MIXTURE SOCIAL WELFARE FUNCTIONS

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ABSTRACT

We show that every binary and Paretian method for passing from preference profiles to lotteries over preferences is associated with a subadditive function on the set of coalitions of individuals. This function gives the power of each coalition to secure its preference for any x over any y.

*The idea of studying mixture social welfare functions grew out of our familiarity with the work of A. Gibbard [2]. In fact, our initial attempts involved trying to obtain a result of the type communicated here as a corollary to his analysis.

INTRODUCTION

A generalization of Arrow's Theorem on the possibility of social welfare functions [1] is established. Arrow considered methods for passing from social preference profiles to social preference relations. Here we study methods for passing from social preference profiles to lotteries over social preference relations. This enlarges the set of objects which are permitted to serve as outcomes of preference aggregation, while at the same time holding fixed the domain over which individuals have preferences.

Consider the following method for resolving conflict. Individuals one and two have different preferences. They agree to flip a coin; if heads comes up one's preferences are followed, and if tails comes up, two's preferences are followed. There is no natural Arrowian dictator. The procedure can be forced into the framework considered by Arrow by requiring individuals to have preferences over lotteries on social preferences. Then, if the social welfare function is binary and Paretian, there will be an individual who dictates the social lottery--at least among those lotteries for which a free triple condition is satisfied. But such an analysis does not capture the intuitively neutral quality of the above procedure.

We adopt the view that admitting as social preferences lotteries on preferences over the basic alternatives increases the possibility for satisfactory preference aggregation. Nevertheless, we will show that the dimensions of this increase are severely limited, and that the force of Arrow's Theorem, which obtains as a corollary to the present analysis, is not diminished. In fact, we view the result as expanding the Arrow conclusion to cover a somewhat more general setting. Loosely speaking, we show that to every binary and Paretian method for passing from preference profiles to lotteries over preferences (called a mixture social welfare function), there can be associated a subadditive function on the set of coalitions of individuals which gives the power of each coalition to secure its preference for any x over any other y. In other words, allowing lotteries as outcomes, while it permits a wide variety of nondictatorial procedures, cannot involve "using a coin to decide on whose preferences to follow when some standard procedure, such as majority voting, does not work." Rather, it must be associated with dividing up the dictatorial power in a subadditive manner.

The introduction is followed by four sections and a postscript. In section one we take up the case of strict (asymmetric) preferences; indifference is not admitted in either the expression of individual preference or in the (mixed) social preference. It is shown that to every binary and Paretian mixture social welfare function, there can be associated a subadditive distribution of power among coalitions of individuals which gives the power of each coalition to secure its preference for any x over any other y. The second section extends the analysis to the classical case of preferences which are represented by weak orderings. In the third section, Arrow's General Possibility Theorem is obtained as a corollary to the preceding analysis. The fourth section is concerned with the method of proof. Although the result we present represents a substantial strengthening of Arrow's Theorem, the astute reader will quickly realize that our proof follows closely the standard proof of that theorem (see; e.g., [1], pp. 97-100). Briefly, the statement "the group E can insure x over y" is replaced by the statement "the group E can insure x over y with probability p"; with this adjustment the steps of the standard proof are followed in turn. This parallel is developed more completely in the fourth section. Finally, it is not hard to show that there exist binary Paretian mixture social welfare functions that give rise to distributions of power which are sub-additive but not additive. Gerard Butters recently showed that the converse of the theorem is true when the set of alternatives is less than or equal to 5. Building on the result presented here, Andrew McLennan demonstrated that, for the case of six or more alternatives, every binary and Paretian mixture social welfare function gives rise to an additive distribution of power, and also that the converse statement obtains.

