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I. INTRODUCTION

This paper unifies the two leading classical concepts of equilibrium for an economy: Walras equilibrium and Cournot equilibrium. The theory provides a fresh setting for the study of competitive markets, and leads to a description of economic equilibrium which differs in substance from the one offered by modern formal competitive theory (see, eg, Debreu [3]).

Because modern formal competitive theory does not permit free entry, the number of firms is essentially fixed. Further, it is posited that firms behave in one of two ways. In one case, perfect competition is assumed: firms act as if they have no effect on price; they maximize profit taking prices as given. In the other case, imperfect competition is assumed: firms recognize and act on their ability to influence price. The model we present here proceeds in quite a different manner. Several distinctive features of classical economic analysis, in particular free entry, occupy a central role in the development.

Specifically, we study a model in which the number of firms is determined endogenously; firms enter when it is profitable to enter, and the entry of firms is a driving force in the explanation of value. The presence of fixed costs (more precisely, the fact that the efficient scale of firms is bounded away from zero) places a limit on the number of firms which are "active" in an equilibrium. When efficient scale is small relative to the size of the market, equilibrium will be

characterized by a large number of firms and small profits for each firm. In this case, firms will have very little effect on price when confined to the range in which they make non-negative profit. (We do not study the case of large efficient scale; however, in that situation firms make large profits and their actions may have a substantial effect on price. The resulting allocations will typically lack the efficiency properties associated with perfect competition.) In contrast to modern formal theory, a change in the specification of tastes (and/or technology) will change not only the actions of firms present in the market, but typically it will alter as well the number of firms in each industry.

Price taking producer behavior obtains only as an idealization; the primitive solution concept employed is Cournot-Nash equilibrium with quantity choosing firms and entry. Firms choose a quantity action given the actions of other firms; they evaluate the profitability of their actions according to the demand function of the price taking (consumer) sector. More precisely, prices are identified by the condition that consumer excess demand matches the aggregate action of firms. Cournot-Nash equilibrium is defined by the following two conditions; no firm in the market can increase profit by altering production, and no firm absent from the market can enter, and achieve a positive profit.

There are a countable infinity of firms of a variety of basic types available; however, since efficient scale is bounded away from zero, only a finite number of them are active in an equilibrium. Since there are only a finite number of active firms,

each firm is significant and affects price: thus, iso-profit manifolds are not linear. Furthermore, profit functions are not concave. The assumption that efficient scale is bounded away from zero could be defended on grounds of realism alone. However, the need for this assumption goes deeper, and is in fact dictated by the requirement that the number of firms be endogenous and determined by the opportunities for profit. For if production becomes increasingly efficient as output converges to zero, then the size of firms will be indeterminate (arbitrarily small) and the number of firms likewise indeterminate (arbitrarily large). Non-convex technology and free entry are intimately related. No theory of economic equilibrium which requires non-increasing returns to scale throughout can provide a satisfactory explanation of the number of firms in a market.

Now suppose that efficient scale is very small relative to the size of the market. Assume that there are a countable infinity of firms of a variety of types available. Profit functions may not be concave and production sets are not convex; this suggests the possible non-existence of Cournot-Nash equilibrium with entry (see; eg, Roberts and Sonnenschein [12]). But, if an equilibrium with entry exists, it is highly likely that it will approximate a Walras equilibrium of the economy obtained by viewing there to be available, for each firm type, a continuum of price taking infinitesimal sized firms.

(Formally one replaces the given technology by a cone technology which includes all multiples of possible productions and convex combinations of such points.) Although generally

loosely presented, reasoning to support this belief is not uncommon in the partial equilibrium analysis of a single market. The availability of a continuum of price taking infinitesimal sized firms is formalized by a horizontal supply curve the height of which is given by minimum average cost. The perfectly competitive (Walras) output is determined by the intersection of that supply curve with the demand curve. Cournot-Nash equilibrium output cannot fall short of the perfectly competitive output by more than the cost minimizing output (efficient scale) of a single firm; otherwise, there would be entry. Thus, if the scale at which firms are most efficient becomes small relative to demand, then the deviation of Cournot-Nash output from perfectly competitive equilibrium is small. (Similar reasoning is no doubt behind a conjecture of Samuelson [14]. "As the size of the market grows relative to the size of the minimum scale at which unit costs are at their lowest, the system approaches the perfectly competitive equilibrium.") But the above argument is possibly vacuous unless one can show that the notion of Cournot-Nash equilibrium with entry is viable when efficient scale is small relative to the market. Novshek [10] provided the required argument for the case of the partial equilibrium analysis of a single market. The central achievement of this paper is to extend the analysis to the case of general economic equilibrium. We next make the preceding argument more explicit.

Consider an Arrow-Debreu private ownership economy \mathcal{E} which satisfies sufficient conditions for the existence of Walras equilibrium -- with one exception: production sets are not

convex. (As a leading case, consider the zero vector union a "standard" and convex production possibility set which is displaced by a vector (of fixed costs) $w \leq 0$ ($\neq 0$).) If firms are infinitesimal relative to the market, and if a continuum of each type of firm is available, it is natural to consider the economy $\hat{\mathcal{E}}$ in which each firm is replaced by the smallest convex cone which contains the production possibility set of that firm. The cone so defined is viewed as an industry; and the implied industry supply curve is the counterpart of the perfectly elastic supply curve of partial equilibrium analysis. (Ownership shares are most conveniently translated into industry ownership shares; however, this is not essential.) Since $\hat{\mathcal{E}}$ has convex production sets, Walras equilibrium exists, and an equilibrium of $\hat{\mathcal{E}}$ is viewed as a Walras equilibrium of \mathcal{E} in which firms both must and do take prices as given. Associated with every equilibrium is a measure of firms in each industry.

Alternatively, for each α ($0 < \alpha \leq 1$), consider the economy $\mathcal{E}(\alpha)$ obtained from the original economy \mathcal{E} by replacing each firm Y by a sequence of firms, referred to as an industry, each with production set $Y(\alpha) = \alpha Y$. When $1/\alpha$ is an integer, $Y(\alpha)$ is viewed as a representation of Y in per capita terms in an economy in which each consumer has been replicated $1/\alpha$ times. As α becomes small, efficient scale becomes small relative to the size of the market. As small becomes infinitesimal; ie, as α approaches zero, it is natural to think of $\mathcal{E}(\alpha)$ as converging to $\hat{\mathcal{E}}$. This is because (in per capita terms) the

sum of the production sets of each industry in $\mathcal{E}(\alpha)$ converges to the production set of the corresponding industry in $\hat{\mathcal{E}}$.

We prove that it is a generic property of economies \mathcal{E} that there exists $\bar{\alpha} > 0$ such that Cournot-Nash equilibrium with free entry exists in $\mathcal{E}(\alpha)$ for all $\alpha \leq \bar{\alpha}$. The number of firms in the market is determined endogenously. Firms who choose to produce, typically have positive profit, and all firms with positive expected profit adopt pure strategies. Pure strategies are not required as a condition for equilibrium, rather they arise as a characteristic of equilibrium. Firms are permitted mixed strategies. Marginal firms are firms which maximize profit by choosing strategies of the form "produce y with probability π and stay out of the market with probability $(1-\pi)$ "; by profit maximization, they must make zero profit. For $\alpha \leq \bar{\alpha}$ only pure and marginal firm strategies are profit maximizing in the equilibria which are exhibited; furthermore, the number of marginal firms is uniformly bounded in α . Thus, as $\alpha \rightarrow 0$, the proportion of firms which are marginal and the aggregate output of marginal firms converges (per capita) to zero. Marginal firms arise naturally in the theory and are a proper analog of the entity which bears their name in less formal analysis. Without marginal firms, Cournot-Nash equilibrium will in general not exist and non-existence will be "robust".

The existence theorem requires as one of its hypotheses the presence of a Walras equilibrium for $\hat{\mathcal{E}}$ at which no industry can increase profit by increasing the scale of its operation. From the present point of view, equilibria of $\hat{\mathcal{E}}$ which fail to have this property are not proper Walras equilibria. Without

this property, firms of arbitrarily small size will profit from entering the industry, and upset equilibrium. Thus, a notion of downward sloping demand (DSD) arises naturally as a requirement for Walras equilibrium with entry, and equilibria which do not satisfy this condition are artifacts of a specification of the competitive model which requires that firms be infinitesimal. While it is natural to think of these equilibria as failing a "stability test", the considerations involved are in fact more basic. If α is sufficiently small, but not zero (ie, if efficient scale is small, but not infinitesimal), then Cournot-Nash equilibria of $\mathcal{E}(\alpha)$ cannot exist close to a point where DSD fails. For at any proposed equilibrium of $\mathcal{E}(\alpha)$ near such a point, profits in every industry must be non-negative, and an inactive firm can make positive profit by entering an industry in which profit increases with scale.

Having settled the problem of existence, it becomes meaningful to inquire whether the set of Cournot-Nash equilibria approaches the set of (DSD) Walras equilibria as efficient scale (measured per capita) approaches zero. Two questions are distinguished. First, whether an arbitrary (DSD) Walras equilibrium of $\hat{\mathcal{E}}$ can be obtained as the limit (as $\alpha \rightarrow 0$) of a sequence of Cournot-Nash equilibria, one point in the sequence for each $\mathcal{E}(\alpha)$. Second, whether every limit of Cournot-Nash equilibria (as $\alpha \rightarrow 0$) is a DSD Walras equilibrium of $\hat{\mathcal{E}}$. Both questions are answered in the affirmative. Since the

Walras equilibria of $\hat{\xi}$ coincide with the Cournot-Nash equilibria of $\hat{\xi}$, the preceding questions might be rephrased as pertaining to the lower and upper - hemi continuity of the Cournot-Nash correspondence at the point of infinitesimal (per-capita) efficient scale. We note here that Cournot's approach of replicating firms with a fixed "demand sector" does not lead to prices which approach Walras equilibrium prices. With efficient scale bounded away from zero and non-negative profit, prices diverge from those associated with efficient scale (competitive prices) as the number of active firms is increased. This is because the output of some firms will necessarily approach zero, and this will require prices higher than minimum average cost so that profits do not become negative. And, of course, equilibrium does not in general exist (Roberts and Sonnenschein [12]). For a partial equilibrium analysis, see Novshek, [10] and Ruffin [13].

