OPTIMAL CONTROL OF STOCHASTIC DIFFERENTIAL EQUATION SYSTEMS

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This paper is a tutorial exposition of the basic techniques of optimal control for stochastic systems in continuous time and of several important applications in economics. Its main purpose is to introduce these techniques to readers who already have some familiarity with the techniques and applications of stochastic control in discrete time, as can be found in Chow (1975) for example, but wish to avoid a heavy investment in the highly mathematical treatments of the subject currently available. The basic idea is to construct a stochastic model in discrete time and let the time intervals between successive measurements become very small. For readers already familiar with the analysis and control of stochastic models in discrete time, we will study continuous-time models using similar tools.

To begin with, we will introduce a system of linear stochastic differential equations and study its dynamic properties, applying the notions developed for a system of linear stochastic difference equations. We then describe the method of dynamic programming in continuous time, formulate a system of nonlinear stochastic differential equations and derive Ito's differentiation rule for a scalar function of stochastic processes which satisfy a system of stochastic differential equations. The major areas of application of continuous time models are in economic theory rather than econometric analysis. We will use for illustration the problem of optimal consumption and portfolio selection over time studied by Robert Merton (1969) and (1971). Dynamic programming will be applied to solve this optimal stochastic control problem in continuous time. An extension of the basic model due to Merton (1973) to explain the prices of capital assets will be discussed. The next application will be the pricing of stock options originally studied by Black and Scholes (1973). We will then return to the control problem of minimizing the expected value of a quadratic loss function subject to the constraint of a system of linear stochastic differential equations. As the last application, we will study the problem of optimal exploitation of a limited natural resource when the probability distribution of its reserves is unknown.

1. Linear Stochastic Differential Equations

As in the formulation of a system of linear stochastic difference equations, consider the evolution of a vector time series $y(t)$ from time t to time $t+h$, where h need no longer be an integer and, in fact, will be assumed to take as small a value as we please. A reasonable extension of the discrete-time model is

$$y(t+h) - y(t) = A(t)y(t)h + v(t+h) - v(t) \tag{1}$$

where the random residual $v(t+h) - v(t)$, as in the discrete-time formulation, is assumed to have mean zero and to be statistically independent through time whatever the choice of h. If the residual were absent, we could divide both sides of (1) by $h = dt$, let dt approach zero and obtain a system of linear differential equations in the limit. We now have the random term in (1) and need to specify it further.

For $h = 1$, let the covariance matrix of the vector of residuals be Σ. For smaller h, we divide the time interval between t and $t+1$ into n segments of equal length, i.e., let $h = \frac{1}{n}$ time units. Because the n successive increments $v(t+h) - v(t)$, $v(t+2h) - v(t+h)$, ..., are assumed to be statistically independent and identically distributed, the co-variance matrix Σ of their sum equals n times the covariance matrix of each increment, implying

$$\text{Cov}[v(t+h) - v(t)] = \frac{1}{n}\Sigma = h\Sigma \tag{2}$$

Σ being the covariance matrix of $v(t+1) - v(t)$. A vector time series $v(t)$

whose successive differences, however divided up, are statistically independent is called a stochastic process with independent increments. If, in addition, the successive differences are normally distributed it is called a Wiener process or Brownian motion. It has been shown in (2) that the covariance matrix of the increment $v(t+h) - v(t)$ is proportional to the time h. This means that the standard deviation of each component of the vector $v(t+h) - v(t)$ is proportional to $\sqrt{h}$. This property is important because terms involving the squares of the elements of $v(t+h) - v(t)$ are of order h and not of order h^2; they do not vanish as h becomes very small. As h becomes small, we write h as dt and rewrite (1) as

$$dy = A(t)ydt + dv \tag{3}$$

where $E(dv) = 0$ and $Cov(dv) = \Sigma(t)dt$. (3) is a system of linear stochastic differential equations. The covariance matrix Σ can be a function of t in the more general case, as we write $\Sigma(t)$ in (3). Since $Cov(dv) = \Sigma dt$, the covariance matrix of $\frac{dv}{dt}$ is $\Sigma(dt)^{-1}$ which increases without bound as dt approaches zero. Therefore, the derivative $\frac{dv}{dt}$ does not exist and one cannot divide equation (3) by dt to obtain an equation explaining the derivative $\frac{dy}{dt}$ by $\frac{dv}{dt}$.

To find the solution $y(t)$ of (3) given $y(t_0)$, we divide the time interval between t_0 and t into n equal segments at points $t_1<t_2<\ldots<t_n = t$ and let the length of each segment be h, which can be made as small as we please by increasing n. If we define $dy(t_i)$ as $y(t_i+h) - y(t_i)$ or $y(t_{i+1}) - y(t_i)$ and $dv(t_i)$ as $v(t_{i+1}) - v(t_i)$, (1) or (3) implies

$$\begin{aligned} y(t_n) &= [I+A(t_{n-1})h]y(t_{n-1}) + dv(t_{n-1}) \qquad (4) \\ &= [I+A(t_{n-1})h][I+A(t_{n-2})h]y(t_{n-2}) \\ &\quad + [I+A(t_{n-1})h]dv(t_{n-2}) + dv(t_{n-1}) \end{aligned}$$

By repeated substitutions of $y(t_{n-2})$ by $y(t_{n-3})$, etc., and by defining the state transition matrix

$$\Phi(t_i,t_j) = \prod_{k=j}^{i-1} [I+A(t_k)h] \qquad i \geq j+1 \tag{5}$$

$$\Phi(t_i,t_i) = I$$

we can rewrite (4) as

$$y(t_n) = \Phi(t_n,t_o)y(t_o) + \sum_{j=0}^{n-1} \Phi(t_n,t_{j+1})dv(t_j) \tag{6}$$

which is a solution to (1). Note that by the definition (5) the state transition matrix satisfies the difference equation

$$d\Phi(t_i,t_o) \equiv \Phi(t_{i+1},t_o) - \Phi(t_i,t_o) = A(t_i)\Phi(t_i,t_o)h \tag{7}$$

The solution (6) provides a heuristic argument for the following solution to the linear stochastic differential equation (3) as $h \to 0$,

$$y(t) = \Phi(t,t_o)y(t_o) + \int_{t_o}^{t} \Phi(t,s)dv(s) \tag{8}$$

The integral in (8) involving the stochastic process $v(s)$ is a stochastic integral. Following Ito, a stochastic integral of a deterministic or stochastic function f is defined as the limit of the sum

$$\int_{t_o}^{t} f(s)dv(s) = \lim_{n\to\infty} \sum_{j=1}^{n} f(t_j)[v(t_{j+1}) - v(t_j)] \tag{9}$$

where the limit of a sequence of random variables g_n is defined by convergence in mean square, i.e.,

$$\lim_{n\to\infty} g_n = g \quad \Longleftrightarrow \quad \lim_{n\to\infty} E|g_n - g|^2 = 0 \tag{10}$$

This integral has the property that the operations of taking mathematical expectation and integration can be interchanged. If $f(t_j)$ in (9) is stochastic, say being $f(v(t_j),t_j)$, it makes a difference in taking the limit whether $f(t_j)$ is weighted by the forward difference $v(t_{j+1}) - v(t_j)$ according to Ito or by the backward difference $v(t_j) - v(t_{j-1})$. When $f(t_j)$ is a deterministic function, defined as the limit of a sequence of piecewise constant functions which are constant over intervals (t_j, t_{j+1}), as $\Phi(t_n, t_j)$ in (6), the resulting integral is the same no matter whether a forward or backward difference is taken. An exposition of this point can be found in Astrom (1970), Chapter 3, Section 5.

It follows from (7) that the state transition matrix $\Phi(t,t_o)$ satisfies the differential equation

$$\frac{d\Phi(t,t_o)}{dt} = A(t)\Phi(t,t_o) \; . \tag{11}$$

In the special case $A(t) = A$, we will solve this differential equation by iteration. Let $t_o = 0$, and let the successive iterations be $\Phi_0(t,0)$, $\Phi_1(t,0)$, ... We have $\Phi_0(t,0) = I$ and

$$\Phi_1(t,0) = I + \int_0^t A\Phi_0(s,0)\,ds = I + At$$

$$\Phi_2(t,0) = I + \int_0^t A\Phi_1(s,0)\,ds = I + At + A^2\frac{t^2}{2}$$

$$\Phi_i(t,0) = I + \int_0^t A\Phi_{i-1}(s,0)ds = I + At + A^2 \frac{t^2}{2} + \ldots + A^i \frac{t^i}{i!}$$

As i increases, $\Phi_i(t,0)$ converges to

$$\Phi(t,0) = I + At + A^2 \frac{t^2}{2} + A^3 \frac{t^3}{3!} + \ldots = e^{At}$$

The solution to (11) is therefore

$$\Phi(t,t_o) = e^{A(t-t_o)} \tag{12}$$

2. Mean and Covariance of Solution to Linear Stochastic Differential Equations

To find the mean path $Ey(t) \equiv \bar{y}(t)$ of the solution (8), we take expectations of both sides of (8). Interchanging expectation and integration on the right-side and using $Edv(s) = 0$, we obtain

$$\bar{y}(t) = \Phi(t,t_o)\bar{y}(t_o) \tag{13}$$

This result could also be obtained by taking expectations of both sides of (3) to yield

$$d\bar{y} = A(t)\bar{y}dt \tag{14}$$

the solution of which is (13). Thus the mean satisfies the deterministic differential equation (14) after dropping the stochastic term in (3).