I. THE THEOREM FOR STRICT PREFERENCES - NOTATION AND DEFINITIONS

Let $A = \{x,y,z,\ldots\}$ be the set of <u>alternatives</u> and $N = \{1,2,\ldots,n\}$ be the set of <u>individuals</u>. We assume throughout that A and N are finite and that the number of elements in A (denoted $\#A$), is at least three. Let $\mathcal{B} = \{B, B', \bar{B}, \ldots\}$ be the set of <u>individual strict preference relations</u> (complete, asymmetric, and transitive binary relations) on A, and $\mathcal{B}^n$, the n-fold cartesian product of $\mathcal{B}$, be the set of social preference <u>profiles</u>. A generic profile is denoted by $\underset{\sim}{B} = (B_1, B_2, \ldots, B_n)$.

A lottery on $\mathcal{B}$ is a probability measure on $\mathcal{B}$; i.e., a function $\ell : 2^{\mathcal{B}} \to [0,1]$ such that $\ell(\phi) = 0$, $\ell(\mathcal{B}) = 1$, and $\ell(S \cup T) = \ell(S) + \ell(T) - \ell(S \cap T)$ for all $S, T \subset \mathcal{B}$. Let $L(\mathcal{B})$ be the set of lotteries on $\mathcal{B}$. For $x, y \in A$, let $\mathcal{B}(xBy) = \{B \in \mathcal{B} : xBy\}$. For $\ell \in L(\mathcal{B})$, let $p(\ell, xBy) = \ell(\mathcal{B}(xBy))$. Clearly, for all distinct x, y, $p(\ell, xBy) + p(\ell, yBx) = 1$. Given $\underset{\sim}{B} \in \mathcal{B}^n$, let $\varphi(\underset{\sim}{B}, xBy) = \{i \in N : xB_i y\}$.

A <u>Strict Mixture Social Welfare Function</u> (SMSWF) is a function $g: \mathcal{B}^n \rightarrow L(\mathcal{B})$. A SMSWF g is <u>binary</u> if for any $x,y \in A$, and any two profiles $\underset{\sim}{B}, \underset{\sim}{B}' \in \mathcal{B}^n$,

$$[\varphi(\underset{\sim}{B}, xBy) = \varphi(\underset{\sim}{B}', xBy)] \text{ implies } [p(g(\underset{\sim}{B}), xBy) = p(g(\underset{\sim}{B}'), xBy)] .$$

A SMSWF is <u>Paretian</u> if for any $x,y \in A$, and any $\underset{\sim}{B} \in \mathcal{B}^n$,

$$[\varphi(\underset{\sim}{B}, xBy) = N] \text{ implies } [p(g(\underset{\sim}{B}), xBy) = 1] .$$

Finally, the real-valued function f defined on subsets of X is subadditive if $f(S \cup T) \leqq f(S) + f(T)$ for all $S,T \subset X$.

<u>THEOREM 1</u>

Given any binary and Paretian SMSWF g, there exists a subadditive function $\mu_g : 2^N \rightarrow R$, such that

(1) $\mu_g(\varphi(\underset{\sim}{B}, xBy)) = p(g(\underset{\sim}{B}), xBy)$,

for all $\underset{\sim}{B} \in \mathcal{B}^n$ and $x,y \in A$; furthermore,

(2) $\mu_g(\emptyset) = 0$,

(3) $\mu_g(C) + \mu_g(N-C) = 1$, for all $C \subset N$, and

(4) $\mu_g(C) \geqq \mu_g(C')$, whenever $C' \subset C$.

The proof of the theorem follows two lemmas.

<u>Lemma 1</u> (Neutrality)

Let g be a binary and Paretian SMSWF. Then for any $\underset{\sim}{B}, \underset{\sim}{B}' \in \mathcal{B}^n$, and $x,y,z,w \in A$,

(5) $[\varphi(\underset{\sim}{B}, xBy) = \varphi(\underset{\sim}{B}', zBw)]$ implies $[p(g(\underset{\sim}{B}), xBy) = p(g(\underset{\sim}{B}'), zBw)]$.

