient $\rho > 0$, if

$$|x-y| > \rho |u-v| .$$

An expenditure system $\mathcal{E}$ is thus a kind of infinite expenditure configuration, represented by a single-valued function $x = x(u)$, subject to $u'x = 1$, mapping a region in the balance space into the composition space, subject to the balance condition, thus:

$$\mathcal{E} : B_o \rightarrow C_o \ (u \rightarrow x;\ u'x = 1).$$

The responsivity condition is that there is always a finite change in composition in response to any finite change in balance; and is the same as invertibility. It requires the mapping to be one-to-one, and determines balance inversely as a function $u(x)$ of composition. Uniform responsivity is a stronger form of the responsivity condition, assuring that the response to a movement of balance through a certain distance will be by a movement of composition through a distance at least a certain fixed multiple of that distance. It implies the inverse continuity of the system, that balance is a continuous function $u(x)$ of composition. Moreover, if a system is invertible and differentiable, in a closed region, then it is uniformly responsive in that region. Thus uniform responsivity is a condition intermediate between the conditions obtained by taking invertibility together with continuity and with differentiability.

Given an expenditure system $\mathcal{E}$ on (B_o, C_o) any configuration $\mathcal{F} = [U;X]$ such that $U \subset B_o$, $X \in C_o$ is said to belong to $\mathcal{E}$ if $\mathcal{F} \subset \mathcal{E}$, that is every figure of $\mathcal{F}$ is a figure of $\mathcal{E}$; and in this case the system $\mathcal{E}$ is said to be a completion of the configuration $\mathcal{F}$. Thus a system is a completion of every configuration which belongs to it.

The preference relation $P_{\mathcal{E}}$ of an expenditure system $\mathcal{E}$ is defined by $P_{\mathcal{E}} = \vec{Q}_{\mathcal{E}}$, where

$$xQ_{\mathcal{E}}y \equiv x \nmid y \wedge u'y \leq 1 .$$

Accordingly, if $\mathcal{F} \subset \mathcal{E}$, then $P_{\mathcal{F}} \subset P_{\mathcal{E}}$; and the preference relation of an expenditure system is the smallest transitive relation with this property, that it contains the preferences of all its configurations. The consistency of $\mathcal{E}$ is defined by the consistency of $Q_{\mathcal{E}}$, which is that $P_{\mathcal{E}}$ be an order. Since $P_{\mathcal{E}}$ is by construction transitive, this just requires that it be irreflexive.

A normal expenditure system is defined to be one which is uniformly responsive and consistent. Any normal expenditure system has the property that there exists a differentiable function φ, called a gauge of the system, such that

$$\varphi(x) > \varphi(y) \iff xP_{\xi}y \ .$$

The gauge can always be chosen to be convex. Any function which is a gauge for a normal expenditure system will be called a <u>normal preference function</u>. It is an increasing function with convex levels.

A relation R which is antisymmetric, and whose negation $\overline{R}$ is transitive is called a <u>scale</u>. Since it follows from these properties that it must itself be transitive, and also irreflexive, since any antisymmetric relation is irreflexive, it is also an order. It has the further property that its complete negation $\tilde{R} = \overline{R} \cap \overline{R'}$, where R' stands for the converse of R , is an equivalence, being reflexive, symmetric and transitive. The classes determined by $\tilde{R}$ are to be called the <u>equivalence classes</u> in the scale R . A scale is finally characterized by the property that it reduces to a complete order of its equivalence classes. Thus, if ρ_x stands for the equivalence class with representative x , then

$$xRy \iff \rho_x \mathcal{R} \rho_y$$

where $\mathcal{R}$ is a complete order, that is, an order whose complete negation is identity:

$$\rho \tilde{\mathcal{R}} \sigma \iff \rho = \sigma \ .$$

Any relation R with the property that it is <u>measured</u> by some numerical function φ , that is, such that

$$xRy \iff \varphi(x) > \varphi(y) \ ,$$

must necessarily be a scale.

If ξ is any expenditure system, to suppose P_{ξ} is an order is not necessarily to have it as a scale. Thus, the <u>indifference relation</u> $\tilde{P}_{\xi}$ is not then necessarily transitive. However, if ξ is uniformly responsive, then this further condition requires that P_{ξ}

cannot be an order without also being a scale. In this case the indifference relation is an equivalence; and <u>indifference classes</u> may then be defined as equivalence classes in the preference scale.

Thus the preference relation of a normal expenditure system is not merely an order, but a scale. Any relation which is the preference relation of a normal expenditure system may now be called a <u>normal preference scale</u>. It is measured by a <u>normal preference function</u>.