To find the autocovariance matrix, we subtract (13) from (8) and consider followed from t on. (In the notation of Chow (1975), this function was written as $\hat{V}$.) By the principle of optimality of dynamic programming, we have

$$V(y_t, t) = \underset{x_t}{\text{Max}}\ E_{t-1}\{u(y_t, x_t, t) + V(y_{t+1}, t+1)\} \tag{21}$$

This maximization problem could be solved beginning from period T and proceeding backward in time to period t_0. The method for optimization in continuous time can be derived by letting the time interval h between successive decisions become as small as one pleases. The above relation for small h is

$$\begin{aligned} V(y(t),t) &= \underset{x_t}{\text{Max}}\ E_t\left\{\int_t^{t+h} u(y,x,s)ds + V(y(t+h),\ t+h)\right\} \\ &= \underset{x_t}{\text{Max}}\ \{u(y(t),x(t),t)h + E_t V(y(t+h),\ t+h)\} \end{aligned} \tag{22}$$

If we let $dV(t)$ denote $V(y(t+h),\ t+h) - V(y(t),t)$, the above equation can be written as

$$V(y,t) = \underset{x_t}{\text{Max}}\ \{u(y,x,t)h + V(y,t) + E_t dV\} \tag{23}$$

Dividing through by $h = dt$, we obtain

$$0 = \underset{x}{\text{Max}}\ \{u(y,x,t) + E_t \frac{1}{dt} dV\} \tag{24}$$

which is to be solved. To solve the optimization problem thus requires the evaluation of the stochastic differential dV where V is a function of y and t and y satisfies a given stochastic differential equation. To evaluate dV, a differentiation rule developed by Ito will be used. Furthermore, we would need to evaluate the expectation $E_t \frac{1}{dt} dV$. For this purpose, a differential generator related to Ito's differential for dV will be derived. These are the subjects of the next section.

4. Ito's Differentiation Rule

We will have occasions to consider nonlinear stochastic differential equations of the form

$$dy = f(y,t)dt + dv = f(y,t)dt + S(y,t)dz \qquad (25)$$

where the vector function $f(y,t)$ replaces the linear function $A(t)y$ in (3) and the covariance matrix of dv may be a function of y and is written as $\Sigma(y,t)dt$. If we let z be a Wiener process with Rdt as the covariance matrix for its increment, we can write dv as $S(y,t)dz$ where $SRS' = \Sigma$. Formally, the solution of (2) is

$$y(t) = y(t_0) + \int_{t_o}^{t} f(y(s),s)ds + \int_{t_o}^{t} S(y(s),s)dz(s) \qquad (26)$$

where the stochastic integral was defined by (9). Although we may not need to express the solution in explicit form, we would like to study the properties of a stochastic process which is a scalar function of y, as exemplified by the function V in the last section on the method of dynamic programming. Let $F = F(y,t)$ be such a function, assumed to be continuously differentiable in t and twice continuously differentiable in y. We wish to derive a stochastic differential equation for F.

We expand the function F in a Taylor series, with $h = dt$,

$$dF = F(y(t+h),t+h) - F(y(t),t)$$

$$= \frac{\partial F}{\partial t}\, dt + \left(\frac{\partial F}{\partial y}\right)' dy + \frac{1}{2}(dy)' \frac{\partial^2 F}{\partial y \partial y'}\, dy + o(dt) \tag{27}$$

where $o(dt)$ denotes terms of order smaller than dt. Denoting the matrix of second partial derivatives of F with respect to y by F_{yy}, using (25) for dy and noting that dv is of order $\sqrt{dt}$, we have

$$(dy)'F_{yy}dy = (dv)'F_{yy}dv + o(dt) = tr(F_{yy}dvdv') + o(dt) \tag{28}$$

Substituting (25) and (28) into (27) gives

$$dF = [\frac{\partial F}{\partial t} + \left(\frac{\partial F}{\partial y}\right)' f]dt + \frac{1}{2}\, tr(F_{yy}dv\; dv') + \left(\frac{\partial F}{\partial y}\right)' dv + o(dt) \tag{29}$$

implying, together with $Edvdv' = \Sigma$,

$$E(dF) = [\frac{\partial F}{\partial t} + \left(\frac{\partial F}{\partial y}\right)' f + \frac{1}{2} tr\,(F_{yy}\Sigma)]dt \tag{30}$$

and

$$var(dF) = E[dF - E(dF)]^2$$

$$= E[\frac{1}{2}\, tr(F_{yy}dv\; dv') - \frac{1}{2}\, tr(F_{yy}\Sigma)dt + \left(\frac{\partial F}{\partial y}\right)' dv + o(dt)]^2$$

$$= E[\left(\frac{\partial F}{\partial y}\right)' dv]^2 + o(dt) = \left(\frac{\partial F}{\partial y}\right)' \Sigma \left(\frac{\partial F}{\partial y}\right) dt + o(dt) \tag{31}$$

(30) and (31) provide a justification for Ito's differentiation rule:

$$dF = [\left(\frac{\partial F}{\partial t}\right) + \left(\frac{\partial F}{\partial y}\right)' f + \frac{1}{2} \operatorname{tr}(F_{yy} \Sigma)] dt + \left(\frac{\partial F}{\partial y}\right)' dv \tag{32}$$

where $F = F(y,t)$ and dy is given by (25). Note that (32) remains valid if the functions f and Σ (or S) have a third argument x, i.e., they become $f(y,x,t)$ and $\Sigma(y,x,t)$. Here x can be viewed as a vector of parameters of the functions f and Σ. It can be used as a vector of exogenous variables or control variables for the system of stochastic differential equations (25).

A related concept to Ito's differential dF is the operation

$$\lim_{h \to 0} E_t \left[\frac{dF}{h}\right] \equiv \mathcal{L}_y [F(y,t)] \tag{33}$$

where dF is defined by the first line of (27) and E_t is the conditional expectation given $y(t)$. The result gives the expected rate of change through time of the function $F(y,t)$ as induced by the stochastic process y. The operator $\mathcal{L}_y$ so defined is the <u>differential generator</u> of the stochastic process $y(t)$. Formally, it can be obtained by $\frac{1}{dt} E_t(dF)$, using (32) for dF; i.e., by taking the expectation of (32) and dividing the result by dt. This gives

$$\mathcal{L}_y [F] = \{ \frac{\partial}{\partial t} + f' \frac{\partial}{\partial y} + \frac{1}{2} \operatorname{tr}(\Sigma \frac{\partial^2}{\partial y \partial y'}) \} [F] \tag{34}$$

The stochastic differential (32) and the differential generator (34) will be applied to solve optimal control problems by the method of dynamic programming in the following sections.