Taken together, the existence and convergence results unify the concepts of Cournot and Walras equilibrium. For economies in which efficient scale is per capita small and entry is free, the assumption of price taking behavior (in $\hat{\xi}$) leads to an outcome which approximates the result obtained with strategic behavior. Free entry, absent from the current formal analysis of perfect competition, is a driving force in the analysis. Firms do not wish to take price as given, but with entry, as efficient scale becomes small in per capita terms, the percentage difference between price taking and strategic behavior becomes small. The concept of Walras equilibrium is derived from a primitive concept of strategic behavior; it is appropriate when the data of the economy

indicates small (per capita) efficient scale and there is free entry. Finally, only the equilibria of $\hat{\xi}$ which satisfy DSD are true economic equilibria. Most of this is of course very classical; what we offer is an adequate formal setting.

Novshek's Theorem ([10], p.8) on the existence of Cournot-Nash equilibrium with entry in a single market suggested the possibility of the general equilibrium analysis presented here. That work is surveyed in Appendix I. Prior to the completion of our paper, we had access to a manuscript by O. Hart ([6]). While Hart's analysis does not include the main concern of this paper - the existence of Cournot-Nash equilibrium with entry - he does provide a convergence result which is similar in spirit to our Theorem 3, but for an economy in which firms choose the commodity they will produce from an infinite set of possible commodities. We heartily recommend this paper to the attention of the reader. Finally, we note that the interplay between significant fixed costs, the number of firms, and the variety of products actually produced, has for many years been a central ingredient of the theory of monopolistic competition. For a particularly interesting modern treatment in partial equilibrium, we recommend a paper by M. Spence ([16]).

The next section introduces the formal model. This is followed by Section III which presents the theorems and two sections (IV and V) devoted to their proof. Section IV is expository and is designed to introduce some of the concepts used.

It is an important part of our presentation, as the formal proof involves an unusually large amount of computation. Remarks and conclusions are contained in Section VI and Section VII.

II. THE MODEL

The notion of a basic private ownership economy \mathcal{E} is standard.

a. An economy $\mathcal{E} = (X_i, \omega_i, \succsim_i, Y_j, \theta_{ij})$ is:

- a1) For each consumer $i=1,2,\dots,m$, a consumption set $X_i \subset \mathbb{R}^l$, an initial endowment vector $\omega_i \in \mathbb{R}^l$, and a preference ordering $\succsim_i \subset X_i \times X_i$,

[Whenever we speak of a collection of consumers $(X_i, \omega_i, \succsim_i)$

the following hypothesis is maintained. Desirability:

for any sequence $\{p^t\} \subset \Delta_\ell = \{(p_1, p_2, \dots, p_\ell) > 0 :$

$\sum p_r = 1\}$, if $p^t \rightarrow \bar{p} \in \bar{\Delta}_\ell \setminus \Delta_\ell$, and if for each i

and t , $h^i(p^t)$ is \succsim_i maximal subject to $p^t x_i \leq p^t \omega_i$, $\frac{1}{\| \sum_i h^i(p^t) \|} \rightarrow \infty$.],

- a2) for each firm j or $k=1,2,\dots,n$, a production set $Y_j \subset \mathbb{R}^l$, [We maintain the hypothesis that for all j , $0 \in Y_j$ and the asymptotic cone of Y_j contains no vectors with positive co-ordinates.]

- a3) for each i and j a non-negative number θ_{ij} which indicates the fraction of firm j owned by individual i . For each j , $\sum_i \theta_{ij} = 1$.

The economies $\mathcal{E}(\alpha)$ are derived from the basic economy \mathcal{E} by replicating the consumer sector ($1/\alpha$ times), viewing production in per capita terms, and for each j , positing the availability of a countable infinity of each of the y_j of \mathcal{E} . In the economy $\mathcal{E}(\alpha)$, we speak of the j 'th industry; for simplicity, ownership shares in \mathcal{E} are translated into industry ownership shares in $\mathcal{E}(\alpha)$.

b. For any economy $\mathcal{E} = (X_i, \omega_i, \lambda_i, Y_j, \theta_{ij})$ and any number

$\alpha > 0$, the economy

$\mathcal{E}(\alpha) = (X_i, \omega_i, \lambda_i, Y_{jt}(\alpha), \theta_{ijt})$ is defined by:

b1) the m consumers are those of \mathcal{E} ,

b2) for each $j \leq n$ and each positive integer t ,

$y_{jt} \in Y_{jt}(\alpha) \equiv Y_j(\alpha)$ if $y_{jt}/\alpha \in Y_j$, and

b3) $\theta_{ijt} \equiv \theta_{ij}$ for all i, j , and t .

The economy $\hat{\mathcal{E}}$ is derived from the basic economy \mathcal{E} by replacing each production set of \mathcal{E} with the smallest convex cone with vertex at the origin which contains it. The economy $\hat{\mathcal{E}}$ is interpreted as the limit of $\mathcal{E}(\alpha)$ as α approaches zero, and corresponds to a view in which the actions of firms in \mathcal{E} are infinitesimal in per capita terms.

c. For any economy $\mathcal{E} = (X_i, \omega_i, \lambda_i, Y_j, \theta_{ij})$, the natural limit of \mathcal{E} is the economy $\hat{\mathcal{E}} = (X_i, \omega_i, \lambda_i, \hat{Y}_j, \theta_{ij})$ defined by:

c1) the m consumers are exactly those of \mathcal{E} ,

- c2) for each firm $j = 1, 2, \dots, n$, \hat{Y}_j is the intersection of all closed convex cones with vertex at the origin which contain Y_j , and
- c3) ownership shares are exactly those of \mathcal{E} .

The definition of a Walras equilibrium and a Walras allocation for economies and pure exchange economies is standard. The Walras correspondence indicates how equilibrium prices vary with initial endowments in a pure exchange economy.

d. For an economy $\mathcal{E} = (X_i, \omega_i, \zeta_i, Y_j, \theta_{ij})$, the triple $(p^*, x^*, y^*) \in R^\ell \times R^{\ell m} \times R^{\ell n}$ is a Walras equilibrium if ^{3/}

$$d1) \quad p^* x_i^* = p^* \omega_i + \sum_j \theta_{ij} p^* y_j^*, \quad i = 1, 2, \dots, m,$$

$$d2) \quad \sum_i x_i^* = \sum_j y_j^* + \sum_i \omega_i,$$

$$d3) \quad x_i \succ_i x_i^* \text{ implies } p^* x_i > p^* \omega_i + \sum_j \theta_{ij} p^* y_j^*, \text{ and}$$

$$d4) \quad p^* y_j > p^* y_j^* \text{ implies } y_j \notin Y_j.$$

If (p^*, x^*, y^*) is a Walras equilibrium for \mathcal{E} , then (x^*, y^*) is called a Walras allocation.

An exchange economy $\mathcal{E} = (X_i, \omega_i, \zeta_i)$ is an economy minus firms and profit shares (a2 and a3) of a). The pair $(p^*, x^*) \in R^\ell \times R^{\ell m}$ is a Walras equilibrium for \mathcal{E} if

$$d5) \quad p^* x_i^* = p^* \omega_i, \quad i=1, 2, \dots, m,$$

$$d6) \quad \sum x_i^* = \sum \omega_i, \text{ and}$$

$$d7) \quad x_i \succ_i x_i^* \text{ implies } p^* x_i > p^* \omega_i.$$

The vector p^* is called an equilibrium price and x^* is called an equilibrium allocation. For (X_i) and (z_i) fixed, the Walras correspondence W assigns to each vector $(\omega_1, \omega_2, \dots, \omega_m)$ the equilibrium prices of $\mathcal{E} = (X_i, \omega_i, z_i)$.

As mentioned in the introduction, the study of equilibrium with entry dictates the requirement that efficient scale occur away from zero output. Since we are ultimately interested in Cournot-Nash equilibria which approximate Walras equilibria of $\hat{\mathcal{E}}$, it is natural to view efficient scale relative to Walras equilibrium prices, p^* of $\hat{\mathcal{E}}$. Not only is it assumed that efficient scale occurs away from zero, but in addition that such efficiency cannot be approximated in a neighborhood of zero output. For simplicity, we add the requirement that efficient scale (relative to p^*) is uniquely attained.

e. Let (p^*, x^*, y^*) be an equilibrium of $\hat{\mathcal{E}}$. We say that \mathcal{E} has efficient outputs bounded away from zero relative to p^* if for all $j \in \{1, 2, \dots, n\}$

e1) $(Y_j \cap \{y : p^* y \geq 0\}) \setminus \{0\}$ is a singleton $\gamma_j y_j^*$ (this defines the scalar γ_j), and

e2) for all $K, \delta \in (0, \infty)$, there exists $\varepsilon > 0$, such that $p^* y / \|y\| < -\varepsilon$ for all $y \in (Y_j \cap N_x(0, K)) \setminus (N_x(\gamma_j y_j^*, \delta) \cup \{0\})$, where $N_x(a, b)$ is the ball in R^l around a with radius b .^{4/}

It is generic that no more than $\ell - 1$ industries satisfy e_1 ,^{5/} and the remaining industries are such that for all prices near p^* , all nonzero outputs in the cones \hat{y}_j yield strictly negative profit. Since the equilibria we construct for Theorems 1 and 2 will have prices converging to p^* , these remaining industries must eventually be entirely inactive, with no profit incentive for entry. Without loss of generality, we ignore these remaining industries, and assume that the number of industries which must be considered, n , is less than ℓ . Of course, the set of industries which must be considered may vary as we look at different equilibria of $\hat{\mathcal{E}}$, but it is a generic property that the number of industries which must be considered is less than ℓ .