<u>Proof of Lemma 1</u>

If $\varphi(\underset{\sim}{B},xBy) = N$, then $p(g(\underset{\sim}{B}),xBy) = 1 = p(g(\underset{\sim}{B}'),zBw)$ since g is Paretian. A similar argument applies if $\varphi(\underset{\sim}{B},xBy) = \emptyset$. Assume now that $\varphi(\underset{\sim}{B},xBy)$ is a proper subset of N. We show first that the Lemma holds when $x = z$. The proof when $y = w$ is identical.

Let $\hat{\underset{\sim}{B}}$ be a profile such that

$$x\hat{B}_j y\hat{B}_j w \quad \text{for } j \in \varphi(\underset{\sim}{B},xBy), \text{ and}$$

$$y\hat{B}_j w\hat{B}_j x \quad \text{for } j \in \varphi(\underset{\sim}{B},yBx) .$$

Since g is Paretian, $p(g(\hat{\underset{\sim}{B}},yBw) = 1$. Thus,

$$(6) \quad p(g(\hat{\underset{\sim}{B}}),xBw) \geqq p(g(\hat{\underset{\sim}{B}}),xBy) .$$

Consider now a profile $\bar{\underset{\sim}{B}}$ such that

$$x\bar{B}_j w\bar{B}_j y \quad \text{for } j \in \varphi(\underset{\sim}{B},xBy), \text{ and}$$

$$w\bar{B}_j y\bar{B}_j x \quad \text{for } j \in \varphi(\underset{\sim}{B},yBx) .$$

Since g is Paretian $p(g(\bar{\underset{\sim}{B}}),wBy) = 1$; thus, $p(g(\bar{\underset{\sim}{B}}),wBx) \geqq p(g(\bar{\underset{\sim}{B}}),yBx)$. Therefore,

$$(7) \quad p(g(\bar{\underset{\sim}{B}}),xBy) \geqq p(g(\bar{\underset{\sim}{B}}),xBw).$$

From (6) and (7) and binarity of g, we conclude $p(g(\underset{\sim}{B}),xBy) = p(g(\underset{\sim}{B}'),xBw)$ as was to be shown.

Consider now the case $x,y,z,$ and w distinct, or x,y,w distinct and $y = z$. (The proof for $w = x$ is identical to the proof for $y = z$.) Let $\bar{\underset{\sim}{B}}$ be a profile such that

$$x\bar{B}_j z\bar{B}_j y\bar{B}_j w \qquad \text{for} \quad j \in \varphi(\underset{\sim}{B}, xBy), \text{ and}$$

$$w\bar{B}_j y\bar{B}_j z\bar{B}_j x \qquad \text{for} \quad j \in \varphi(\underset{\sim}{B}, yBx) .$$

(If $z = y$ replace the middle $\bar{B}_j$'s with equalities.) Since the lemma holds with $x = z$,

$$(8) \quad p(g(\underset{\sim}{\bar{B}}), xBy) = p(g(\underset{\sim}{\bar{B}}), xBw) .$$

Since the lemma holds with $y = w$,

$$(9) \quad p(g(\underset{\sim}{\bar{B}}), xBw) = p(g(\underset{\sim}{\bar{B}}), xBw) .$$

From (8), (9), and binarity of g, we conclude that $p(g(\underset{\sim}{B}), xBy) = p(g(\underset{\sim}{B}'), zBw)$, as was to be shown.

With Lemma 1 proved, it is natural to define the function $\mu_g : 2^N \to R$ promised in the Theorem by $\mu_g(C) = p(g(\underset{\sim}{B}), zBw)$, where z and w are arbitrary distinct alternatives and $\underset{\sim}{B}$ is any profile such that $\varphi(\underset{\sim}{B}, zBw) = C$. Lemma 1 and binarity guarantee that $\mu_g(C)$ is independent of the choice of z,w, as well as the position in $\underset{\sim}{B}$ of alternatives other than z and w .