5. Optimum Consumption and Portfolio Selection over Time

The problem of this and the following section was studied by Merton (1969, 1971), and its discrete version partly by Samuelson (1969). At time t, the individual chooses his rate of consumption $C(t)$ per unit time during period t (between t and $t+h$) and the number $N_i(t)$ of shares to be invested in asset i during period t, given his initial wealth $W(t) = \sum_1^n N_i(t-h)P_i(t)$ and the prices $P_i(t)$ per share of the assets. The prices are assumed to follow the stochastic differential equations

$$\frac{dP_i}{P_i} = \alpha_i(P,t)dt + s_i(P,t)dz_i \tag{35}$$

where P is the vector of asset prices and z_i are components of a multivariate Wiener process, with $E(dz_i) = 0$, $\text{var}(dz_i) = 1$ and $E(dz_i dz_j) = \rho_{ij}$. If α_i and s_i are constants, (35) describes a "geometric Brownian motion" hypothesis for asset prices. If there is no wage income and all incomes are derived from capital gains (dividends being included in changes in asset prices), it can be shown that the change in wealth from t to $t+h$ satisfies the budget constraint

$$dW = \sum_1^n N_i(t)dP_i - C(t)dt \tag{36}$$

Let $w_i(t) = N_i(t)P_i(t)/W(t)$ be the fraction of wealth invested in asset i, with $\sum_i w_i = 1$. We substitute (35) for dP_i in (36) to obtain

$$dW = \sum_1^n w_i W\alpha_i dt - Cdt + \sum_1^n w_i W s_i dz_i \tag{37}$$

If we assume the n^{th} asset to be risk-free, i.e., $s_n = 0$, and denote the instantaneous rate of return α_n of this asset by r , we can write (37) as, with $m = n-1$,

$$dW = \sum_1^m w_i(\alpha_i - r)Wdt + (rW-C)dt + \sum_1^m w_i W s_i dz_i \tag{38}$$

The endogenous or state variables of this problem, corresponding to the vector y of the previous sections, are W and P. They are governed by the stochastic differential equations (37) and (35) which correspond to (25) when the vector of control variables x is added to the arguments of f and S. The control variables are C and $w = (w_1, \ldots, w_n)'$, with $\sum_1^n w_i = 1$. The problem is to maximize

$$E_o \left[\int_o^T U(C(t),t)dt + B(W(T),T) \right]$$

where U is the utility function and B is the bequest function. This model assumes that assets are traded continuously in time and that there are no transaction costs in trading. The latter assumption is unrealistic, but we will consider in section 7 a consequence of dropping this assumption.

We apply the method of dynamic programming to solve this problem. By the result of section 3, this amounts to solving equation (24), which, using the differential generator $\mathcal{L}_y$ of section 4, can be written as

$$\max_{C(t),w(t)} \{U(C,t) + \mathcal{L}_y[V(y,t)]\} = 0 \tag{39}$$

Thus the optimum policy for $C(t)$ and $w(t)$ will be found by solving (39).

Using (37) and (35) as stochastic differential equations for $(W,P) = y$, and (34) for the operator $\mathcal{L}_y$, we find

$$\mathcal{L}_y[V(y,t)] = \frac{\partial V}{\partial t} + [\sum_1^n w_i\alpha_i W - C]\frac{\partial V}{\partial W} + \sum_1^n \alpha_i P_i \frac{\partial V}{\partial P_i}$$

$$+ \frac{1}{2}\sum_{11}^{nn}\sigma_{ij}w_iw_jW^2\frac{\partial^2 V}{\partial W^2} + \frac{1}{2}\sum_{11}^{nn}P_iP_j\sigma_{ij}\frac{\partial^2 V}{\partial P_i\partial P_j} \tag{40}$$

$$+ \sum_{11}^{nn}P_iw_jW\sigma_{ij}\frac{\partial^2 V}{\partial P_i\partial W}$$

where $\sigma_{ij} = \rho_{ij}s_is_j$. We perform the maximization (39) by differentiating with respect to C and w the Lagrangian expression

$$L = U(C,t) + \mathcal{L}_y[V] + \lambda[1-\sum_1^n w_i] \tag{41}$$

and obtain the first-order conditions, with subscripts denoting partial derivatives and $V_{jw} = \partial^2 V/\partial P_j\partial W$,

$$L_C(C,w) = U_c(C,t) - V_w = 0 \tag{42}$$

$$L_{w_k}(C,w) = W^2 V_{ww}\sum_1^n \sigma_{kj}w_j - \lambda + WV_w\alpha_k + W\sum_1^n P_j\sigma_{kj}V_{jw} = 0$$

$$(k = 1, \ldots, n) \tag{43}$$

$$L_\lambda(C,w) = 1 - \sum_1^n w_i = 0 \tag{44}$$

Equations (42) - (44) can be solved to obtain the optimal C and w as functions of the partial derivatives of V. These functions can be substituted for C and w in (40) and, by (39), we need to solve the resulting partial differential equation $U(C,t) + \mathcal{L}_y[V(y,t)] = 0$ for the function $V(y,t)$.

6. Consumption and Portfolio Selection When Asset Prices are Log-normal

Interesting results can be obtained for the special case when the asset prices follow a geometric Brownian motion, i.e., when α_i and s_i in (35) are constants. In this case, current prices P provide no information on the relative rates of change in the prices according to (35) and the maximum expected future utility V is a function of W and t only, and not of P. The terms involving P_i drop out in (40). (42) and (43) become respectively

$$U_C(C,t) - V_W = 0 \tag{45}$$

$$W^2 V_{WW} \sum_1^n \sigma_{kj} w_j - \lambda + W V_W \alpha_k = 0 \tag{46}$$

Defining the inverse function $G = [U_C]^{-1}$, we solve (45) to obtain the optimum consumption

$$\hat{C} = G(V_W, t) \tag{47}$$

To obtain the optimal portfolio w, we solve (46) and (44) for w and λ, or solve

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} & 1 \\ & & & & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn} & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \\ \mu \end{bmatrix} = -\frac{V_W}{WV_{WW}} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} \tag{48}$$

where $\mu = -\lambda / W^2 V_{WW}$. By partitioning the bordered matrix in equation 48, we find the first n rows of its inverse to be

$$\begin{bmatrix} \sigma^{11} & \sigma^{12} & \ldots & \sigma^{1n} \\ & \cdot & \cdot & \cdot \\ \sigma^{n1} & \sigma^{n2} & \ldots & \sigma^{nn} \end{bmatrix} - \Gamma^{-1} \begin{bmatrix} (\Sigma\sigma^{1j})(\Sigma\sigma^{1j}) & (\Sigma\sigma^{1j})(\Sigma\sigma^{2j}) & \ldots & (\Sigma\sigma^{1j})(\Sigma\sigma^{nj}) \\ & & & \\ (\Sigma\sigma^{1j})(\Sigma\sigma^{nj}) & (\Sigma\sigma^{2j})(\Sigma\sigma^{nj}) & \ldots & (\Sigma\sigma^{nj})(\Sigma\sigma^{nj}) \end{bmatrix}, \quad \Gamma^{-1} \begin{bmatrix} \Sigma\sigma^{1j} \\ \vdots \\ \Sigma\sigma^{nj} \end{bmatrix} \tag{49}$$

where $(\sigma_{ij})^{-1} = (\sigma^{ij})$ and $\Gamma = \sum_{ij}\Sigma\sigma^{ij}$. Therefore the optimal portfolio rules are

$$\begin{aligned} \hat{w}_k &= \sum_{\ell}[\sigma^{k\ell} - \Gamma^{-1}(\sum_j\sigma^{kj})(\sum_i\sigma^{\ell i})]\left[\frac{-V_w}{WV_{ww}}\alpha_\ell\right] + \Gamma^{-1}\sum_j\sigma^{kj} \\ &= \Gamma^{-1}\sum_j\sigma^{kj} - \frac{V_w}{WV_{ww}}\left[\sum_\ell\sigma^{k\ell}\alpha_\ell - \Gamma^{-1}\sum_j\sigma^{kj}\sum_{\ell i}\Sigma\sigma^{\ell i}\alpha_\ell\right] \\ &= h_k + m(W,t)\cdot g_k \qquad (k = 1, \ldots, n) \end{aligned} \tag{50}$$

where we have defined

$$h_k = \Gamma^{-1}\sum_j\sigma^{kj} \tag{51}$$

$$m(W,t) = -\frac{V_w}{WV_{ww}} \tag{52}$$

$$g_k = \sum_j\sigma^{kj}\left[\alpha_j - \Gamma^{-1}\sum_{\ell i}\Sigma\sigma^{\ell i}\alpha_\ell\right] \tag{53}$$

implying $\sum_1^n h_k = 1$ and $\sum_1^n g_k = 0$.

The first component h_k of the optimal fraction $\hat{w}_k$ invested in asset k is proportional to the elements σ^{kj} in the k^{th} row of the inverse of the covariance matrix of the relative rates of returns $\frac{dP_j}{P_j}$ stipulated by (35). The factor g_k in the second component of $\hat{w}_k$ is a weighted average, using σ^{kj} as weights, of the difference between the expected rate of return α_j for asset j and the average expected rate of return $\Gamma^{-1}\sum_{\ell}\sum_{i}\sigma^{\ell i}\alpha_\ell$ for all assets. If the covariances σ_{ij} were zero for $i \neq j$, g_k would become $\sigma_{kk}^{-1}[\alpha_k - \sum_{\ell}\sigma_{\ell\ell}^{-1}\alpha_\ell]$, thus measuring the expected rate of return α_k for asset k as compared with the average expected rate $\sum_{\ell}\sigma_{\ell\ell}^{-1}\alpha_\ell$ for all assets. The first component h_k recommends investment proportional to the inverses of the variances and covariances. Since $V_W > 0$ and $V_{WW} < 0$ for maximum V , $m(W,t) > 0$ by (52). The second component $m(W,t)g_k$ recommends investment in asset k proportional to the expected rate of return α_k for k (and to the expected rates α_j for other assets correlated with it), as compared with the average expected rate for all assets. The factors h_k and g_k are determined entirely by the means, variances, and covariances of the relative rates of returns of the assets, and not by the utility function, the amount of wealth, and the time horizon. $m(W,t)$ certainly depends on the wealth and the utility function of individual i making the decision.