Production sets are required to have one of two forms in a neighborhood of efficient scale. Either they are finite polyhedra, or their boundaries are smooth manifolds. While we do not require that the manifolds have maximal dimension $(\ell - 1)$, we consider a formulation which rules out (mixed) cases in which, for example, inputs can only be combined in fixed proportion but yield varying marginal returns. While cases of this type could be included in the analysis, their treatment increases notational complexity.

f. If there exists a cube C with center at $\gamma_j y_j^*$ such that $C \cap Y_j$ is a finite polyhedron, then Y_j is called locally polyhedral, at $\gamma_j y_j^*$.

Let $S_j(y)$ be the largest subset of $\{1, 2, \dots, l\}$ for which there exists a neighborhood $N(y)$ such that the projection of the set of efficient productions of Y_j in $N(y)$ on to the $S_j(y)$ co-ordinate subspace is a singleton.

$L \setminus S_j(y)$ gives the co-ordinates in which production is locally variable by the j^{th} firm at y . If there exists a neighborhood $N(\gamma_j y^*_j)$ such that the set of efficient productions of Y_j in $N(\gamma_j y^*_j)$ is a smooth manifold of dimension $\#(L \setminus S_j(\gamma_j y^*_j)) - 1$, then Y_j is called smooth at $\gamma_j y^*_j$. (We maintain the hypothesis that $1 \in L \setminus \bigcup_j S_j(\gamma_j y^*_j)$, where the union is taken over j such that Y_j is smooth at $\gamma_j y^*_j$.) Y_j is regular smooth if $D^2 \hat{g}(\gamma_j y^*_j)$ is negative definite where \hat{g} gives the first co-ordinate of efficient points in $N(\gamma_j y^*_j)$ as a function of the remaining $\#(L \setminus S_j(\gamma_j y^*_j)) - 1$ smoothly variable co-ordinates.

A conceptual problem must be addressed prior to the definition of Cournot-Nash equilibrium in $\mathcal{E}(\alpha)$. Suppose that several firms assert a quantity action. In general, there will be more than one price consistent with their actions; ie, there may be more than one price which generates a demand that matches the aggregate action of firms. To see this, consider the exchange economy $\tilde{\mathcal{E}}$ obtained by distributing the actions of firms in $\mathcal{E}(\alpha)$ according to profit shares (and adding these amounts to initial commodity endowments). It is clear that each equilibrium price of $\tilde{\mathcal{E}}$ generates in $\mathcal{E}(\alpha)$ a demand which balances the asserted quantity actions of the firms (Rader [11]).

Unless one price is singled out, it is impossible to evaluate the actions of each firm, and so a Cournot-Nash game is not well defined. Since our methods demand that price varies smoothly with the quantity actions of firms, we develop a terminology to apply to situations in which there exists locally a twice continuously differentiable selection from an appropriate Walras correspondence. Balasko [2] has demonstrated that the existence of such a selection is generic in an appropriate space.

Prices are expressed relative to the price of the first commodity; ie, $p_1 = 1$. We note that in non-competitive theory the choice of numeraire typically affects the profit maximizing action of firms.^{6/} Our formalization requires that there is a salient commodity in which all firms measure profit.

g. Let p^* be an equilibrium price for the exchange economy $\mathcal{E} = (X_i, \omega_i, \lambda_i)$. The pair (F, p^*) is a p^* based inverse demand function if

- g1) F is a selection from the Walras correspondence of the the exchange economy \mathcal{E} with $p_1 \equiv 1$, and
- g2) F is twice continuously differentiable in a neighborhood of $\omega = (\omega_1, \omega_2, \dots, \omega_m)$ and has value p^* at ω .

Two notions of equilibrium for an economy $\mathcal{E}(\alpha)$ are defined. The first corresponds to a Cournot-Nash equilibrium in mixed (quantity) strategies, where pay-offs are defined relative to some selection from the appropriate Walras correspondence. The second, called Cournot equilibrium,

corresponds to equilibria of the above type in which optimal mixed strategies are either pure quantity strategies (typically indicating positive expected profit) or "produce y with probability π and stay out with probability $(1 - \pi)$ ", (indicating zero expected profit). Equilibria of the latter form, Cournot equilibria, are economically more meaningful and their existence is demonstrated in Theorem 1. Despite the fact that arbitrary mixed strategies are available in a Cournot equilibrium, optimizing action leads to only marginal firms and pure strategy firms.

h. Let α be a positive number. Consider an economy $\mathcal{E} = (X_i, \omega_i, \lambda_i, Y_j, \theta_{ij})$ and the associated economy $\mathcal{E}(\alpha)$.

A Cournot equilibrium in mixed strategies for $\mathcal{E}(\alpha)$ is a non-negative n vector η , with integer components, a function μ from $\{(j,v): j \leq n \text{ and } v \leq \eta_j\}$ into the set of probability measures on R^l , and a p based inverse demand function (F,p) for the exchange economy $\tilde{\mathcal{E}} = (X_i, \omega_i + \sum_j \theta_{ij} (\sum_{v \leq \eta_j} I(1, d\mu(j,v))), \lambda_i)$ such that

h1) for all (j,v) in the domain of μ , the support of $\mu(j,v) \subset Y_j(\alpha)$,

h2) for all (\bar{j}, \bar{v}) in the domain of μ , and all probability distributions ν with support in $Y_{\bar{j}}(\alpha)$,

$$I(F'(\sum_{v \leq \eta_j} y_{jv})_{Y_{\bar{j}\bar{v}}}, d\mu) \geq I(F'(\sum_{v \leq \eta_j} y_{jv})_{Y_{\bar{j}\bar{v}}}, d\mu(\nu \text{ in } \bar{j}\bar{v})),$$

where μ is the product measure of the $\mu(j,v)$, $(v \leq \eta(j), j \leq n)$, and $\mu(\nu \text{ in } \bar{j}, \bar{v})$ is μ with the \bar{j}, \bar{v} co-ordinate replaced by ν .

h3) for all $k \leq n$, and all v with support in $Y_k(\alpha)$,

$$I(F(\sum_{j,v} Y_{jv}) Y_{k\eta(k)+1}, d(\mu \times v)) \leq 0,$$

$$v \leq \eta(j) + \delta_{jk}$$

where $\mu \times v$ is μ with the additional co-ordinate (the $k, \eta(k)+1$ co-ordinate) v . The probability measure ρ on $R^{\ell n}$ is the Cournot production for $\xi(\alpha)$ corresponding to the above Cournot equilibrium in mixed strategies if for all μ -measurable $A \subset R^{\ell n}$, $\rho(A)$ is the μ probability that for all j , $\sum_{v \leq \eta(j)} Y_{jv}$ belongs to the projection of A onto the

$(\lambda(j-1)+1, \dots, \lambda(j-1)+\lambda)$ co-ordinates subspace of $R^{\ell n}$. The measure ρ on $R^{\ell n}$ induces a measure ξ on $R^{\ell-1}$ by $\xi(P) = \rho\{y \in R^{\ell n} : F(y) \in \{1\} \times P\}$. The measure ξ is the Cournot price distribution corresponding to the above Cournot equilibrium in mixed strategies.

A Cournot equilibrium for $\xi(\alpha)$ is a Cournot equilibrium in mixed strategies $(\mu, \eta, (F, p))$ such that for all j and v , either

h4) $\mu(j, v)$ is degenerate, or

h5) $\mu(j, v)$ is of the form "0 with probability π , ($\pi > 0$) $Y_j \in Y_j(\alpha)$ with probability $(1-\pi)$."

Firms of the type described in h5) are referred to as marginal firms.^{9/} Since $\pi > 0$, profit maximization implies zero expected profit.

We now formalize the requirement of downward sloping demand.^{10/} The economy exhibits downward sloping demand at (p^*, x^*, y^*, F) if F has the property that $\lambda y_j^* F(y^* + \lambda y_j^*) < 0$ for $\lambda \neq 0$, but small; ie, profit of the action y_j^* decreases (from zero) as λ increases from zero. It is convenient at the same time to introduce a measure of the effect on the price of commodity r of a "one unit" increase in the use (output) of commodity s by industry j ; for this we introduce the matrices B_j .^{11/} Recall that in the introduction we argued that equilibria of $\hat{\mathcal{E}}$ which fail to satisfy DSD cannot be achieved as a limit of Nash-Cournot equilibria.

i. Let $\mathcal{E} = (X_i, \omega_i, \xi_i, Y_j, \theta_{ij})$ be given. Assume (p^*, x^*, y^*) is a Walras equilibrium of $\hat{\mathcal{E}}$ and (F, p^*) is a p^* based inverse demand function for the exchange economy

$$\mathcal{E}(y^*) = (X_i, \omega_i + \sum_j \theta_{ij} y_j^*, \xi_i).$$

For $j \in \{1, 2, \dots, n\}$, let B_j be the $l \times l$ matrix with generic entry

$$\left[\sum_{t=1}^m \theta_{tj} \frac{\partial F}{\partial \omega_{ts}} \left((\omega_i + \sum_{j=1}^n \theta_{ij} y_j^*) \right)_i \right]_{rs}$$

The economy \mathcal{E} exhibits downward sloping demand (DSD) at (p^*, x^*, y^*, F) if $y_j^{*'} B_j y_j^* < 0$ for all j .