<u>Lemma 2</u> (Nonperversity)

Let g be a binary and Paretian SMSWF. Then, for any $x, y \in A$ and any $\underset{\sim}{B}, \underset{\sim}{B}' \in \mathcal{B}^n$,

$$(10) \quad [\varphi(\underset{\sim}{B}, xBy) \supset \varphi(\underset{\sim}{B}', xBy)] \text{ implies } [p(g(\underset{\sim}{B}), xBy) \geqq p(g(\underset{\sim}{B}'), xBy)].$$

Proof of Lemma 2

Since g is Paretian, (10) holds when $\varphi(\underset{\sim}{B}, xBy) = N$. Thus, it is sufficient to show that, it holds when $\#\varphi(\underset{\sim}{B},xBy) \neq n$, and $\#\varphi(\underset{\sim}{B}',xBy) = \#\varphi(\underset{\sim}{B},xBy) + 1$. Let $\{i\} = \varphi(\underset{\sim}{B}',xBy) - \varphi(\underset{\sim}{B},xBy)$. Consider $\underset{\sim}{\bar{B}}$ such that for some z

$$x\bar{B}_j y\bar{B}_j z \quad \text{for} \quad j \in \varphi(\underset{\sim}{B}',xBy) - \{i\},$$

$$y\bar{B}_j z\bar{B}_j x \quad \text{for} \quad j \in \varphi(\underset{\sim}{B}',yBx), \text{ and}$$

$$y\bar{B}_i x\bar{B}_i z .$$

By neutrality and binarity of g

$$(11) \quad p(g(\underset{\sim}{\bar{B}}),xBz) = p(g(\underset{\sim}{B}'),xBy) .$$

Since g is Paretian $p(g(\underset{\sim}{\bar{B}}),yBz) = 1$; thus,

$$(12) \quad p(g(\underset{\sim}{\bar{B}}),xBz) \geqq p(g(\underset{\sim}{\bar{B}}),xBy) .$$

From (11), (12), and binarity of g, we conclude that $p(g(\underset{\sim}{B}'),xBy) \geqq p(g(\underset{\sim}{B}),xBy)$ as was to be demonstrated.

Proof of Theorem 1 (A remark of G. Butters showed that one-third of an earlier proof was unnecessary.)

It is clear from the definition of μ_g which follows Lemma 1 that μ_g satisfies (1), (2), and (3) of the Theorem. Lemma 2 guarantees that (4) is satisfied. It remains to show that μ_g is subadditive. Because of (2), (3), and (4), it is sufficient to show that for all disjoint and nonempty C_1 and C_2 which do not exhaust N, $\mu_g(C_1 \cup C_2) \leqq \mu_g(C_1) + \mu_g(C_2)$.

Let $\underset{\sim}{B}$ be a profile such that

$$xB_iyB_iz \quad \text{for} \quad i \in C_1,$$

$$yB_izB_ix \quad \text{for} \quad i \in C_2, \text{ and}$$

$$zB_ixB_iy \quad \text{for} \quad i \in C_3 = N - (C_1 \cup C_2) .$$

Then,

$$p(g(\underset{\sim}{B}),xBy) = \mu_g(C_1 \cup C_3)$$

$$p(g(\underset{\sim}{B}),yBz) = \mu_g(C_1 \cup C_2)$$

$$p(g(\underset{\sim}{B}),xBz) = \mu_g(C_1) .$$

The set of rankings where x appears over y and y over z will get at least probability $\mu_g(C_1 \cup C_3) + \mu_g(C_1 \cup C_2) - 1$ under the lottery $g(\underset{\sim}{B})$. Thus,

$$p(g(\underset{\sim}{B}),xBz) = \mu_g(C_1) \geqq \mu_g(C_1 \cup C_3) + \mu_g(C_1 \cup C_2) - 1, \text{ or}$$

$$1 \geqq \mu_g(C_1 \cup C_3) + \mu_g(C_1 \cup C_2) - \mu_g(C_1)$$

Thus, $1 \geqq 1 - \mu_g(C_2) + \mu_g(C_1 \cup C_2) - \mu_g(C_1)$, which gives the result.