If φ is a normal preference function with gradient $g = \varphi_x$, the associated normal expenditure system is determined from the <u>equilibrium condition</u>

$$g = u\lambda ,$$

where the multiplier λ has the value

$$\lambda = x'g ,$$

since $u'x = 1$.

A <u>normal expenditure configuration</u> is defined to be one for which there exists a normal completion. Thus a configuration $\mathcal{F}$ is normal if there exists a normal expenditure system $\mathcal{E}$ containing it.

11. Levels and multipliers

A set of number pairs (φ_r, λ_r) define <u>normal levels</u> and <u>multipliers</u> for a configuration $\mathcal{F} = \{u_r; x_r\}$ if there exists a normal preference function φ, which is a gauge for some normal completion $\mathcal{E}$ of $\mathcal{F}$ and is such that

$$\varphi(x_r) = \varphi_r , \quad \lambda(x_r) = \lambda_r .$$

Then if g is the gradient of φ, and

$$g(x_r) = g_r ,$$

the equilibrium condition provides that

$$g_r = u_r \lambda_r .$$

Now the convexity of φ requires that

$$(x_r - x_s)' g_s > \varphi_r - \varphi_s .$$

Therefore,

$$(x_r - x_s)' u_s \lambda_s > \varphi_r - \varphi_s ,$$

or equivalently

$$\lambda_s D_{sr} > \varphi_r - \varphi_s .$$

Moreover, since φ must be an increasing function, it is required that $\lambda_r > 0$.

Let any solution (φ_r, λ_r) of the system of normal inequalities

$$\lambda_r > 0 , \ \lambda_s D_{sr} > \varphi_r - \varphi_s$$

define coherent levels and multipliers of the configuration $\mathfrak{F}$ with cross-structure $D_{\mathfrak{F}} = \{D_{rs}\}$; and let $\mathfrak{F}$ be called a coherent configuration if this system of inequalities is consistent. Any normal expenditure system having a convex gauge, the following has been proved:

THEOREM. A configuration is coherent if it is normal; and any set of normal levels and multipliers is also a set of coherent levels and multipliers.

The converse is also true, but will be proved elsewhere.

12. Coherence and concordance

Any solution of the system of inequalities $\lambda_r > 0$, $\lambda_r D_{rs} + \lambda_s D_{st} + \cdots + \lambda_q D_{qr} > 0$ when $r, s, t, \ldots, p$ are distinct elements from $1, \ldots, k$ defines a set of concordant multipliers for the configuration $\mathfrak{F}$ with cross-structure $D_{\mathfrak{F}} = \{D_{rs}\}$; and $\mathfrak{F}$ is defined to be a concordant configuration if these inequalities are consistent.

THEOREM. A configuration is coherent if and only if it is concordant; and any multipliers form a coherent set for the configuration if and only if they form a concordant set.

If $\Lambda = \{\lambda_r\}$ is a coherent set of multipliers, let $\Phi = \{\varphi_r\}$ be an associated set of coherent levels, so that $\lambda_r > 0$, and for any distinct elements $r,s,t,\ldots,q$

$$\lambda_r D_{rs} > \varphi_s - \varphi_r$$

$$\lambda_s D_{st} > \varphi_t - \varphi_s$$

$$\lambda_q D_{qr} > \varphi_q - \varphi_r ,$$

so that, by addition,

$$\lambda_r D_{rs} + \lambda_s D_{st} + \ldots + \lambda_q D_{qr} > 0 ,$$

showing the multiplier set Λ to be concordant. Conversely, if the concordance condition holds for $\Lambda = \{\lambda_r\}$ then the consistency theorem for a system of inequalities of the form $a_{rs} > x_r - x_s$ (Res. Mem. No. 18) shows the existence of a level set $\Phi = \{\varphi_r\}$ such that $\lambda_s D_{sr} > \varphi_r - \varphi_s$; whence Λ is coherent. It follows now that the coherence and concordance conditions for the configuration are equivalent; and also that in order to construct the coherent sets of levels and multipliers it is possible first to find the coherent multiplier sets separately, these being identical with the concordant sets, and then, with any such multiplier set, to determine all the level sets which can be formed with it under the condition of coherence.

13. Concordance and consistency

THEOREM I. An expenditure configuration is concordant if and only if it is consistent.

Firstly, it is obvious that concordance implies consistency. For if the configuration is concordant, there exists a multiplier set $\Lambda = \{\lambda_r\}$ such that

$$\lambda_r > 0,\ \lambda_r D_{rs} + \lambda_s D_{st} + \dots + \lambda_p D_{pr} > 0$$

for every distinct $r, s, t, \dots, p$ and this implies the impossibility of

$$D_{rs} \leq 0,\ D_{st} \leq 0, \dots, D_{pr} \leq 0 ,$$

which is equivalent to consistency.