Since an individual's relative demand $\hat{w}_k$ for the k^{th} asset has only one parameter $m(W,t)$ which is affected by his wealth and his utility function, the demand can be satisfied by selection from shares of only two "mutual funds," the first holding a fraction δ_k of its value in asset k and the second a fraction λ_k , with

$$\begin{aligned} \delta_k &= h_k + (a-b)g_k \qquad (k = 1, \ldots, n) \\ \lambda_k &= h_k - bg_k \qquad (k = 1, \ldots, n) \end{aligned} \tag{54}$$

where a and b are arbitrary constants. Any value of $m(W,t)$ for an individual can always be met by a suitable linear combination of δ_k and λ_k , i.e., by

$$m = \theta(a-b) + (1-\theta)(-b) = \theta a - b$$

or by investing a fraction $\theta = (m+b)/a$ in the first fund and the remainder $(1-\theta)$ in the second fund. This is known as a <u>mutual fund theorem</u>.

If the n^{th} asset is riskless, $s_n = 0$, $\alpha_n = r$ and equation (38) replaces (37). We need only to solve for $m = n-1$ optimal control equations for $\hat{w}_k$, $k = 1, \ldots, m$, with $\hat{w}_n = 1 - \sum_1^m \hat{w}_k$. The Lagrangian multiplier in (41) and (46) disappears. In our derivations, m replaces n, $(\alpha_i - r)$ replaces α_i and (48) becomes

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1m} \\ & \cdot & \cdot & \cdot \\ \sigma_{m1} & \sigma_{m2} & \cdots & \sigma_{mm} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix} = = \frac{-V_W}{WV_{WW}} \begin{bmatrix} \alpha_1 - r \\ \alpha_2 - r \\ \vdots \\ \alpha_m - r \end{bmatrix} \qquad (55)$$

the solution of which is

$$\hat{w}_k = m(W,t) g_k \qquad (k = 1, \ldots, m) \qquad (56)$$

where

$$g_k = \sum_{j=1}^{m} \sigma^{kj} [\alpha_j - r] \qquad (k = 1, \ldots, m) \qquad (57)$$

To satisfy the demands $\hat{w}_k$ for any individual, there need be only two mutual funds, the first holding a fraction $\delta_k = (a-b) g_k$ of its value in asset k $(k = 1, \ldots, m)$ while the second holding a fraction $\lambda_k = -bg_k$. To achieve any $m(W,t) g_k$ desired, the individual again invests a fraction $\theta = (m+b)/a$ of his wealth in the first fund. We can let $b = 0$, so that the second fund holds only the riskless asset n and no risky assets. Thus only one mutual fund holding the risky assets by proportions ag_k, with $\sum_{k=1}^{m} ag_k = 1$, and a second fund holding only the riskless asset will satisfy the demand of any individual.

Let the n^{th} asset be money, with $r = 0$. The relative holdings of the risky fund are, by (57),

$$\delta_k = ag_k = a\sum_{j=1}^{m}\sigma^{kj}\alpha_j \qquad (k = 1, \ldots, m) \qquad (58)$$

where, to insure $\sum_{k=1}^{m} ag_k = 1$, $a = (\sum_{kj}\sum\sigma^{kj}\alpha_j)^{-1}$. The holdings (58) are in agreement with the traditional Tobin-Markowitz mean-variance analysis. We find the portfolio $\delta = (\delta_1, \ldots, \delta_m)'$ for the fund which minimizes the variance $\sigma^2 = \delta'\Sigma\delta$ of the rate of return for a given mean rate of return $m = \delta'\alpha$, with $\alpha = (\alpha_1, \ldots, \alpha_m)'$, yielding a function $\sigma(m)$. In the (m,σ) diagram, if we draw a line going through the origin and tangential to the curve $\sigma(m)$, we will find the portfolio δ given by (58).

The above mutual fund theorem and generalization of the mean-variance portfolio analysis were obtained without using a specific form for the utility function and without deriving the function $V(y,t)$ explicitly. The reader is referred to Merton (1969) and (1971) for explicit solutions of $V(y,t)$ and for further discussions of the economics of this problem, and to Rosenberg and Ohlson (1976) for a critique of the assumption that the rates of return are stationary and serially independent.

7. Capital Asset Pricing With Shifts in Investment Opportunities

One variation of the model of section 5 suggested by Merton (1973) is to introduce a new vector $X = (x_1, \ldots, x_N)$ of N state variables to replace the vector P of asset prices. X may include all, some or none elements of P.

It may include the parameters α_i in equation (35) which will themselves be assumed to satisfy the stochastic differential equations

$$d\alpha_i = a_i dt + b_i dq_i \tag{59}$$

where dq_i are Wiener processes with unit variance. Let us write the stochastic differential equations for the elements of this new vector as

$$dx_i = f_i(X)dt + g_i^*(X)dq_i \qquad (i = 1, \ldots, N) \tag{60}$$

Let $E(dq_i dz_j) = \eta_{ij}$ and $E(dq_i dq_j) = v_{ij}$, dz_j being defined for (35), with $E(dz_i dz_j) = \rho_{ij}$. We further let the n^{th} asset be "instantaneously riskless" in the sense of $s_n = 0$ and $\alpha_n = r(t)$ in (35), but $b_n \neq 0$ in (59) for $d\alpha_n = dr$.

Assume that each individual maximizes utility over time as in section 5. The present variation, with $y = (W,X)$, leads to a slight modification of (40),

$$\begin{aligned} \mathcal{L}_y[V(y,t)] = {} & \frac{\partial V}{\partial t} + \{[\sum_1^m w_i(\alpha_i - r) + r]W - c\} V_w + \sum_1^N f_i V_i \\ & + \frac{1}{2}\sum_1^m\sum_1^m \sigma_{ij} w_i w_j W^2 V_{ww} + \frac{1}{2}\sum_1^N\sum_1^N g_i^* g_j^* v_{ij} V_{ij} \\ & + \sum_{i=1}^N \sum_{j=1}^m g_i^* w_j W s_j \eta_{ij} V_{iw} \end{aligned} \tag{61}$$

where $m = n-1$ as before and V_i denotes partial derivative with respect to the i-th element of X. Equation (43) becomes

$$WV_{ww}\sum_1^m \sigma_{kj} w_j + V_w(\alpha_k - r) + \sum_1^N g_j^* s_k \eta_{jk} V_{jw} = 0 \qquad (k = 1, \ldots, m) \tag{62}$$

The solution of this linear system of equations for w_k yields

$$\hat{w}_k W = A \sum_{i=1}^{m} \sigma^{ki}(\alpha_i - r) + \sum_{i=1}^{m} \sum_{j=1}^{N} H_j g_j^* s_i \eta_{ji} \sigma^{ki} \qquad (k = 1, \ldots, m) \qquad (63)$$

where $A = -V_W/V_{WW}$ and $H_j = -V_{jW}/V_{WW}$. The first term of this demand function is the same as given by (57). To interpret the second term, note that $V_W = U_C$ by (42) and hence $V_{WW} = U_{CC}\frac{\partial C}{\partial W}$ and $V_{jW} = U_{CC}\frac{\partial C}{\partial x_j}$, implying

$$H_j = -\frac{\partial C}{\partial x_j} \Big/ \frac{\partial C}{\partial W} \qquad (64)$$

If the j-th state variable has a negative or "unfavorable" effect on consumption, i.e., $\frac{\partial C}{\partial x_j} < 0$, H_j will be positive. Since $(g_j^* s_i \eta_{ji})$ is the covariance between dx_j and dP_i, the expression $H_j \sum_{i=1}^{m} (g_j^* s_i \eta_{ji})\sigma^{ki}$ measures the investment in asset k to hedge against the unfavorable effect of state variable j acting through its correlation with P_i for all $i = 1, \ldots, m$.