II. EXTENSION TO THE CASE WHERE PREFERENCES ARE REPRESENTED BY WEAK ORDERINGS

NOTATION AND DEFINITIONS

As in the previous section A is the set of alternatives and N is the set of individuals. Let $\mathcal{R} = \{R, R', \ldots\}$ be the set of <u>individual preference relations</u> (complete, reflexive and transitive binary relations) on A, and $\mathcal{R}^n$ the n-fold cartesian product of $\mathcal{R}$, be the set of social preference <u>profiles</u>. A generic profile is denoted by $\underset{\sim}{R} = (R_1, R_2, \ldots, R_n)$. A lottery on $\mathcal{R}$ is a probability measure on $\mathcal{R}$. Let $L(\mathcal{R})$ be the set of lotteries on $\mathcal{R}$. Given $R \in \mathcal{R}$, let the associated strict preference B and indifference I relations be defined as usual.

For $x,y \in A$, let $\mathcal{R}(xRy) = \{R \in \mathcal{R} : xRy\}$,

$$\mathcal{R}(xBy) = \{R \in \mathcal{R} : xBy\} \equiv \{R \in \mathcal{R} : xRy \text{ and not } yRx\}, \text{ and}$$

$$\mathcal{R}(xIy) = \{R \in \mathcal{R} : xIy\} \equiv \{R \in \mathcal{R} : xRy \text{ and } yRx\} .$$

For $\ell \in L(\mathcal{R})$, let $p(\ell, xRy) = \ell(\mathcal{R}(xRy))$,

$$p(\ell, xBy) = \ell(\mathcal{R}(xBy)), \text{ and}$$

$$p(\ell, xIy) = \ell(\mathcal{R}(xIy)) .$$

Clearly, for all distinct x,y,

$$p(\ell, xBy) + p(\ell, yBx) + p(\ell, xIy) = 1 = p(\ell, xRy) + p(\ell, yBx) .$$

Given $\underset{\sim}{R} \in \mathcal{R}^n$, let $\varphi(\underset{\sim}{R}, xRy) = \{i \in N : xR_i y\}$,

$$\varphi(\underset{\sim}{R}, xBy) = \{i \in N : xB_i y\}, \text{ and}$$

$$\varphi(\underset{\sim}{R}, xIy) = \{i \in N : xI_i y\} .$$

A <u>Mixture Social Welfare Function</u> (MSWF) is a function $g : \mathcal{R}^N \to L(\mathcal{R})$. A MSWF is <u>binary</u> if for any $x,y \in A$, and any two profiles $\underset{\sim}{R}, \underset{\sim}{R}' \in \mathcal{R}^n$,

$$[\varphi(\underset{\sim}{R}, xBy) = \varphi(\underset{\sim}{R}', xBy) \text{ and } \varphi(\underset{\sim}{R}, yBx) = \varphi(\underset{\sim}{R}', yBx)] \text{ implies}$$

$$[p(g(\underset{\sim}{R}), xBy) = p(g(\underset{\sim}{R}'), xBy) \text{ and } p(g(\underset{\sim}{R}), yBx) = p(g(\underset{\sim}{R}'), yBx)]$$

A MSWF is <u>Paretian</u> if for any $x,y\epsilon A$, and any $\underset{\sim}{R} \in \mathcal{R}^n$,

$$[\varphi(\underset{\sim}{R},xBy) = N] \text{ implies } [p(g(\underset{\sim}{R}),xBy) = 1] .$$

THEOREM 2

Given any binary and Paretian MSWF g, there exists a subadditive function $\mu_g : 2^N \rightarrow R$, such that