A variety of non-degeneracy conditions are necessary to the analysis. Appendix II establishes the genericity of these assumptions in an appropriate space.

j. Let $\mathcal{E} = (X_i, \omega_i, \lambda_i, Y_j, \theta_{ij})$ be given, and let (p^*, x^*, y^*) be a Walras equilibrium of $\hat{\mathcal{E}}$. Suppose that (F, p^*) is a p^* based inverse demand function for the exchange economy $\mathcal{E}(y^*) = (X_i, \omega_i + \sum_j \theta_{ij} y_j^*, \lambda_i)$ and that for all j , Y_j is either locally polyhedral or regular smooth at $\gamma_j y_j^*$. Let P_j be the $\ell \times \ell$ matrix corresponding to the projection onto the space of commodities (excluding 1) which are smoothly variable by firms of type j at $\gamma_j y_j^*$, and let G_j be the $\ell \times \ell$ pseudo inverse of $D^2 g_j(\gamma_j y_j^*)$ which satisfies $P_j G_j P_j = G_j$ (where g_j gives commodity 1 as a function of the other commodities, on the efficient production surface near $\gamma_j y_j^*$

and the s, t entry of $D^2 g_j(\gamma_j y_j^*)$ is defined to be zero if s or t is 1 or corresponds to a non smoothly variable component). If Y_j is locally polyhedral at $\gamma_j y_j^*$,

$G_j = [0]$. Also define $\bar{p} = p^*_1$; ie, $\bar{p}' = (p_2^* p_3^* \dots p_\ell^*) = (F_2(y^*) F_3(y^*) \dots F_\ell(y^*))$. The economy \mathcal{E} satisfies the condition ND (non-degeneracy) at (p^*, x^*, y^*, F) provided:

$$j1) \begin{bmatrix} \frac{1}{\delta_1} & \begin{bmatrix} 0 & -\bar{p}' \\ 0 & I \end{bmatrix} G_1 & & & 0 \\ & & \ddots & & \\ & 0 & & \frac{1}{\delta_n} & \begin{bmatrix} 0 & -\bar{p}' \\ 0 & I \end{bmatrix} G_n \\ & & & & \end{bmatrix} \begin{bmatrix} B_1 & B_2 & \dots & B_n \\ \cdot & \cdot & \cdot & \cdot \\ B_1 & B_2 & \dots & B_n \end{bmatrix} = H_1$$

$\ell n \times \ell n$ $\ell n \times \ell n$

does not have eigenvalue -1 (G_j and B_j are $\ell \times \ell$, I is $(\ell-1) \times (\ell-1)$).

Theorem 1

Let $\xi = (X_i, \omega_i, \lambda_i, Y_j, \theta_{ij})$ be given. Assume that $\hat{\xi}$ has a Walras equilibrium (p^*, x^*, y^*) , and that there exists a p^* based inverse demand function (F, p^*) [for the exchange economy $\xi(y^*) = (X_i, \omega_i + \sum_j \theta_{ij} y_j^*, \lambda_i)$], which exhibits downward sloping demand at (p^*, x^*, y^*, F) . Assume ξ has efficient outputs bounded away from zero relative to p^* , and for all j, Y_j is either locally polyhedral or regular smooth at $\gamma_j y_j^*$. Assume in addition that the non-degeneracy conditions ND are satisfied. Then, there exists $\bar{\alpha} > 0$ such that for all $\alpha \leq \bar{\alpha}$ there is a Cournot equilibrium for the economy $\xi(\alpha)$. Furthermore, there exists a fixed number N such that these equilibria may be chosen so that for all $\alpha \leq \bar{\alpha}$, the number of marginal firms is less than N .

It follows that as α approaches zero, the maximum of the sum of the outputs of the marginal firms in each industry becomes arbitrarily small (in euclidean norm) relative to the sum of the outputs of all firms in that industry.

Theorem 2 is a corollary to the proof of Theorem 1. It establishes that an arbitrary (DSD) Walras equilibrium of $\hat{\xi}$ can be obtained as the limit (as $\alpha \rightarrow 0$) of a sequence of Cournot-Nash equilibria, one point in the sequence for each $\xi(\alpha)$. Furthermore, the rate of convergence is α .

Theorem 2

Let the conditions of Theorem 1 be satisfied. Then there exists ρ , from $(0, 1]$ to the set of probability measures on

\mathbb{R}^n with the following property: there exist $S_1, S_2, \bar{\alpha} \in (0, \infty)$ such that for all $\alpha \leq \bar{\alpha}$

2.1) $\rho(\alpha)$ is the Cournot production associated with some Cournot equilibrium $(\mu(\alpha), \eta(\alpha), (F(\alpha), \rho(\alpha)))$ of $\xi(\alpha)$,

2.2) $\text{Max}_{y \in \text{support } \rho(\alpha)} \|y^* - y\| < \alpha S_1$, and

2.3) $\text{Max}_{p \in \text{support } \xi(\alpha)} \|\bar{p} - p\| < \alpha S_2$, where $\xi(\alpha)$ is the Cournot price distribution corresponding to $(F(\alpha), \rho(\alpha))$, and $\bar{p}' = (p_2^*, p_3^* \dots p_l^*)$.

Theorem 3 establishes upper hemi continuity of the Cournot equilibrium correspondence at $\alpha = 0$: any sequence of Cournot equilibria with mixed strategies, one for each $\alpha_k (\alpha_k \rightarrow 0)$, for which expected output converges, converges to a DSD Walras equilibrium of $\hat{\xi}$ (which is a Cournot equilibrium of $\xi(0)$). Global analogs of e1) and e2) are introduced and guarantee that efficient outputs are bounded away from zero at all relevant prices.

e1') For all j , if $t \in \Omega$ is such that $t'y \leq 0$ for all $y \in Y_j$ then there is at most one $0 \neq y(t) \in Y_j$ such that $t'y(t) = 0$.

e2') For all j , with t and $y(t)$ as in e1', given $K, \delta \in (0, \infty)$, there exists $\epsilon > 0$ such that for all $s \in N(t, \epsilon)$, $[(y \in Y_j \mid s'y \geq 0) \cap N(0, K)] \subset [(0) \cup N(y(t), \delta)]$.

e3') The cones $\hat{Y}_1, \hat{Y}_2, \dots, \hat{Y}_n, -\Omega$ are positively semi independent, and for each j , 0 is an exposed point of \hat{Y}_j .

Theorem 3

Let $\mathcal{E} = (X_i, \omega_i, \lambda_i, Y_j, \theta_{ij})$ be given. Suppose that the Y_1, Y_2, \dots, Y_n satisfy e1', e2', and e3' and that there exist:

- (i) $F : R^{\ell m} \rightarrow \{1\} \times R^{\ell-1}$, a selection from the Walras correspondence for exchange economies with X_i, λ_i fixed;
- (ii) a sequence $(\alpha_k)_{k=1}^{\infty}$ of strictly positive reals; and
- (iii) a function ρ from $\{\alpha_1, \alpha_2, \alpha_3, \dots\}$ to the set of probability measures on $R^{\ell n}$, such that
- (iv) $\alpha_k \rightarrow 0$ as $k \rightarrow \infty$,
- (v) for all k , $\rho(\alpha_k)$ is the Cournot production associated with some Cournot equilibrium in mixed strategies $(\mu(\alpha_k), \eta(\alpha_k), (F, \rho(\alpha_k)))$ of $\mathcal{E}(\alpha_k)$ and
- (vi) there exists $y^* \in R^{\ell n}$ such that $I(y, d\rho(\alpha_k)) \rightarrow y^*$ as $k \rightarrow \infty$.

Then

3.1) for all $\varepsilon > 0$, $\lim_{k \rightarrow \infty} \rho(\alpha_k) \{ N(y^*, \varepsilon) \} = 1$.

Furthermore, if F is continuous at y^* , and we let $\xi(\alpha_k)$ denote the Cournot price distribution corresponding to $(\rho(\alpha_k), F)$,

3.2) there exists $\bar{p} \in R^{\ell-1}$ such that for all $\varepsilon > 0$,

$\lim_{k \rightarrow \infty} \xi(\alpha_k) \{ N(\bar{p}, \varepsilon) \} = 1$, and

3.3) there exists $x^* \in R^{\ell m}$ such that $((1, \bar{p}), x^*, y^*)$, is a Walras equilibrium of $\hat{\mathcal{E}}$.

Finally, if F is twice continuously differentiable in a neighborhood of y^* [more precisely in a neighborhood of $(\omega_i + \sum_j \theta_{ij} y_j^*)_i$] and B_j is defined as in i for each $j = 1, 2, \dots, n$, then

$$3.4) \quad y_j^{*'} B_j y_j^* \leq 0 \text{ for each } j = 1, 2, \dots, n.$$

IV. INTRODUCTION TO THE PROOF OF THEOREM 1

The purpose of this section is to develop some of the concepts used in the proof of Theorem 1. Our framework is partial equilibrium: a single market with U-shaped average cost curve firms and downward sloping demand. The reader will profit from turning first to Appendix I, where that framework is explicitly introduced. Let y^* be the competitive industry output, αy^* be the "competitive output" for a firm of size α , and $EIS(\alpha)$ be the largest y for which there is a non-zero optimal response, $y(\alpha)$. The following "residual demand" diagram (figure 1) is standard; the average cost function AC_α^1 and demand function F are drawn relative to the action $y^* - \alpha y^*$ by other firms.

We note:

- 1) The profit maximizing response of a single firm is less than αy^* and profit is strictly positive.
- 2) As the origin of the average cost function (equal to the aggregate action of other firms) moves to the right, maximum profit declines, and at aggregate output of others, $EIS(\alpha)$, maximum profit is zero and is achieved at both zero output and output $y(\alpha)$.