Consider the special case when the vector X of state variables consists only of $\alpha_n = r$ which affects the mean rates of return $\alpha_i(X)$ of the assets $i = 1, \ldots, m$. (63) becomes

$$\begin{aligned} \hat{w}_k W &= A \sum_{i=1}^{m} \sigma^{ki}(\alpha_i - r) + H_r \sum_{i=1}^{m} \mathrm{Cov}(dr, dP_i/P)\sigma^{ki} \\ &= A g_k + H_r d_k \qquad (k = 1, \ldots, m) \qquad (65) \end{aligned}$$

and $\hat{w}_n = 1 - \sum_{1}^{m} \hat{w}_k$, where g_k and d_k are independent of the individual's utility function and wealth. (65) is a generalization of the asset demand functions (56)-(57). Since there is an additional term H_r which depends on the individual's utility function and wealth, any individual's demand can be satisfied by <u>three</u>

mutual funds. Let the first fund hold a fraction $\delta_k = ag_k$ of its value in asset k, $k = 1, \ldots, m$. Let the second fund hold only the "instantaneously riskless" asset n. Let the third fund hold a fraction cd_k in asset k. The demand functions (65) for any individual will be satisfied by investing proportions θ_1, $(1-\theta_1-\theta_3)$ and θ_3 in the three funds respectively, where

$$\theta_1 ag_k + \theta_3 cd_k = \frac{A}{W} g_k + \frac{H_r}{W} d_k \qquad (k = 1, \ldots, m)$$

or $\theta_1 = A/Wa$ and $\theta_3 = H_r/Wc$. Summing the above equations over k, we get $\theta_1 + \theta_3 = \sum_1^m \hat{w}_k = 1-\hat{w}_n$, which insures that the demand $\hat{w}_n$ for the "instantaneously riskless" asset n can be met by investing the remaining proportion $1-\theta_1-\theta_3$ of the individual's wealth in the second mutual fund. Comparison of (63) and (65) shows that if there are two state variables shifting the mean rates of return or investment opportunities which one would wish to hedge against, i.e., $N = 2$ in the model, there will be one extra term in (65) and four mutual funds will be required.

This analysis provides a theory of mutual funds. The first type of funds holds a portfolio δ_k proportional to g_k in the demand function (65). The second holds an instantaneously riskless asset like a short-term government bond. Each of the remaining funds holds a collection of capital assets to hedge against one type of contingencies. If there were no transaction costs, the individual could make up the collection himself. Since there are transaction costs, each fund provides a service in offering the required collection of assets.

This analysis also provides an equilibrium theory of market prices of the m capital assets, interpreted as securities of m firms. Let the demand functions (65) for asset k by individual i be written as

$$D_k^i = A^i \sum_{j=1}^{m} \sigma^{kj}(\alpha_j - r) + H_r^i d_k = A^i g_k + H^i d_k \qquad (k = 1, \ldots, m) \qquad (66)$$

The market demand for the asset of firm k is the sum of above over all individuals i, i.e.,

$$D_k = \sum_i D_k^i = (\sum_i A^i) g_k + (\sum_i H^i) d_k = A g_k + H d_k \qquad (67)$$

If we redefine w_k to be the ratio of the value of the assets of firm k to the total market value M of the assets of all firms, then in equilibrium $D_k = w_k M$. Given $D_k = w_k M$, we use (67) to solve for the equilibrium expected rates of return α for assets k, noting $g_k = \sum_{j=1}^{m} \sigma^{kj}(\alpha_j - r)$ in (67). The solution of these linear equations is

$$\alpha_k - r = \left(\frac{M}{A}\right) \sum_{j=1}^{m} w_j \sigma_{kj} - \frac{H}{A} \sum_{j=1}^{m} d_j \sigma_{kj} \qquad (k = 1, \ldots, m) \qquad (68)$$

which provides an explanation of the expected rate of return of an asset k. Since $\sum_j w_j \sigma_{kj}$ is the covariance of the (instantaneous) rate of return of asset k and the aggregate of the rates of return of all assets in the market (i.e., the aggregate rate of the market portfolio), this covariance being known as the "beta" of the k^{th} asset in the finance literature, the first term of (68) requires a higher expected rate of return for asset k insofar as its price change varies with those of the entire collection of risky assets in the market. Recall that d_j represents the portfolio of the third mutual fund which can be used to hedge against the shifts in expected returns. $\sum_j d_j \sigma_{kj}$ is thus the covariance of the rate of return for asset k and the rate for this fund. The second term of (68) justifies a lower expected rate of return for asset k insofar as it serves the hedging function provided by the third mutual fund. For further discussion of equilibrium capital asset pricing, the reader may refer to Long (1974).

(72) and (70) yield the differential equation for $w(P,t)$

$$\frac{\partial w}{\partial t} = rw - rP\frac{\partial w}{\partial P} - \frac{1}{2}s^2P^2\frac{\partial^2 w}{\partial P^2} \tag{73}$$

The solution of (73) will give the function w for the pricing of call options. The boundary condition of this problem is $w(P,T) = P - c$ for $P \geq c$ and $w(P,T) = 0$ for $P < c$, c being the exercise price.

As pointed out by Cox and Ross (1976), the solution can be obtained alternatively by assuming that there exist risk neutral investors, so that the price of the stock will follow (69) with $\alpha = r$. The option price at time t will be its expected price at time T discounted back to t, namely,

$$w(P,t) = e^{-r(T-t)}Ew(P,T) \tag{74}$$

Let $X = \ell nP$. Using (69) and Ito's differentiation rule, we have

$$dX = \left(\frac{d\log P}{dP}rP + \frac{1}{2}\frac{d^2\log P}{dP^2}s^2P^2\right)dt + \frac{d\log P}{dP}sPdz = \left(r - \frac{1}{2}s^2\right)dt + sdz \tag{75}$$

Given $X_t = \log P_t$, (75) implies that the distribution of X_T is normal with mean $X_t + (r - \frac{1}{2}s^2)(T-t)$ and variance $s^2(T-t)$. The price of the option at time T will be zero if $P_T < c$, and it will be $P_T - c$ if $P_T \geq c$. Therefore, the expected price of the option at T is

$$Ew(P,T) = \int_c^{\infty}(P_T - c)\,pdf(P_T)dP_T \tag{76}$$

where pdf stands for the probability density function.

Since the pdf of $X_T = \log P_T$ is normal with mean and variance given above, (76) can be written as

$$Ew(P,T) = \int_{\log c}^{\infty} (e^x - c)\frac{1}{\sqrt{2\pi}\, s\sqrt{T-t}} \exp\left\{-\frac{1}{2}\cdot\frac{[x - x_t - (r-\frac{1}{2}s^2)(T-t)]^2}{s^2(T-t)}\right\} dx \qquad (77)$$

Substituting (77) into (74) and simplifying, we obtain the solution

$$w(P,t) = e^{-r(T-t)} \int_{\log c}^{\infty} \frac{1}{\sqrt{2\pi}\, s\sqrt{T-t}} \exp\left\{-\frac{1}{2}\cdot\frac{[x-x_t-(r+\frac{1}{2}s^2)(T-t)]^2}{s^2(T-t)} + x_t + r(T-t)\right\} dx$$

$$- e^{-r(T-t)}\, c \int_{\log c}^{\infty} \frac{1}{\sqrt{2\pi}\, s\sqrt{T-t}} \exp\left\{-\frac{1}{2}\cdot\frac{[x-x_t-(r-\frac{1}{2}s^2)(T-t)]^2}{s^2(T-t)}\right\} dx$$

$$= P_t N\left(\frac{\ell n[P_t/c]+[r+\frac{1}{2}s^2][T-t]}{s\sqrt{T-t}}\right) - ce^{-r(T-t)} N\left(\frac{\ell n[P_t/c]+[r-\frac{1}{2}s^2][T-t]}{s\sqrt{T-t}}\right) \qquad (78)$$

where N stands for the cumulative unit normal distribution function. Black and Scholes (1973) have pointed out that by considering corporate liabilities as combinations of options, the pricing formula (78) can be applied to corporate liabilities such as common stock, corporate bonds and warrants.