(15a) $\mu_g(\varphi(\underset{\sim}{R},xBy)) \leq p(g(\underset{\sim}{R}),xBy)$

for all $\underset{\sim}{R} \in \mathcal{R}^n$ and $x,y\epsilon A$,

and

(15b) $\mu_g(\varphi(\underset{\sim}{R},xBy)) = p(g(\underset{\sim}{R}),xBy)$

for all $\underset{\sim}{R} \in \mathcal{R}^n$ and $x,y\epsilon A$ such that

$\varphi(\underset{\sim}{R},xIy) = \emptyset$; furthermore,

(16) $\mu_g(\emptyset) = 0$,

(17) $\mu_g(C) + \mu_g(N-C) = 1$, for all $C \in 2^N$, and

(18) $\mu_g(C) \geqq \mu_g(C')$, whenever $C' \subset C$.

The proof of Theorem 2 follows a Lemma.

Lemma 3

Let g be a binary and Paretian MSWF. Then, for all $\underset{\sim}{R} \in \mathcal{R}^n$, and all $x,y\epsilon A$,

$$[\varphi(\underset{\sim}{R},xIy) = \emptyset] \text{ implies } [p(g(\underset{\sim}{R}),xIy) = 0] .$$

<u>Proof of Lemma 3</u>

Consider any profile $\underset{\sim}{R}' \in \mathcal{R}^n$ such that, for some $z \in A$ $(x \neq z \neq y)$ and all $i \in N$,

$(xB_i'zB_i'y)$ if and only if (xB_iy), and

$(zB_i'yB_i'x)$ if and only if (yB_ix) .

Since g is Paretian, $p(g(\underset{\sim}{R}'), zBy) = 1$. Thus,

$$(19) \quad p(g(\underset{\sim}{R}'),zBx) \geqq p(g(\underset{\sim}{R}'),yBx) + p(g(\underset{\sim}{R}'),yIx) .$$

Choose $\underset{\sim}{R}''$ such that, for all $i \in N$,

$(xB_i''yB_i''z)$ if and only if (xB_iy), and

$(yB_i''zB_i''x)$ if and only if (yB_ix) .

Since g is Paretian, $p(g(\underset{\sim}{R}''),yBz) = 1$. Thus,

$$(20) \quad p(g(\underset{\sim}{R}''),xRz) \geqq p(g(\underset{\sim}{R}''),xBz) \geqq p(g(\underset{\sim}{R}''),xBy) + p(g(\underset{\sim}{R}''),xIy) .$$

Since g is binary, (19) and (20) yield

$$(21) \quad 1 = p(g(\underset{\sim}{R}'),zBx) + p(g(\underset{\sim}{R}'),xRz) \geqq p(g(\underset{\sim}{R}),yBx) + p(g(\underset{\sim}{R}),xBy) + 2p(g(\underset{\sim}{R}),xIy) .$$

Since $p(g(\underset{\sim}{R}),yBx) + p(g(\underset{\sim}{R}),xBy) + p(g(\underset{\sim}{R}),xIy) = 1$, and $p(g(\underset{\sim}{R}),xIy) \geqq 0$, it must be that $p(g(\underset{\sim}{R}),xIy) = 0$, as we wanted to show.

<u>Proof of Theorem 2</u>

Consider g restricted to $\mathcal{B}^n = \{\underset{\sim}{R} \in \mathcal{R}^n: \varphi(\underset{\sim}{R}, xIy) = \emptyset$ for all $x,y \in A\}$. By the previous lemma the image of g so restricted is a subset of $L(\mathcal{B})$. Thus Theorem 1 applies and there exists a subadditive function μ_g which satisfies (1), (16), (17), and (18). Since g is binary, μ_g satisfies (15b) as well, and to complete the proof it is sufficient to show that $\mu_g(\varphi(\underset{\sim}{R}, xBy)) \leqq p(g(\underset{\sim}{R}), xBy)$ for all $\underset{\sim}{R} \in \mathcal{R}^n$ and all pairs $x,y \in A$.