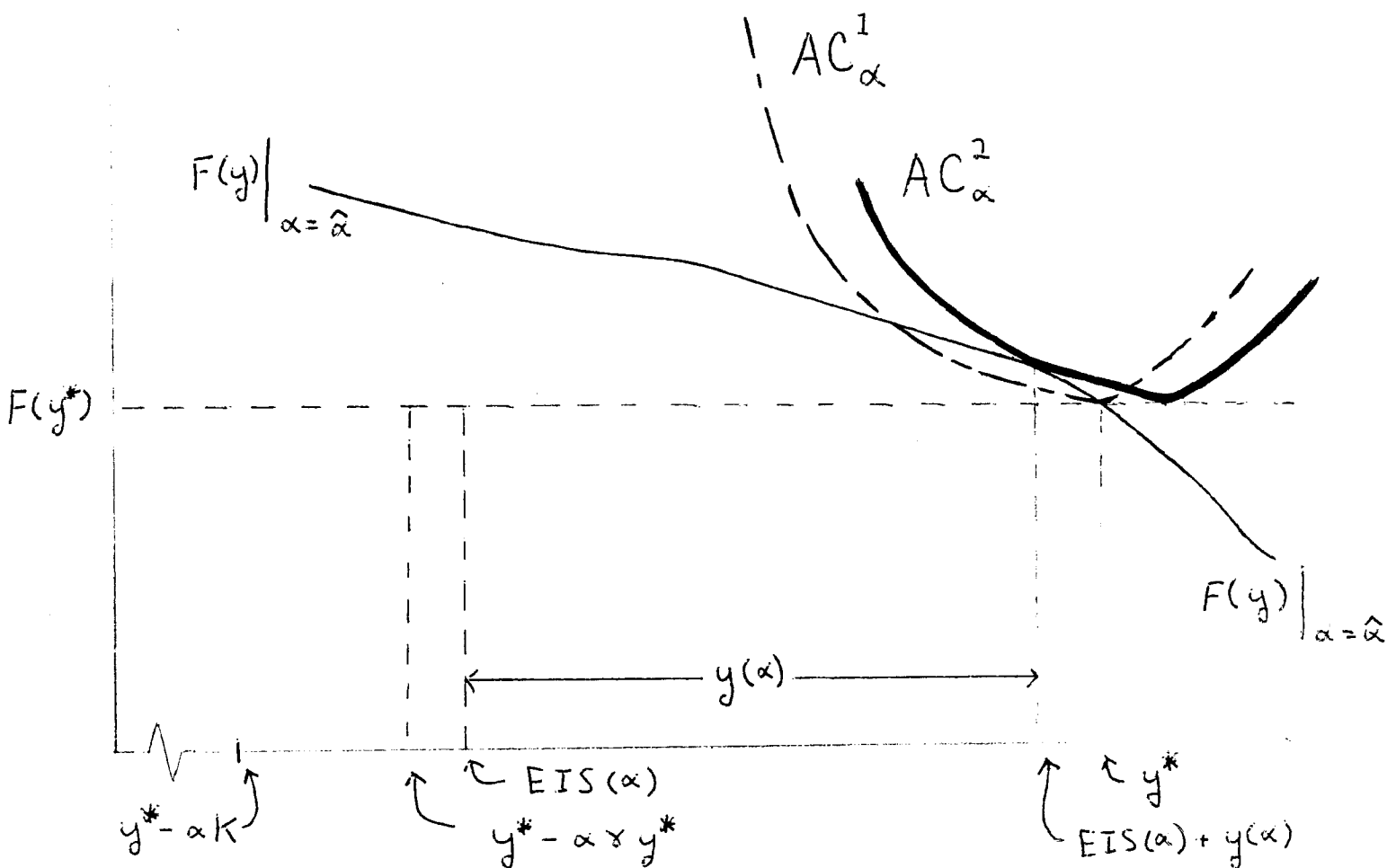


Figure 1

AC_{α}^1 is the average cost curve positioned with respect to residual demand when the output of other firms is $y^* - \alpha y^*$.

AC_{α}^2 is the average cost curve positioned with respect to residual demand when the output of other firms is $EIS(\alpha)$.

(Note that αK converges to zero as α converges to zero. However, in this diagram $[y^* - \alpha K, y^*]$ is independent of α and F varies with α , because the horizontal axis is relabeled as α varies.)

3) By adopting the convention that αK is a fixed distance on the horizontal axis, the shape of AC_α remains fixed in our diagram as α varies. For α sufficiently small, $[EIS(\alpha) - (y^* - \alpha \delta y^*)]/\alpha$ and $[\alpha \delta y^* - y(\alpha)]/\alpha$ are non-negative and converge to zero, since F becomes flatter in the diagram with standard unit $\alpha \delta y^*$.

The backward map $b(y, \alpha)$ [$b(y, \alpha) \neq y$] indicates what the aggregate industry output must have been for the non zero action of one maximizing firm to be $y - b(y, \alpha)$ yielding aggregate output y . For α small, and $y \in [EIS(\alpha), EIS(\alpha) + y(\alpha)]$, $b(y, \alpha)$ is a well defined continuous function. Define the intervals $B(\alpha, N) = \{y - N(y - b(y, \alpha)) \mid y \in [EIS(\alpha), EIS(\alpha) + y(\alpha)]\}$, and note that $B(\alpha, N+1)$ is left of $B(\alpha, N)$; however, they may intersect.

If $0 \in B(\alpha, N)$ for some N , then there exists $y \in [EIS(\alpha), EIS(\alpha) + y(\alpha)]$ with $y - N(y - b(y, \alpha)) = 0$, so there is a symmetric N firm equilibrium with aggregate output y , each operating firm acting optimally and earning non-negative profit, and no incentive for entry (since 0 is an optimal response to $y \geq EIS(\alpha)$).

If $0 \notin B(\alpha, N)$ for all N , consider the backward mapping for a marginal firm. For expected output $y \in [EIS(\alpha), EIS(\alpha) + y(\alpha)]$ a marginal firm must have come from $EIS(\alpha)$ with action $y(\alpha)$ and probability $[y - EIS(\alpha)]/y(\alpha)$. (A marginal firm is also profit maximizing, so it can only come from $EIS(\alpha)$, where it is indifferent between action $y(\alpha)$ and 0 . For any $y \neq EIS(\alpha)$, a pure strategy strictly dominates

every mixed strategy.) As y varies through $[EIS(\alpha), EIS(\alpha) + y(\alpha)]$, the expected output of the marginal firm varies through $[0, y(\alpha)]$.

Let N be such that $B(\alpha, N + 1) < 0 < B(\alpha, N)$. Using N regular firms and one marginal firm (and ignoring here a minor correction [which vanishes as $\alpha \rightarrow 0$] necessary because expected price is not equal to price of expected output), the total backward map from $EIS(\alpha) + y(\alpha)$ goes to $z \in B(\alpha, N + 1) < 0$ since the marginal firm has the same action as a regular firm, with probability equal to 1, while the total backward map from $EIS(\alpha)$ goes to $z \in B(\alpha, N) > 0$ since the marginal firm has action 0 with probability equal to 1. The map is continuous, so there exists an equilibrium with N regular firms and one marginal firm, with expected output $y \in (EIS(\alpha), EIS(\alpha) + y(\alpha))$. In the equilibrium, all firms are profit maximizing, regular firms earn strictly positive expected profit, the marginal firm earns zero expected profit, and there is no incentive for entry. (It should be noted that in the partial equilibrium model marginal firms are not needed for small α ; ie, $0 \in B(\alpha, N)$ for some N (see Appendix I).)

The essential ingredients of the above argument are: first, to surround zero by the intervals $B(\alpha, N)$ and $B(\alpha, N + 1)$ [for some N], and then if necessary ($0 \notin B(\alpha, N)$), to introduce marginal firms to capture zero. For situations in which there is more than one commodity (and/or more than one industry), the corresponding problem of surrounding and capturing zero (starting at a point of non-negative profit and no entry) becomes more complex. The non-degeneracy assumptions $(j1 - j3)$ play a central role in the argument. Appendix II establishes the genericity of these assumptions.

V. PROOFS

The proof of Theorem 1 follows a sequence of lemmas. Let $z_j(Z, \alpha)$ denote the set of profit maximizing actions for a firm with production set $Y_j(\alpha)$ when the action of other firms is $y^* + \alpha Z$. Figure 2 is an analog of Figure 1. We have represented the production possibilities of the j^{th} firm with $\| \alpha K \|$ a fixed distance in the diagram. If the action of "other firms" is $y^* + \alpha Z$ (for some fixed α), iso-profit manifolds are defined for the firm. Since the axes are re-labeled as α varies, the relation between y^* and $y^* + \alpha Z$ remains fixed in the diagram. As α approaches zero, the actions of "other firms" approach the competitive output, and in Figure 2 (with $\| \alpha K \|$ a fixed distance), iso-profit manifolds "converge uniformly" to the iso-profit hyperplanes H_{p^*} associated with the competitive equilibrium price p^* . (The dependence of the iso-profit manifolds on α in Figure 2 is analogous to the dependence of the inverse demand function F on α in Figure 1.) In the limit, the profit maximizing actions for the j^{th} firm are 0 and $\alpha \delta_j y_j^*$. Lemmas 1-4 show that for α sufficiently small, the profit maximizing actions of a firm are either 0 or an efficient production in a neighborhood of $\alpha \delta_j y_j^*$.

Lemma 1: Assume the hypotheses of Theorem 1 are satisfied. Given $K < \infty$, there exists $M < \infty$ such that $\|x\| < M\alpha$ for all $x \in z_j(Z, \alpha)$ for all $(Z, \alpha) \in N(0, K) \times (0, 1] \subset \mathbb{R}^{1n} \times \mathbb{R}^1$.