9. Optimal Control of a Linear System with Quadratic Loss

The continuous-time version of the optimal linear-quadratic control problem assumes a linear model

$$dy = A(t)ydt + C(t)xdt + dv \qquad (79)$$

where $dv = Sdz$ has covariance matrix $\Sigma dt = SS'dt$, z being a multivariate

Wiener process with Idt as its incremental covariance matrix, and a quadratic loss function

$$W(y,x,t) = \frac{1}{2}[(y-a_1)'K_1(t)(y-a_1) + (x-a_2)'K_2(t)(x-a_2)]$$

$$= \frac{1}{2}y'K_1y - y'K_1a_1 + \frac{1}{2}x'K_2x - x'K_2a_2 + \frac{1}{2}a_1'K_1a_1 + \frac{1}{2}a_2'K_2a_2$$

$$= \frac{1}{2}y'K_1y - y'k_1 + \frac{1}{2}x'K_2x - x'k_2 + d(t) \qquad (80)$$

The problem is to find

$$V(y,t) = \underset{x}{\text{Min}}\, E_t\left[\int_t^T W(y,x,t)dt + \frac{1}{2}y'(T)K_0y(T) - y'(T)k_0 + d_0\right] \qquad (81)$$

Applying the optimality condition derived from dynamic programming as stated in (24) we have

$$\underset{x}{\text{Min}}\,\{\frac{1}{2}y'K_1y - y'k_1 + \frac{1}{2}x'K_2x - x'k_2 + d + \mathcal{L}_y[V(y,t]\} = 0 \qquad (82)$$

We use the differential generator (34) for $\mathcal{L}_y[\ ]$ in (82) to obtain

$$\underset{x}{\text{Min}}\{\frac{1}{2}y'K_1y - y'k_1 + \frac{1}{2}x'K_2x - x'k_2 + d + \frac{\partial V}{\partial t} + (\frac{\partial V}{\partial y})'[A(t)y + C(t)x] + \frac{1}{2}\text{tr}\left(\Sigma\frac{\partial^2 V}{\partial y\partial y'}\right)\} = 0 \qquad (83)$$

Differentiating the expression in curly brackets with respect to x yields

$$\frac{\partial\{\ \}}{\partial x} = K_2x - k_2 + C'\frac{\partial V}{\partial y} = 0$$

which implies the optimal control equation

$$x = -K_2^{-1}C' \frac{\partial V}{\partial y} + K_2^{-1}k_2 \tag{84}$$

When (84) is substituted for x in (83), we have

$$\frac{1}{2} y'K_1y - \frac{1}{2}\left(\frac{\partial V}{\partial y}\right)' CK_2^{-1}C'\left(\frac{\partial V}{\partial y}\right) - y'k_1 + \left(\frac{\partial V}{\partial y}\right)' CK_2^{-1}k_2 - \frac{1}{2}k_2'K_2^{-1}k_2$$
$$+ d + \left(\frac{\partial V}{\partial y}\right)' Ay + \frac{1}{2}\mathrm{tr}\left(\Sigma \frac{\partial^2 V}{\partial y \partial y'}\right) = - \frac{\partial V}{\partial t} \tag{85}$$

The partial differentiation equation (85) is to be solved.

From knowledge of the solution for V in the discrete-time formulation of this problem, let us try the quadratic function

$$V = \frac{1}{2}y'H(t)y - y'h(t) + c(t) \tag{86}$$

The appropriate derivatives of (86) can be substituted into (85), giving

$$\frac{1}{2} y'K_1y - \frac{1}{2} y'HCK_2^{-1}C'Hy + y'HCK_2^{-1}C'h - \frac{1}{2}h'CK_2^{-1}C'h - y'k_1 + y'HCK_2^{-1}k_2$$
$$- h'CK_2^{-1}k_2 - \frac{1}{2}k_2'K_2^{-1}k_2 + d + \frac{1}{2}y'HAy + \frac{1}{2}y'A'Hy - h'Ay + \frac{1}{2}\mathrm{tr}(\Sigma H)$$
$$= -\frac{1}{2}y'\left(\frac{dH}{dt}\right)y + y'\frac{dh}{dt} - \frac{dc}{dt} \tag{87}$$

(87) implies the following differential equations for H, h, and c,

$$-\frac{dH}{dt} = K_1 - HCK_2^{-1}C'H + HA + A'H \tag{88}$$

$$\frac{dh}{dt} = HCK_2^{-1}C'h - A'h - k_1 + HCK_2^{-1}k_2 \tag{89}$$

$$\frac{dc}{dt} = \frac{1}{2}h'CK_2^{-1}C'h + h'CK_2^{-1}k_2 + \frac{1}{2}k_2'K_2^{-1}k_2 - d - \frac{1}{2}tr(\Sigma H) \tag{90}$$

These differential equations are to be solved with the boundary conditions $H(T) = K_0$, $h(T) = k_0$ and $c(T) = d_0$. Having found the parameters $H(t)$, $h(t)$ and $c(t)$ of V, we can evaluate $\frac{\partial V}{\partial y}$ for the optimal control equation (84) as $H(t)y - h(t)$.

10. Optimum Use and Exploration of a Natural Resource

A classic paper of Hotelling (1931) deals with the optimum rate of consuming an exhaustible resource over time when the total reserve of the resource is known. This section attempts to find a solution to the optimum use and extraction of an exhaustible resource when the amount of the reserve is unknown.

In the certainty case, let $x(t)$ be the rate of consumption and $y(t)$ be the known stock of reserves at time t. The differential equation is

$$dy = -\,xdt \tag{91}$$

Given a utility function $u(x,t)$ the problem is to find

$$V(y,t) = \max_x \left[\int_t^T u(x,s)ds + B(y(T),T) \right] \tag{92}$$

By the method of dynamic programming, we need to solve

$$\max_{x} \{ u(x,t) + \frac{\partial V}{\partial t} - x \frac{\partial V}{\partial y} \} = 0 \tag{93}$$

Differentiation yields the first-order condition

$$\frac{\partial u}{\partial x} = \frac{\partial V}{\partial y} \tag{94}$$

which equates the marginal utility of consuming the resource at each point in time and the shadow price of the stock of reserve. Denoting the function $\frac{\partial u}{\partial x} = u_x$ by $G(x,t)$, we write the solution of (94) as

$$x = G^{-1} (\frac{\partial V}{\partial y} , t) = G^{-1} (V_y,t) \tag{95}$$

When (95) is substituted into (93), the resulting partial differential equation can be solved for $V(y,t)$.

To illustrate, let $u = e^{-\rho t}(x - \frac{1}{2} \beta x^2)$. This problem becomes a special case of the problem of Section 9, with $K_1 = 0$, $k_1 = 0$, $A = 0$, $C = -1$, $K_2 = e^{-\rho t}\beta$, $k_2 = e^{-\rho t}$, $d = 0$ and $\Sigma = 0$. The loss function W of (80) is $-u$. Therefore, if we redefine the function V to measure total expected utility instead of total expected loss, but retain the definitions for H, h and c, we change the sign of the right-hand side of (86):

$$V(y,t) = - \frac{1}{2} H(t)y^2 + h(t)y - c(t) \tag{96}$$

By (88), (89) and (90), the differential equations for H, h and c are

$$- \frac{dH}{dt} = -\beta^{-1}e^{\rho t}H^2 \tag{97}$$

$$\frac{dh}{dt} = \beta^{-1}e^{\rho t}Hh - \beta^{-1}H \tag{98}$$

$$\frac{dc}{dt} = \frac{1}{2}\beta^{-1}e^{\rho t}h^2 - \beta^{-1}h + \frac{1}{2}\beta^{-1}e^{-\rho t} \tag{99}$$

The terminal utility is assumed to be

$$-\frac{1}{2}y'(T)K_0y(T) + y'(T)k_0 - d_0 = -\frac{1}{2}\lambda y^2(T) \tag{100}$$

with $K_0 = \lambda$, $k_0 = 0$ and $d_0 = 0$. We will let λ be extremely large to penalize any non-zero $y(T)$ and to insure that the resource is used up at time T.

The solution to (97) is $H(t) = -\beta\rho(e^{\rho t}+\alpha)^{-1}$ where α is a constant of integration. To determine α, we use the condition $H(T) = K_0 = \lambda$, yielding $\alpha = -e^{\rho T}-\beta\rho/\lambda$. The solution to (97) is therefore

$$H(t) = \beta\rho(e^{\rho T} - e^{\rho t} + \beta\rho/\lambda)^{-1} \tag{101}$$

Similarly, the solution to (98) is

$$h(t) = \rho(T-t)(e^{\rho T} - e^{\rho t} + \beta\rho/\lambda)^{-1} \tag{102}$$

which satisfies the terminal condition $h(T) = k_0 = 0$. Having found the coefficients $H(t)$ and $h(t)$ of the quadratic function V, we can use (94) to obtain the optimum consumption function

$$x(t) = \beta^{-1}\{1 + e^{\rho t}[H(t)y(t) - h(t)]\} \tag{103}$$

which is linear in the stock of reserve y. The shadow price of the reserve stock can be obtained as $\frac{\partial V}{\partial y} = h(t)-H(t)y$.