Given any profile $\underset{\sim}{R}$, let $\hat{\underset{\sim}{R}}$ be such that, for some $z \in A$ $(x \neq z \neq y)$,

$(x\hat{B}_j z \hat{B}_j y)$ if and only if $(xB_j y)$,

$(y\hat{B}_j x \hat{B}_j z)$ if and only if $(yB_j x)$, and

$(x\hat{I}_j y\hat{B}_j z)$ if and only if $(xI_j y)$.

Since $\varphi(\hat{\underset{\sim}{R}}, yIz) = \emptyset$ and $\varphi(\underset{\sim}{R}, xBy) = \varphi(\hat{\underset{\sim}{R}}, zBy)$, we have $p(g(\hat{\underset{\sim}{R}}), zBy) = \mu_g(\varphi(\hat{\underset{\sim}{R}}, zBy)) = \mu_g(\varphi(\underset{\sim}{R}, xBy))$.

Since g is Paretian, $p(g(\hat{\underset{\sim}{R}}), xBz) = 1$. Thus $p(g(\underset{\sim}{R}), xBy) = p(g(\hat{\underset{\sim}{R}}), xBy) \geq p(g(\hat{\underset{\sim}{R}}), zBy) = \mu_g(\varphi(\underset{\sim}{R}, xBy))$.

III. THE GENERAL POSSIBILITY THEOREM AS A COROLLARY

A <u>Social Welfare Function</u> (SWF) w is a MSWF such that for every $\underset{\sim}{R} \in \mathcal{R}^n$, there exists $R \in \mathcal{R}$, such that $[w(\underset{\sim}{R})](\{R\}) = 1$. A SWF w is <u>dictatorial</u> if there exists a $j \in N$ such that for all $x,y \in A$ and all $\underset{\sim}{R} \in \mathcal{R}^n$,

$[xB_j y]$ implies $[p(w(\underset{\sim}{R}), xBy) = 1]$.

<u>Corollary</u> (Arrow's Theorem)

If w is a binary and Paretian SWF, then it is dictatorial.

<u>Proof</u>: By Theorem 2 there exists a subadditive function μ_w satisfying (15)-(18). Since μ_w is subadditive, there exists $j \in N$ such that $\mu_w(\{j\})$ is positive. Let $\underset{\sim}{R}$ be an arbitrary profile such that $xB_j y$. By (15) and (18), $p(w(\underset{\sim}{R}), xBy) > 0$; but since w is a SWF, $p(w(\underset{\sim}{R}), xBy) = 1$.

IV. PARALLEL WITH THE PROOF OF ARROW'S THEOREM

For simplicity we restrict attention to the case of strict preferences as considered in II.

Outline of the Standard Proof of the General Possibility Theorem (see; e.g., [1], pp. 97-100)	Outline of the Proof of Theorem 1 and Arrow's Theorem as a Corollary
Let g be a binary and Paretian SWF.	Let g be a binary and Paretian MSWF.
(A1) Let $\underset{\sim}{B}$ and x and y be given, and let $C = \varphi(\underset{\sim}{B}, xBy)$. Write $d_g(C) = 0$ or 1 accordingly as x is preferred or inferior to y in the social preference relation $g(\underset{\sim}{B})$. (We write $xg(\underset{\sim}{B})y$, $y\, g(\underset{\sim}{B})x$, etc.) It is necessary to show that d_g is a well-defined function on 2^N, the set of coalitions. By binarity,	(B-S1) Let $\underset{\sim}{B}$ and x and y be given and let $C = \varphi(\underset{\sim}{B}, xBy)$. Define $\mu_g(C)$ to be the $g(\underset{\sim}{B})$ probability of the event xBy: $\mu_g(C) = p(g(\underset{\sim}{B}), xBy)$. It is necessary to show that μ_g is a well-defined function on 2^N. By binarity, $p(g(\underset{\sim}{B}), xBy) = p(g(\underset{\sim}{B}'), xBy)$ whenever $\varphi(\underset{\sim}{B}, xBy) = \varphi(\underset{\sim}{B}', xBy)$. Thus, it is sufficient to show