Now, conversely, suppose the configuration is consistent, so that $P_{\mathfrak{F}}$ is an order; and take any complete order refinement R of $P_{\mathfrak{F}}$. Then R is a complete order with the property that

$$D_{rs} \leq 0 \Rightarrow x_r R x_s \quad (r \neq s) .$$

Without loss in generality it can be supposed that the elements of the base set $X = \{x_1, \dots, x_k\}$ of the configuration are already given in the order R so that

$$x_r R x_s \Leftrightarrow r < s ,$$

and hence

$$D_{rs} \leq 0 \Rightarrow r \leq s ,$$

or equivalently,

$$r > s \Rightarrow D_{rs} > 0 .$$

The proof will now be completed by induction, by showing that the concordance of the sub-configuration $\mathfrak{F}_{m-1} = \{E_r\}_{r<m}$ of $\mathfrak{F}$ implies that of $\mathfrak{F}_m$. Thus, it is assumed there have been found multipliers $\{\lambda_r\}_{r<m}$ such that

$$\lambda_r D_{rs} + \lambda_s D_{st} + \cdots + \lambda_q D_{qr} > 0 \qquad (r,\ldots,q < m) .$$

Now

$$D_{mr} > 0 \quad (r < m) .$$

Therefore, it is possible to define

$$\mu_m = - \min_{r,\ldots,q<\omega} \frac{\lambda_r D_{rs} + \lambda_s D_{st} + \cdots + \lambda_q D_{qm}}{D_{mr}} .$$

Then $\{\lambda_1,\ldots,\lambda_m\}$ is a concordant multiplier set for every $\lambda_m > \mu_m$. In particular, it is possible to choose $\lambda_m > 0$. Hence the configuration $\mathfrak{F}_m$ is concordant. The case $m = 2$ is obvious. Therefore, by induction, $\mathfrak{F}_k = \mathfrak{F}$ is concordant.

It appears from this proof that it is possible, moreover, to find a multiplier set such that

$$\lambda_1 < \lambda_2 < \ldots\ldots$$

Accordingly:

THEOREM II. Corresponding to every complete order which refines the order of descending preference of a consistent expenditure configuration there exists concordant sets of multipliers with that complete order for their order of ascending magnitude; and any initial sequence of such multipliers in this order determines a lower bound for the next multiplier, which can be chosen arbitrarily large.

14. Consistent and concordant scales

If an expenditure configuration $\mathfrak{F}$ is consistent, its preference relation $P_{\mathfrak{F}}$ is an order, and any scale applied to the configuration elements, which is a refinement of this order, will be called a consistent scale. Since any order can be refined to a complete order, which is the same as a complete scale, it follows that consistent scales exist for every consistent configuration, even if, in the case of $P_{\mathfrak{F}}$ being a complete order, the only instance may be $P_{\mathfrak{F}}$ itself. But when $P_{\mathfrak{F}}$ is a partial order, it will have a variety of scale refinements.

Now with any set $\Phi = \{\varphi_r\}$ of concordant levels, there is determined a scale S, with the definition

$$x_r S x_s \equiv \varphi_r > \varphi_s .$$

Any such scale, belonging to a set of concordant levels, will be called a <u>concordant scale</u> of the configuration.

It may now be asked if there is any distinction between the thus defined consistent scale and concordant scales. That there is none is seen as follows:

THEOREM. <u>For any expenditure configuration, consistency is equivalent to concordance; and the consistent scales are equivalent to the concordant scales.</u>

It has been shown that consistency is equivalent to coherence and coherence to concordance, so the first part is proved.

Now assume $\mathfrak{F}$ to be a consistent, and therefore also a concordant configuration. Then, as part of this condition,

$$D_{rs} \leq 0 \Rightarrow D_{sr} > 0 .$$

Let $\{\varphi_r, \lambda_r\}$ be concordant levels and multipliers, so that

$$\lambda_r > 0 , \; \lambda_s D_{sr} > \varphi_r - \varphi_s > -\lambda_r D_{rs} .$$

If S is any consistent scale, then

$$D_{rs} \leq 0 \Rightarrow \varphi_r - \varphi_s \geq 0 .$$

Hence any concordant scale is a consistent scale. It now has to be shown that any consistent scale is a concordant scale.