The above solution is provided partly to illustrate the method of section 9. Note, however, that it ignores the restrictions that $y(t) \geq 0$. An alternative method of solving this problem is to utilize the optimality condition that the marginal utility of consumption should be the same at all time, i.e.,

$$\frac{du}{dx} = k = e^{-\rho t}(1 - \beta x) \tag{104}$$

where k is const nt through time and is chosen to exhaust all the resource at T. Hence the rate of consumption is

$$x(t) = \beta^{-1}(1 - ke^{\rho t}) \qquad (105)$$

and total consumption from t to T is

$$\int_t^T x(s)\,ds = \beta^{-1}[T-t-k\rho^{-1}(e^{\rho T}-e^{\rho t})] = y(t) \qquad (106)$$

yielding

$$k = \rho(e^{\rho T} - e^{\rho t})^{-1}[T-t-\beta y(t)]$$

and accordingly the optimal consumption function

$$x(t) = \beta^{-1}\{1 + e^{\rho t}\rho(e^{\rho T} - e^{\rho t})^{-1}[\beta y(t) - (T-t)]\} \qquad (107)$$

which agrees with (103). The function $V(y,t)$ can be obtained as $\int_t^T u(x(s))\,ds$.

When the total stock of reserves in the ground is unknown, a simple assumption is that it is distributed at random over the surface of the earth, with an expected number λ of hidden reserves per square mile. The reserves are assumed to take only discrete values. The quantity z to be discovered in n square miles is assumed to follow a Poisson distribution

$$f(z|\lambda,n) = \frac{e^{-n\lambda}(n\lambda)^z}{z!} \qquad (108)$$

The parameter λ is unknown.

To provide a model of learning about λ, let us consider for the moment that decisions on consumption and exploration of the resource are made in discrete time. We assume for analytical convenience that the prior distribution of λ_t at the beginning of period t is gamma with parameters s_t and r_t

(time subscript to be omitted when understood).

$$g(\lambda | s, r) = \frac{e^{-s\lambda} \lambda^{r-1} s^r}{(r-1)!} \tag{109}$$

The conditional distribution of λ given z units being discovered in n squared miles is, by the Bayes theorem,

$$g(\lambda | z; s,r,n) = \frac{f(z|\lambda,n) g(\lambda|s,r)}{\int_0^{\infty} \text{numerator } d\lambda} \tag{110}$$

The marginal distribution of z is

$$\int_0^{\infty} f(z|\lambda,n) g(\lambda|s,r) d\lambda = \frac{n^z}{z!} \cdot \frac{s^r}{(r-1)!} \cdot \int_0^{\infty} e^{-(s+n)\lambda} \lambda^{r+z-1} d\lambda$$

or

$$\hat{f}(z|s, r, n) = \frac{n^z}{z!} \cdot \frac{s^r}{(r-1)!} \cdot \frac{(r+z-1)!}{(s+n)^{r+z}} \tag{111}$$

Note that the function g given by (109) is a natural conjugate prior density function for the parameter λ of the Poisson distribution. The former distribution has parameters s_t and r_t . After n_t square miles are explored and z_t units of the resource are found, the posterior density function of λ as given by (110) has the same form, but has new parameters

$$s_{t+1} = s_t + n_t \tag{112}$$

and

$$r_{t+1} = r_t + z_t \tag{113}$$

Thus s_t and r_t can be interpreted respectively as the total number of square miles explored and the total quantity of the resource discovered up to time t. The marginal distribution function (111) of z, rewritten below, is a negative binomial distribution

$$\hat{f}(z|s,r,n) = \frac{(z+r-1)!}{z!\ (r-1)!}\left(\frac{n}{s+n}\right)^{z}\left(\frac{s}{s+n}\right)^{r} \tag{114}$$

Having considered the problem of resource use and exploration in discrete time, we will reformulate the problem in continuous time by letting the time interval h between successive decisions become small. Let $x_2(t)$ be the number of square miles to be explored per unit time at time t. In a small time interval h, $x_2(t)h$ is the number n_t of square miles explored; n_t is small compared with the total square miles s_t having been explored up to that point in history. According to (114), the probability of discovering no resource during the time interval h is

$$P(z=0) = \left(\frac{s}{s+x_2h}\right)^{r} = \left(1+\frac{x_2h}{s}\right)^{-r} = [e^{\frac{x_2h}{s}} + o(h)]^{-r}$$

$$= e^{-\frac{r}{s}x_2h} + o(h) = 1 - \frac{r}{s}x_2h + o(h) \tag{115}$$

The probability of discovering one unit of the resource during time h is, again by (114),

$$P(z=1) = r\left(\frac{x_2h}{s+x_2h}\right)\left(\frac{s}{s+x_2h}\right)^{r}$$

$$= r\left(\frac{x_2h}{s} + o(h)\right)\left(1 - \frac{r}{s}x_2h + o(h)\right)$$

$$= \frac{r}{s} x_2 h + o(h) \tag{116}$$

Thus in a very small time interval h, the probability of finding no resource is $1 - \frac{r}{s} x_2 h$; the probability of finding one unit of the resource is $\frac{r}{s} x_2 h$; and the probability of finding two or more units can be ignored. We have just specified a Poisson process dv with parameter $\lambda x_2 = \frac{r}{s} x_2$. This process generates an outcome "one" with probability $\lambda x_2 dt$ and an outcome zero with probability $1-\lambda x_2 dt$ during a time interval $dt = h$ if an exploratory effort x_2 is applied.

Our model so far has three state variables: the quantity $y_1(t)$ of known reserves in stock, the amount $s(t)$ of land already explored, and the total quantity $r(t)$ of the resource ever discovered up to time t. The two control variables are the rate $x_1(t)$ of consumption and the rate $x_2(t)$ of exploratory effort. We will replace the second state variable by $y_2(t) = L - s(t)$, where L is total explorable land; $y_2(t)$ thus denotes the amount of land as yet unexplored. The state variables satisfy the stochastic differential equations

$$\begin{aligned} dy_1 &= -x_1 dt + dv \\ dy_2 &= -x_2 dt \\ dr &= dv \end{aligned} \tag{117}$$

The objective is to find

$$V(y,t) = \max_x E_t \left[\int_t^T u(y,x,s)ds + B(y(T),T) \right] \tag{118}$$

where y denotes the vector of state variables and x the vector of control variables.

Since the stochastic differential equations involve the Poisson process dv , we will derive the optimality condition for a model of the form

$$dy = f(y,x,t)dt + g(y,x,t)dv \tag{119}$$

where f and g are vector functions. In the special case of the model (117),

$$f = \begin{bmatrix} -x_1 \\ -x_2 \\ 0 \end{bmatrix} \qquad g = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \tag{120}$$

By the method of dynamic programming, we need to solve

$$\max_x \{u(y,x,t) + E_t(\frac{1}{h}\, dV)\} = 0 \tag{121}$$

where

$$\begin{aligned} dV &= V(y(t+dt),t+dt) - V(y(t),t) \\ &= \frac{\partial V}{\partial t} dt + \left(\frac{\partial V}{\partial y}\right)' dy + o(dt) \\ &= \frac{\partial V}{\partial t} dt + \left(\frac{\partial V}{\partial y}\right)' fdt + \left(\frac{\partial V}{\partial y}\right)' gdv + o(dt) \end{aligned} \tag{122}$$

Note that dv has probability $\lambda x_2 dt$ of being one (or being W in a more general formulation with W having some given probability distribution) and probability $(1 - \lambda x_2 dt)$ of being zero. Hence

$$E_t\left(\frac{1}{dt}dV\right) = \frac{\partial V}{\partial t} + \left(\frac{\partial V}{\partial y}\right)' f + \lambda x_2[V(y(t)+g,t) - V(y(t),t)] \tag{123}$$

When (123) is substituted into (121) for our model (120), we have

$$\max_{x} \{u(x,t) + \frac{\partial V}{\partial t} - x_1 \frac{\partial V}{\partial y_1} - x_2 \frac{\partial V}{\partial y_2} + \lambda x_2[V(y_1+1, y_2, r+1, t) - V(y,t)]\} = 0 \tag{124}$$

We will study a simplified version of the problem (124) by ignoring the third state variable r which is the quantity of the resource discovered up to t. This state variable helps us construct the estimate $\lambda = \frac{r}{s} = \frac{r}{L-y_2}$. In other words, we are ignoring the possibility of active learning about λ in the future while utilizing only the current value of $\lambda(t)$ as if it would remain constant in the future. The problem (124) will then become

$$\max_{x} \{u(x,t) + \frac{\partial V}{\partial t} - x_1 \frac{\partial V}{\partial y_1} - x_2 \frac{\partial V}{\partial y_2} + \lambda x_2 \Delta V_1\} = 0 \tag{125}$$

where $\Delta V_1 = V(y_1+1, y_2, t) - V(y_1, y_2, t)$. Compare (125) with (93).