$xg(\underset{\sim}{B})y$ is equivalent to $xg(\underset{\sim}{B}')y$ whenever $\varphi(\underset{\sim}{B},xBy) = \varphi(\underset{\sim}{B}',xBy)$. Thus, it is sufficient to show that the number $d_g(C)$ is independent of the choice of x and y; i.e., for all $\underset{\sim}{B},\underset{\sim}{B}' \in \mathcal{B}^n$ and $x,y,z,w \in A$, $[\varphi(\underset{\sim}{B},xBy)=\varphi(\underset{\sim}{B},zBw)]$ implies $[xg(\underset{\sim}{B})y$ is equivalent to $zg(\underset{\sim}{B}')w]$. This is established. If $d_g(C) = 1$, C is called a weakly decisive coalition; if $d_g(C) = 0$, C is called a losing coalition.

that the number $\mu_g(C)$ is independent of the choice of x and y; i.e., for all $B,B' \in \mathcal{B}^n$ and $x,y,z,w \in A$, $[\varphi(\underset{\sim}{B},xBy) = \varphi(\underset{\sim}{B}',zBw)]$ implies $[p(g(\underset{\sim}{B}),xBy) = p(g(\underset{\sim}{B}'),zBw)]$. This is Lemma 1.

(A2) It is demonstrated that if $d_g(C) = 1$ and $C \subset C'$, then $d_g(C') = 1$; alternatively, $C \subset C'$ implies $d_g(C) \leqq d_g(C')$.

(B-S2) It is demonstrated that if $C \subset C'$, then $\mu_g(C) \leqq \mu_g(C')$. This is Lemma 2.

(A3) It is demonstrated that the union of disjoint losing coalitions is losing; $C \cap C' = \emptyset$ and $d_g(C) = d_g(C') = 0$ implies $d_g(C \cup C') = 0$. Alternatively, $d_g(C) + d_g(C') \geqq d_g(C \cup C')$ for all $C,C' \subset N$.

(B-S3) The function μ_g is subadditive; i.e., $u_g(C) + \mu_g(C') \geqq \mu_g(C \cup C')$ for all $C,C' \subset N$. This is the final step in the proof of Theorem 1.

(A4) The proof of the theorem is completed as follows. The function d_g takes only the values zero and one. Since g is Paretian, $d_g(N) = 1$. The fact that N is the finite union of its elements together with (A3) implies $d_g(\{j\})=1$ for some j. By (A2), j is a dictator.

(B-S4) (Arrow's Theorem as a Corollary) Since g is Paretian, $\mu_g(N)=1$. The fact that N is the finite union of its elements together with (B-S3) implies $\mu_g(\{j\}) > 0$ for some j. Since g is a SWF, $\mu(\{j\})=1$, and so by (B-S2), j is a dictator.

Finally we observe a further parallel with the Arrow Theorem. An Arrowian dictator who places x indifferent to y is <u>not</u> guaranteed that x will be indifferent to y socially; i.e., if g is binary and Paretian on $\mathcal{R}^n$ and if "j" is an Arrowian dictator, then for some $\underset{\sim}{R}$ it is possible that xI_jy and yet x is socially preferred to y. Similarly for the case of binary and Paretian MSWF's; if a coalition places x indifferent to y it will not in general be the case that the event xIy will have probability equal to the power of that coalition. As (15a) indicates, any coalition C (with power $\mu_g(C)$) can assure x over y with probability at least $\mu_g(C)$; however, the probability of x over y may exceed $\mu_g(C)$.

CONCLUSION

Allowing preference lotteries for social preferences does not diminish the force of Arrow's Theorem. All binary and Paretian MSWF's are associated with subadditive distributions of the dictatorial power.

References

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