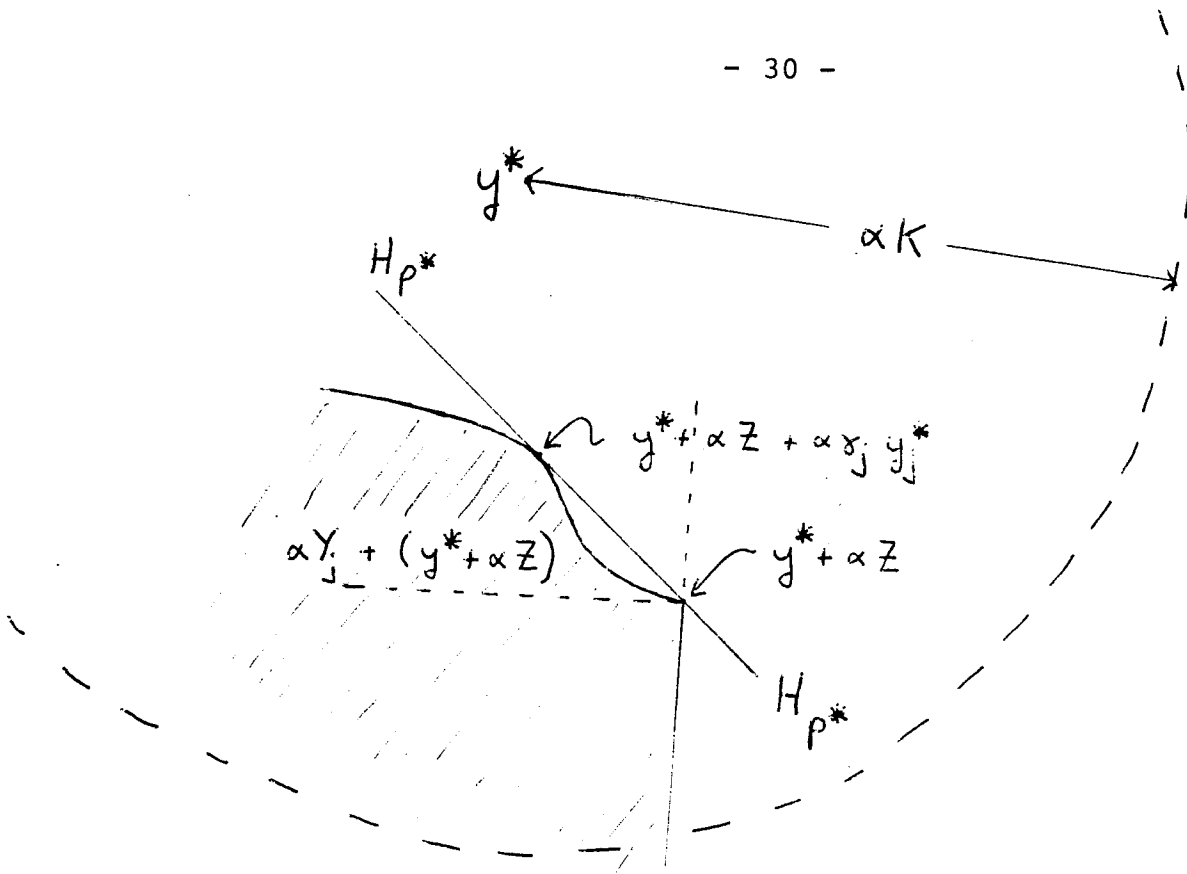


Figure 2

H_{p^*} is the set of actions by a firm of type j which earn zero profit when the action of other firms is $y^* + \alpha Z$ and prices are fixed at p^* .

Proof of Lemma 1:

The set of aggregate (over industries as well as firms) productions

$$S = \left[\bigcup_{\substack{\alpha \leq 1 \\ Z \in N(O, K)}} (Y_j(\alpha) + \{ \sum_t (y_t^* + \alpha Z_t) + \sum_i \omega_i \}) \right] \cap \Omega^c$$

is bounded above. By desirability (see a1) and the upper hemicontinuity of the Walras correspondence,

$\{F(y) \mid y = y_j(\alpha) + y^* + \alpha Z \in \text{Domain of } F \text{ for some}$

$(y_j(\alpha), \alpha, Z) \in Y_j(\alpha) \times (0, 1] \times N(O, K)$

such that $y_j(\alpha) + \sum_t (y_t^* + \alpha Z_t) + \sum_i \omega_i \in S\}$ is contained in a

closed cone C in the interior of Ω^k (this set contains all

prices that could result from a feasible action in $Y_j(\alpha)$ when

others' actions are $y^* + \alpha Z$, $(Z, \alpha) \in N(O, K) \times (0, 1]$). Since the

asymptotic cone of Y_j is contained in $-\Omega^k$, there exists $M < \infty$

such that $px < 0$ for all $p \in C \setminus \{0\}$, $x \in Y_j(1)$, $\|x\| > M$. Noting

$0 \in Y_j(\alpha) = \alpha Y_j(1)$ completes the proof. ||

Let $EY_j(\alpha)$ be the set of technologically efficient productions in $Y_j(\alpha)$:

$$EY_j(\alpha) = \{y \in Y_j(\alpha) \mid \nexists 0 \neq x \in \Omega^k \text{ such that } y + x \in Y_j(\alpha)\}.$$

Lemma 2: Assume that the hypotheses of Theorem 1 are

satisfied. Given $K < \infty$ there exists $\bar{\alpha} > 0$ such that

$x \in EY_j(\alpha)$ for all $x \in z_j(Z, \alpha)$, for all $(Z, \alpha) \in N(O, K) \times (0, \bar{\alpha}]$.

Proof of Lemma 2:

The profit of a firm in the j^{th} industry which chooses x given that the actions of other firms are $y^* + \alpha Z$ is

$\Pi(x|Z, \alpha) = x'F(y^* + \alpha Z + x)$. By the previous lemma $\|x\| \rightarrow 0$ as $\alpha \rightarrow 0$, and by g2 F is continuously differentiable at $y^* + \alpha Z + x$ for all α sufficiently small. Thus,

$\frac{\partial \Pi}{\partial x_i} = F_i(y^* + \alpha Z + x) + x' \left(\frac{\partial F}{\partial y_{ji}}(y^* + \alpha Z + x) \right)$. Since $F_i(y^*) > 0$ for all i , it follows that $\frac{\partial \Pi}{\partial x_i} > 0$ for α sufficiently small, and $x \in EY_j(\alpha)$ for all $x \in Z_j(Z, \alpha)$. ||

Lemma 3: Assume that the hypotheses of Theorem 1 are satisfied. Given $K, \delta \in (0, \infty)$ there exists $\bar{\alpha} > 0$ such that $x \in N(\alpha Y_j, y_j^*, \alpha \delta) \cup \{0\}$ for all $x \in Z_j(Z, \alpha)$, for all $(Z, \alpha) \in N(0, K) \times (0, \bar{\alpha}]$.

Proof of Lemma 3:

By e2 and the continuity of F in a neighborhood of y^* , there exists $\beta(M) > 0$ such that $y'F(W) / \|y\| < -\beta(M)$ for each $y \in [Y_j \cap N(0, M)] \setminus [N(\delta_j y_j^*, \delta) \cup \{0\}]$ and W close to y^* . Setting M equal to the M value guaranteed by Lemma 1, for α small, $\frac{1}{\alpha} x'F(\alpha Z + y^* + x) / \|\frac{1}{\alpha} x\| < -\beta(M)$ for $\frac{1}{\alpha} x \in [Y_j \cap N(0, M)] \setminus [N(\delta_j y_j^*, \delta) \cup \{0\}]$. Noting that $\frac{1}{\alpha} Z_j(Z, \alpha) \subset N(0, M)$ for $(Z, \alpha) \in N(0, K) \times (0, 1]$ and that the zero action yields zero profit completes the proof. ||

Lemma 4: Assume that the hypotheses of Theorem 1 are satisfied and that Y_j is locally polyhedral at $\delta_j y_j^*$. Given $K < \infty$ there exists $\bar{\alpha} > 0$ such that $z_j(Z, \alpha) \subset \{0, \alpha \delta_j y_j^*\}$ for all $(Z, \alpha) \in N(O, K) \times (0, \bar{\alpha}]$.

Proof of Lemma 4:

Let $T_j = \{t \in \mathbb{R} \mid \|t\| = 1 \text{ and there exist } y_j \in Y_j, \lambda > 0 \text{ such that } t = \lambda(\delta_j y_j^* - y_j) \text{ and } \mu \delta_j y_j^* + (1-\mu)y_j \in Y_j \text{ for all } \mu \in [0, 1]\}$. If $T_j = \emptyset$, Lemma 3 yields the result. If not, there exists $\delta > 0$ such that

$$N(\delta_j y_j^*, \delta) \cap Y_j = \{\delta_j y_j^* - at \mid t \in T_j, a \in [0, \delta)\}.$$

By Lemma 3, $\frac{1}{\alpha} z_j(Z, \alpha) \subset (\{0\} \cup [N(\delta_j y_j^*, \delta) \cap Y_j])$ for all $Z \in N(O, K)$, for all α sufficiently small.

For any $x \in N(\delta_j y_j^*, \delta) \cap Y_j$,

$$\Pi(\alpha x \mid Z, \alpha) = \alpha x' F(y^* + \alpha Z + \alpha x) \text{ is differentiable for } \alpha \text{ small, and } \frac{\partial \Pi}{\partial x}(\alpha x \mid Z, \alpha) = \alpha F(y^* + \alpha Z + \alpha x) + \alpha^2 \left[\frac{\partial F_i}{\partial y_{jk}}(y^* + \alpha Z + \alpha x) \right]_{ik} x.$$

$$\text{For each } t \in T_j, \frac{1}{\alpha} t' \frac{\partial \Pi}{\partial x}(\alpha x \mid Z, \alpha) = t' F(y^* + \alpha Z + \alpha x) + \alpha t' \left[\frac{\partial F_i}{\partial y_{jk}} \right]_{ik} x$$

which is strictly positive for α sufficiently small since $t' F(y^*) > 0$ for all $t \in T_j$, and $\alpha Z + \alpha x \rightarrow 0$. Thus, for α sufficiently small, for all $Z \in N(O, K)$, $x = \delta_j y_j^*$ maximizes $\Pi(\alpha x \mid Z, \alpha)$ over the set $x \in N(\delta_j y_j^*, \delta) \cap Y_j$, and $z_j(Z, \alpha) \subset \{0, \alpha \delta_j y_j^*\}$. ||

The technique of proof involves separating out the determination of how many firms of each type are in the market from what an active firm does. From Lemmas 1-4, if other firms are doing approximately y^* , then the action of a firm which is active in equilibrium (and thus making non-negative profit), will be approximately $\alpha x_j y_j^*$. We therefore study those actions of firms which maximize profit on a neighborhood of $\alpha x_j y_j^*$. As α converges to zero, the number of firms present in an equilibrium grows without bound. Although the deviation of active firms from $\alpha x_j y_j^*$ becomes insignificant, the total deviation of active firms may be significant from the viewpoint of entry of new firms, and is thus important for the analysis. Lemma 5 computes the deviation of an individual firm, and Lemma 6 makes use of the significant aggregate deviation.

For each j , let $\delta_j > 0$ be such that $\overline{N(x_j y_j^*, \delta_j)}$ is a subset of the cube C (if Y_j is locally polyhedral at $x_j y_j^*$) or the neighborhood $N(x_j y_j^*)$ (if Y_j is locally regular smooth at $x_j y_j^*$) which is described in f .