Without loss in generality, it can be assumed that the figures of the configuration are already given in a complete order which refines any desired consistent scale. Then it is required to find concordant levels such that

$$\varphi_r > \varphi_s \iff r < s ,$$

it being given that

$$D_{rs} \leq 0 \Rightarrow r < s .$$

Suppose the solution has been found for the sub-configuration $\mathcal{F}_{m-1}$ formed out of the first $m-1$ figures. Thus:

$$\lambda_s D_{sr} > \varphi_r - \varphi_s > -\lambda_r D_{rs} \quad (r < s < m) .$$

Then it is required to find λ_m, φ_m such that

$$\lambda_m D_{mr} > \varphi_r - \varphi_m > -\lambda_r D_{rm} \quad (r < m) ,$$

$$\varphi_r - \varphi_m > 0 \quad (r < m)$$

where

$$D_{mr} > 0 \quad (r < m)$$

Let

$$A_{mr} = \min_{s,t,\ldots,q} \lambda_m D_{ms} + \lambda_s D_{st} + \ldots + \lambda_q D_{qr}$$

and

$$A_{rm} \quad \min_{s,t,\ldots,q} \lambda_r D_{rs} + \lambda_s D_{st} + \ldots + \lambda_q D_{qm}$$

for $r,s,t,\ldots,q < m$. Then the conditions for a solution are

$$A_{mr} > \varphi_r - \varphi_m > -A_{rm} ,$$

or equivalently,

$$\varphi_r + A_{rm} > \varphi_m > \varphi_r - A_{mr} ,$$

together with

$$\varphi_r > \varphi_m .$$

Now, since $D_{ms} > 0$ $(s < m)$ it is possible to have A_{mr} arbitrarily large, by taking λ_m large. Moreover, A_{rm} is independent of λ_m . Therefore, with any φ_m such that

$$\varphi_m < \min_{r<m} \{\varphi_r + A_{rm}\} \quad \text{and} \quad \varphi_m < \min_{r<m} \varphi_r ,$$

it is possible to choose λ_m, sufficiently large, such that

$$\varphi_m > \max_{r<m} \{\varphi_r - A_{mr}\} ;$$

then λ_m, φ_m have been found, as required. The solution can then be carried a step further, extending the one supposed already found for $\mathfrak{F}_{m-1}$, thus to obtain a solution for $\mathfrak{F}_m$. By induction, the process can be continued to $\mathfrak{F}_k = \mathfrak{F}$. Since the two-figure case is obvious, the proof by induction is complete.

PREVIOUS RESEARCH MEMORANDA

"Beneficiation of United States Manganese Reserves." Research Memorandum No. 1 (January 1957).

"Programming the Supply of Strategic Materials." Research Memorandum No. 2 (September 1957).

"Econometric Analysis of the United States Manganese Problem." Final Report, Part I, Research Memorandum No. 3 (October 1958).

"On the Magnification of Derived Demand." R. G. Davis and W. G. Mellon. Research Memorandum No. 4 (November 1958).

"A General Theory of Measurement-Applications to Utility." J. Pfanzagl. Research Memorandum No. 5 (December 1958).

"Estimates of National and Regional Steel Production in the United States for 1970." R. E. Kuenne. Research Memorandum No. 6 (February 1959).

"Value and Expenditure." S. N. Afriat. Research Memorandum No. 7 (February 1959).

"Structures in Logistics Operations." H. D. Mills. Research Memorandum No. 8 (February 1959).

"Stochastic Properties of Elementary Logistic Components." H. D. Mills. Research Memorandum No. 9 (February 1959).

"Uncertainty and the Expansion of Bank Credit." Daniel Orr and Walter G. Mellon. Research Memorandum No. 10 (November 1959).

"The Concepts of Equivalence, Order and Scale in the Analysis of Preferences." S. N. Afriat. Research Memorandum No. 11 (January 1960).

"First Report of the Time Series Project." Clive W. J. Granger. Research Memorandum No. 12 (February 1960).

"The Axiomatic Concept of a Scale and the Logic of Preferences in the Theory of Consumers' Expenditures." S. N. Afriat. Research Memorandum No. 13 (March 1960).

"Econometric Analysis of the United States Manganese Problem." Final Report, Part II, Research Memorandum No. 14 (March 1960).

"Production Stability and Inventory Variation." Daniel Orr. Research Memorandum No. 15 (May 1960).

"Statistical Projection." S. N. Afriat. Research Memorandum No. 16 (June 1960).

"Preference Structures and the Measurement of Military Worth." Richard G. Davis. Research Memorandum No. 17 (September 1960).

"The System of Inequalities $a_{rs} > X_r - X_s$." S. N. Afriat. Research Memorandum No. 18 (October 1960).

"A New Look at Economic Time Series Analysis." Oskar Morgenstern. Research Memorandum No. 19 (January 1961).

"Gradient Configurations and Quadratic Functions." S. N. Afriat. Research Memorandum No. 20 (January 1961).