If $u(x,t) = e^{-\rho t}u(x)$, ρ being the rate of discount, and if the planning horizon is infinite, we can write $V(y,t)$ as $e^{-\rho t}V(y)$ since the expected total utilities V for a given initial state y at two different points of time differ only by the discounting factor. Substituting $e^{-\rho t}u(x)$ for $u(x,t)$ and $e^{-\rho t}V(y)$ for $V(y,t)$ in (125) gives

$$\max_{x} \{u(x) - x_1 \frac{\partial V}{\partial y_1} - x_2 \frac{\partial V}{\partial y_2} + \lambda x_2 \Delta V_1\} = \rho V(y) \tag{126}$$

We solve (126) by differentiation, using subscripts to denote partial derivatives

$$\frac{\partial\{\ \}}{\partial x_1} = u_1 - V_1 = 0 \tag{127}$$

$$\frac{\partial\{\ \}}{\partial x_2} = u_2 - V_2 + \lambda\Delta V_1 = 0 \qquad (128)$$

V_1 and V_2 are the shadow prices of the known stock of reserves and of unexplored land respectively. The price $V_1 = p_1$ of the reserve is equated to its marginal utility. The price $V_2 = p_2$ of the unexplored land is equated to the sum of its marginal utility u_1 (actually the negative marginal cost of exploration per square mile) and the expected gain $\lambda\Delta V_1$ of discovering new resource. We can write $u(x)$ as the sum $u(x_1) - c(x_2)$ where $c(x_2)$ is the cost of exploring x_2 square miles of land per unit time. The marginal utility $u_1(x_1)$ is a decreasing function of x_1. Denoting the inverse of the function u_1 by u_1^{-1}, the solution of (127) is

$$x_1 = u_1^{-1}(p_1) \qquad (129)$$

The marginal cost $\frac{dc}{dx_2} = -u_2(x_2)$ is assumed to be a nondecreasing function of x_2. The solution of (128) is

$$x_2 = -u_2^{-1}[-p_2 + \lambda p_1] \qquad (130)$$

showing that the exploratory effort x_2 will increase as the price p_2 of land is lower, as the density λ of deposits is higher and as the value p_1 of the resource is higher.

To study the dynamics of the prices following the work of Arrow (1977), we differentiate (126) with respect to y_1 and y_2 respectively, obtaining

$$-x_1V_{11} - x_2V_{12} + \lambda x_2[V_1(y_1+1,y_2) - V_1(y_1,y_2)] = \rho V_1 \qquad (131)$$

$$-x_1V_{12} - x_2V_{22} + \lambda x_2[V_2(y_1+1,y_2) - V_2(y_1,y_2)] = \rho V_2 \qquad (132)$$

In differentiating (126), we treat the control variables x_1 and x_2 as constants. The reader can verify the results (131) and (132) by treating x_1 and x_2 as functions of y_1 and y_2 and utilizing the first-order conditions (127) and (128). Consider the price of the resource at time $t+dt$. It will depend on whether additional resource is discovered during the time interval dt since the argument of the function V_1 takes different values in the two cases:

$$
\begin{aligned}
V_1(t+dt) &= V_1(y,t) + V_{11}dy_1 + V_{12}dy_2 \\
&= V_1(y,t) - V_{11}x_1 dt - V_{12}x_2 dt \quad \text{with prob. } (1 - \lambda x_2 dt) \qquad (133) \\
V_1(t+dt) &= V_1(y_1+1, y_2) \qquad \text{with prob. } \lambda x_2 dt
\end{aligned}
$$

Using (131) to substitute $\lambda x_2[V_1(y_1+1,y_2) - V_1(y_1,y_2)] - \rho V_1$ for $x_1V_{11} + x_2V_{12}$ in (133), one finds the expectation

$$
\begin{aligned}
E\left[\frac{V_1(t+dt) - V_1(t)}{dt}\right] &= (1 - \lambda x_2 dt)[-x_1V_{11} - x_2V_{22}] \\
&\quad + \lambda x_2 dt[V_1(y_1+1,y_2) - V_1(y_1,y_2)]/dt \\
&= \rho V_1 + (\lambda x_2)^2[V_1(y_1+1,y_2) - V_1(y_1,y_2)]dt - \lambda x_2\rho V_1 dt \qquad (134)
\end{aligned}
$$

By taking the limit of (134) as dt approaches zero, one finds that the expected proportional rate of increase in the price of the resource to be the rate of discount ρ. This conclusion generalizes a conclusion of Hotelling (1931) for the case with known quantity of the exhaustible resource.

An explicit solution to this problem can be obtained if we assume a quadratic loss function and a finite time horizon T. Equation (125) then becomes

$$0 = \underset{x}{\text{Min}} \{ \frac{1}{2} y'K_1(t)y - y'k_1(t) + \frac{1}{2} x'K_2(t)x - x'k_2(t) + d(t)$$

$$+ \frac{\partial V}{\partial t} + \left(\frac{\partial V}{\partial y} \right)' (Ay + Cx) + (\Delta V)'Dx\} \qquad (135)$$

(135) is a formmlation of the optimization problem which includes our problem of exhaustible resource as a special case if we assume

$$K_1(t) = 0 \quad k_1(t) = 0 \quad A = 0 \quad C = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\Delta V = \begin{bmatrix} \Delta V_1 \\ \Delta V_2 \end{bmatrix} = \begin{bmatrix} V(y_1+1, y_2) - V(y_1,y_2) \\ V(y_1, y_2+1) - V(y_1,y_2) \end{bmatrix} \quad D = \begin{bmatrix} 0 & \lambda \\ 0 & 0 \end{bmatrix} \qquad (136)$$

Finding the minimum of (135) by differentiation yields

$$\frac{\partial\{\ \}}{\partial x} = K_2x - k_2 + C'\left(\frac{\partial V}{\partial y}\right) + D'\Delta V = 0$$

which gives the optimal feedback control equation

$$x = -K_2^{-1} C'\left(\frac{\partial V}{\partial y}\right) - K_2^{-1} D'\Delta V + K_2^{-1} k_2 \qquad (137)$$

when (137) is substituted for x in (135), we obtain

$$\frac{1}{2} y'K_1y - y'k_1 - \frac{1}{2}\left(\frac{\partial V}{\partial y}\right)' CK_2^{-1}C'\left(\frac{\partial V}{\partial y}\right) - \frac{1}{2}\Delta V'DK_2^{-1}D'\Delta V + \left(\frac{\partial V}{\partial y}\right)' Ay \left(\frac{\partial V}{\partial y}\right)' CK_2^{-1}k_2$$

$$+ \Delta V'DK_2^{-1}k_2 - \left(\frac{\partial V}{\partial y}\right)' CK_2^{-1}D'\Delta V + d - \frac{1}{2} k_2'K_2^{-1}k_2 = -\frac{\partial V}{\partial t} \qquad (138)$$

(138) is a partial differential equation to be solved.

From knowledge of linear-quadratic control theory, we can try a quadratic function for the solution,

$$V = \frac{1}{2} y'H(t)y - y'h(t) + c(t) \tag{139}$$

The derivatives and difference of (139) are

$$\frac{\partial V}{\partial y} = Hy - h \tag{140}$$

$$\Delta V = Hy - h - h_d \tag{141}$$

where h_d is a vector composed of the diagonal elements of H, and

$$\frac{\partial V}{\partial t} = \frac{1}{2} y' \frac{dH}{dt} y - y' \frac{dh}{dt} + \frac{dc}{dt} \tag{142}$$

Substituting these derivatives into (138) and equating coefficients of the quadratic functions on both sides of the resulting equation, we obtain

$$-\frac{dH}{dt} = K_1 - H(CK_2^{-1}C' + DK_2^{-1}D' + CK_2^{-1}D' + DK_2^{-1}C')H + HA + A'H \tag{143}$$

$$\begin{aligned}\frac{dh}{dt} = & -k_1 + H(CK_2^{-1}C' + DK_2^{-1}D' + CK_2^{-1}D' + DK_2^{-1}C')h - A'h \\ & + H(CK_2^{-1}k_2 + DK_2^{-1}k_2 + DK_2^{-1}D'h_d + CK_2^{-1}D'h_d)\end{aligned} \tag{144}$$

$$\begin{aligned}-\frac{dc}{dt} = & -\frac{1}{2} h'CK_2^{-1}C'h - h'CK_2^{-1}k_2 - \frac{1}{2}(h'+h_d')DK_2^{-1}D'(h+h_d) \\ & - k_2'K_2^{-1}D'(h+h_d) - h'CK_2^{-1}Ch - h'CK_2^{-1}D'(h+h_d) + d - \frac{1}{2} k_2'K_2^{-1}k_2\end{aligned} \tag{145}$$

By introducing appropriate terminal conditions analogous to the ones given in the beginning of this section, one can solve these differential equations to obtain the function V of (139).

This section has illustrated the method of dynamic programming as applied to a continuous time model governed by a Poisson process. It has treated some useful methods that are applicable to the optimum use and exploration of an exhaustible resource, while leaving the discussion of many economic issues to be further explored.

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