Let $z_j^*(Z, \alpha)$ denote the set of profit maximizing actions in $Y_j(\alpha) \cap \overline{N(\alpha x_j y_j^*, \alpha \delta_j)}$ when other firms' actions are $y^* + \alpha Z$. By Lemma 3, for α sufficiently small, $z_j^*(Z, \alpha) = z_j(Z, \alpha)$ if $\Pi(x|Z, \alpha) > 0$ for some $x \in z_j^*(Z, \alpha)$.

Lemma 5: Assume the hypotheses of Theorem 1 are satisfied.

Given $K < \infty$ there exists $\bar{\alpha} > 0$ such that for all

$(Z, \alpha) \in N(0, K) \times (0, \bar{\alpha}]$ $z_j^*(Z, \alpha)$ is a singleton and

$$z_j^*(Z, \alpha) = \alpha \delta_j y_j^* - \alpha^2 \begin{bmatrix} 0 & -\bar{p}' \\ 0 & I \end{bmatrix} G_j \begin{bmatrix} 0 & 0 \\ -\bar{p} & I \end{bmatrix} \{ \sum_t B_t Z_t + \delta_j (B_j + B_j') y_j^* \} + O(\alpha^3)$$

(= $\alpha \delta_j y_j^*$ if Y_j is locally polyhedral at $\delta_j y_j^*$)

and

$$\frac{\partial z_j^*(Z, \alpha)}{\partial Z_t'} = -\alpha^2 \begin{bmatrix} 0 & -\bar{p}' \\ 0 & I \end{bmatrix} G_j B_t + O(\alpha^3)$$

(= $[0]$ if Y_j is locally polyhedral at $\delta_j y_j^*$).

[Recall $\bar{p}' = (p_2^* \ p_3^* \ \dots \ p_k^*)$]

Proof of Lemma 5:

The results for locally polyhedral Y_j follow from the proof of Lemma 3.

If Y_j is locally regular smooth at $\delta_j y_j^*$, by Lemma 2 we can restrict attention to the smooth efficient manifold, so the profit maximization problem becomes

$$\max_{x \in P_j(\mathbb{R}^k)} \Pi(\alpha \delta_j y_j^* + \alpha x + \alpha g_j(x) e_1 | Z, \alpha)$$

where g_j gives the

deviation of the first coordinate from $\delta_j y_{j1}^*$ as a function of the deviations x of the other coordinates from $\delta_j y_j^*$ (on the smooth efficient manifold of $Y_j \cap N(\delta_j y_j^*, \delta_j)$).

[P_j is the projection to the space of components other than 1 which are smoothly variable].

Restricting attention to changes in x which are in $P_j(R^k)$, for α sufficiently small the profit function is differentiable, and

$$\left. \frac{\partial \Pi}{\partial x} \right|_{\text{restricted to } P_j(R^k)} = \alpha \{ P_j + [\frac{\partial g_j(x)}{\partial x} \cdot 0] \} \{ F(\cdot) + \alpha [\frac{\partial F_i(\cdot)}{\partial y_{jk}}]_{ik} \{ y_j^* + x + g_j(x) e_1 \} \}$$

where F and $\frac{\partial F_i}{\partial y_{jk}}$ are evaluated at $y^* + \alpha Z + \alpha \{ y_j^* + x + g_j(x) e_1 \}$

and $\frac{\partial g_j(x)}{\partial x}$ is defined to be zero for those components not variable in $P_j(R^k)$.

Since $\|x\| \rightarrow 0$ as $\alpha \rightarrow 0$ for optimal choices,

$$\left. \frac{\partial^2 \Pi}{\partial x \partial x'} \right|_{\text{restricted to } P_j(R^k)} = \alpha \frac{\partial^2 g_j(x)}{\partial x \partial x'} + O(\alpha^2) \text{ where the } s, t \text{ entry of}$$

$\frac{\partial^2 g_j(x)}{\partial x \partial x'}$ is defined to be zero if either x_s or x_t is not variable

in $P_j(R^k)$. y_j is regular smooth at y_j^* so $x' \frac{\partial^2 g_j(0)}{\partial x \partial x'} x < 0$

for all $0 \neq x \in P_j(R^k)$.

Since $\frac{\partial^2 g_j(x)}{\partial x \partial x'}$ converges to $\frac{\partial^2 g_j(0)}{\partial x \partial x'}$ as $x \rightarrow 0$, $z_j^*(Z, \alpha)$ is

a singleton for α sufficiently small.

To find the unique profit maximizing x , set $\frac{\partial \Pi}{\partial x} = 0$,

use the Taylor expansion for $F(\cdot)$ and $\left[\frac{\partial F_i(\cdot)}{\partial y_{jk}} \right]$ (at y^*) and

$\frac{\partial g_j(x)}{\partial x}$ (at 0), note that, as defined, $\frac{\partial g_j(0)}{\partial x} = -P_j F(y^*)$ and solve

to find

$$x = -\alpha G_j \left[\begin{array}{cc} 0 & 0 \\ -P_j & I_j \end{array} \right] \left\{ \sum_t B_t Z_t + \gamma_j (B_j + B_j') y_j^* \right\} + O(\alpha^2).$$

(Recall $F_1 \equiv 1$ so the first row of each B_t is a zero vector.)

With this evaluation of x , $g_j(x) = -F'(y^*) P_j x + O(\alpha^2)$ and

the $z_j^*(Z, \alpha)$ result follows.

The $\frac{\partial z_j^*(Z, \alpha)}{\partial Z_t}$ result follows from implicit differentiation of the first order condition along with the evaluation of x above.

Using the "reaction functions" of Lemma 5, it is possible to find a Cournot "equilibrium" when the number of firms of each type is fixed and allowed to be a non integer and α is small. Lemma 6 computes "equilibrium" aggregate output as a function of α and the number of operating firms of each type, which Lemma 7 shows that for α small, for certain numbers of operating firms, each firm earns positive profit (so, as noted in the discussion before Lemma 5, the "reaction function" gives the true profit maximizing action of the firm) and there is no incentive for entry.

Let $Z(v, \alpha)$, $v \in \mathbb{R}^n$, be the normalized error from competitive output y^* in a symmetric (within industries) "equilibrium" in which there are $\frac{1}{\alpha \delta_j} + v_j$ firms of type j operating and forced to maximize profit over the restricted set $Y_j(\alpha) \cap N(\alpha \delta_j y_j^*, \alpha \delta_j)$ for $j = 1, 2, \dots, n$ (no entry is allowed). The normalization is such that aggregate output in this "equilibrium" is $y^* + \alpha Z(v, \alpha)$. Note that $\frac{1}{\alpha \delta_j} + v_j$ is not restricted to integral values.

Lemma 6: Assume the hypotheses of Theorem 1 are satisfied. Given $K < \infty$ there exists $\bar{\alpha} > 0$ such that for all $(v, \alpha) \in N(0, K) \times (0, \bar{\alpha}] \subset \mathbb{R}^n \times \mathbb{R}^1$, a $Z(v, \alpha)$ exists and

$$Z(v, \alpha) = -\{I + H_1\}^{-1} \begin{pmatrix} -v_j \delta_j y_j^* + \begin{bmatrix} 0 & -\bar{p} \\ 0 & I \end{bmatrix} G_j \begin{bmatrix} 0 & 0 \\ -\bar{p} & I \end{bmatrix} B_j' y_j^* \\ \vdots \\ \vdots \end{pmatrix}_j + O(\alpha)$$

where H_1 is the matrix in $j1$ (so $\{I + H_1\}$ is invertible).

Proof of Lemma 6:

Let $\phi_j(Z, \alpha) = Z + \frac{1}{\alpha} z_j^*(Z, \alpha)$ and define $\psi_j(Z, \alpha)$ by

$\phi_j(\psi_j(Z, \alpha), \alpha) = Z$. By Lemma 5, given $K < \infty$, there exists $\bar{\alpha} > 0$ such that for all $(Z, \alpha) \in N(O, K) \times (O, \bar{\alpha}]$, $\psi_j(Z, \alpha)$ is a well defined function and, by implicit differentiation,

$$\frac{\partial \psi_j(Z, \alpha)}{\partial Z_t} = I + \alpha \begin{bmatrix} 0 & -\bar{p}' \\ 0 & I \end{bmatrix} G_j B_t + O(\alpha^2).$$

Using $\phi_j(\psi_j(O, \alpha), \alpha) = O$ and $\phi_j(Z, \alpha) = Z + \frac{1}{\alpha} z_j^*(Z, \alpha)$, we find $\psi_j(O, \alpha) = -\delta_j y_j^* + \alpha \delta_j \begin{bmatrix} 0 & -\bar{p}' \\ 0 & I \end{bmatrix} G_j \begin{bmatrix} 0 & 0 \\ -\bar{p} & I \end{bmatrix} B_j' y_j^* + O(\alpha^2)$

and, for $Z \in N(O, K)$,

$$\psi_j(Z, \alpha) = Z - \delta_j y_j^* + \alpha \begin{bmatrix} 0 & -\bar{p}' \\ 0 & I \end{bmatrix} G_j \begin{bmatrix} 0 & 0 \\ -\bar{p} & I \end{bmatrix} \left\{ \sum_t B_t Z_t + \delta_j B_j' y_j^* \right\} + O(\alpha^2).$$

For $v \in R^n$ the backward map $\Psi(Z, v, \alpha)$ is defined by

$$\Psi(Z, v, \alpha) = Z + \frac{1}{\alpha} y^* + \sum_j \left(\frac{1}{\alpha \lambda_j} + v_j \right) [\psi_j(Z, \alpha) - Z].$$

To understand the significance of this function, note that for aggregate output in a Cournot equilibrium in pure strategies of (α) to be $y^* + \alpha Z$, each operating firm of type j has action $\alpha(Z - \psi_j(Z, \alpha))$ [as a result of defining $\psi_j(\cdot, \alpha)$ as the inverse of $\phi_j(\cdot, \alpha)$]. If there are $\frac{1}{\alpha \lambda_j} + v_j$ firms of type j operating with pure strategies, their aggregate production is $(\frac{1}{\alpha \lambda_j} + v_j) \alpha (Z - \psi_j(Z, \alpha))$. Hence if $\Psi(Z, v, \alpha) = 0$, then $y^* + \alpha Z = \sum_j \left(\frac{1}{\alpha \lambda_j} + v_j \right) \alpha (Z - \psi_j(Z, \alpha))$, and the aggregate output of the $\frac{1}{\alpha \lambda_j} + v_j$ firms, $j=1, 2, \dots, n$ is exactly the aggregate output needed. Thus, the "equilibrium" error $Z(v, \alpha)$ satisfies $\Psi(Z(v, \alpha), v, \alpha) = 0$.

$$\frac{\partial \Psi(Z, v, \alpha)}{\partial Z'} = I + \sum_j \left(\frac{1}{\alpha x_j} + v_j \right) \left[\frac{\partial \psi_j(Z, \alpha)}{\partial Z'} - I \right] = I + H_1 + O(\alpha),$$

and for $(Z, v) \in N(O, K_1) \times N(O, K_2) \subset \mathbb{R}^{2n} \times \mathbb{R}^n$ converges uniformly to $I + H_1$ as α converges to 0 (where K_1, K_2 are any finite numbers).

$$\text{Also, } \Psi(O, v, \alpha) = \left(-v_j x_j y_j^* + \begin{bmatrix} 0 & -\bar{p}' \\ 0 & I \end{bmatrix} G_j \begin{bmatrix} 0 & 0 \\ -\bar{p} & I \end{bmatrix} B_j y_j^* \right)_j + O(\alpha)$$

so $\Psi(Z, v, \alpha) = \Psi(O, v, \alpha) + (I + H_1)Z + O(\alpha)$ for

$(Z, v) \in N(O, K_1) \times N(O, K_2)$, and

$Z(v, \alpha) = -(I + H_1)^{-1} \Psi(O, v, \alpha) + O(\alpha)$. [Given the K in the statement of the lemma, an appropriate K_1 is determined by $\{Z(v, \alpha) \mid v \in N(O, K)\}$.] ||

Lemma 7: Assume the hypotheses of Theorem 1 are satisfied.

There exist $v^* \in \mathbb{R}^n$, $\xi^* > 0$, $\bar{\alpha} > 0$ such that

$$\Pi(z_j^*(\psi_j(Z(v, \alpha), \alpha), \alpha) \mid \psi_j(Z(v, \alpha), \alpha), \alpha) > 0 \text{ and}$$

$\Pi(z_j^*(Z(v, \alpha), \alpha) \mid Z(v, \alpha), \alpha) < 0$ for all $j = 1, 2, \dots, n$, for all $(v, \alpha) \in \overline{N(v^*, \xi^*)} \times (0, \bar{\alpha}]$.

Proof of Lemma 7:

$$\text{Let } Z(v, 0) = -\{I + H_1\}^{-1} \begin{matrix} -v_j \\ \vdots \\ -v_j \end{matrix} y_j^* + \begin{matrix} 0 & -\bar{p}' \\ 0 & I \end{matrix} G_j \begin{matrix} 0 & 0 \\ -\bar{p} & I \end{matrix} B_j y_j^*$$

Then for any $K < \infty$, $Z(\cdot, \alpha)$ converges uniformly to $Z(\cdot, 0)$, $\alpha Z(\cdot, \alpha)$ converges uniformly to zero, and $\frac{1}{\alpha} z_j^*(\psi_j(Z(\cdot, \alpha), \alpha), \alpha)$ and $\frac{1}{\alpha} z_j^*(Z(\cdot, \alpha), \alpha)$ converge uniformly to y_j^* for $v \in N(0, K)$.

The first expression for profit is

$$\begin{aligned} z_j^*(\psi_j(Z(v, \alpha), \alpha), \alpha) F(y^* + \alpha Z(v, \alpha)) &= \\ \alpha \{Z(v, \alpha) - \psi_j(Z(v, \alpha), \alpha)\}' F(y^* + \alpha Z(v, \alpha)) &= \\ \alpha^2 y_j^* y_j^* \sum_t B_t Z_t(v, \alpha) + (\alpha^3) \text{ (using } F'(y^*) \begin{bmatrix} 0 & -\bar{p}' \\ 0 & I \end{bmatrix} = 0) \end{aligned}$$

so $\frac{1}{\alpha^2}$ times that profit converges uniformly to

$$y_j^* y_j^* \sum_t B_t Z_t(v, 0).$$

The second expression for profit is

$$\begin{aligned} z_j^*(Z(v, \alpha), \alpha) F(y^* + \alpha Z(v, \alpha) + z_j^*(Z(v, \alpha), \alpha)) &= \\ \alpha^2 y_j^* y_j^* (\sum_t B_t Z_t(v, \alpha) + y_j^* B_j y_j^*) + (\alpha^3) \end{aligned}$$

so $\frac{1}{\alpha^2}$ times that profit converges uniformly to

$$y_j^* y_j^* (\sum_t B_t Z_t(v, 0) + y_j^* B_j y_j^*).$$

Thus, it suffices to show that there exists a $v^* \in \mathbb{R}^n$

such that

$$0 << \begin{matrix} y_1^* \\ \vdots \\ y_n^* \end{matrix} [B_1 \dots B_n] Z(v^*, 0) << - \begin{matrix} y_1^* B_1 y_1^* \\ \vdots \\ y_n^* B_n y_n^* \end{matrix}$$

But by j3 $\begin{matrix} y_1^* \\ \vdots \\ y_n^* \end{matrix} [B_1 \dots B_n] \left[\frac{\partial Z(v, 0)}{\partial v'} \right] = - \begin{matrix} y_1^* \\ \vdots \\ y_n^* \end{matrix} [B_1 \dots B_n] \{I + H_1\}^{-1} \begin{matrix} y_1^* & 0 & 0 \\ 0 & 2y_2^* & 0 \\ 0 & 0 & y_n^* \end{matrix}$

has full rank, so such a v^* does exist. By the uniform convergence on compact sets and the openness of the condition just above, the $\delta > 0$ and $\bar{\alpha} > 0$ also exist. ||

There exists $\alpha^* > 0$ such that for all $(v, \alpha) \in \overline{N(v^*, \xi^*)} \times (0, \alpha^*]$, the $Z(v, \alpha)$ "equilibrium" has strictly positive profit for operating firms who are therefore, by the remark before Lemma 5, at global profit maximizing positions ($z_j^* = z_j$) and there is no incentive for entry over the entire production set $Y_j(\alpha)$. It only remains to show that there is an equilibrium with integral numbers of firms in each industry. To this end, we introduce marginal firms, and a backward map for marginal firms (see figures 3 and 4).

As in section IV, a marginal firm must be faced with an aggregate output of other firms which makes it indifferent between entering and staying out of the market, i.e., the output of other firms must be on the marginal firm's entry indifference surface. Given an appropriate aggregate output y , the backward map for a marginal firm gives a point Z on the entry indifference surface for that firm, and a number q between zero and one such that when the firm is faced with output Z by other firms, and is active in the market (at its optimal nonzero output) with probability q , and inactive with probability $1-q$, then expected aggregate output is y . Lemma 8 computes the backward map for marginal firms.

First, for each j , for each $K < \infty$ define the entry indifference surface $EIS_j(K, \alpha) = \{Z \in N(0, K) \mid \Pi(z_j^*(Z, \alpha) \mid Z, \alpha) = 0\}$ and $EIS_j(K) = \{Z \in N(0, K) \mid y_j^* \sum_t B_t Z_t = -\delta_j y_j^* B_j y_j^*\}$. For α sufficiently small, $EIS_j(K, \alpha)$ is a C^1 manifold with boundary (since F and z_j^* are C^1) and there exists $M_1 < \infty$ such that $d(EIS_j(K, \alpha), EIS_j(K)) \leq \alpha M_1$ (where d is the Hausdorff metric).

For $\alpha \in [0, \alpha^*]$ let $Q(\alpha) = \{Z(v, \alpha) \mid v \in \overline{N(v^*, \xi^*)}\}$. Since $Z(\cdot, \alpha)$ converges uniformly to $Z(\cdot, 0)$, there exists $M_2 < \infty$ such that $d(Q(\alpha), Q(0)) \leq \alpha M_2$ for all small α . Thus for K

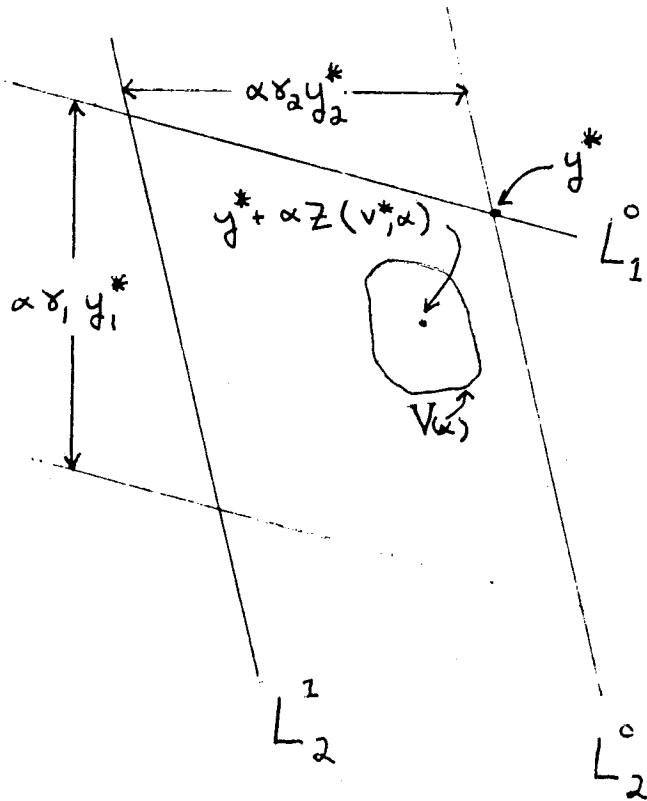


Figure 3

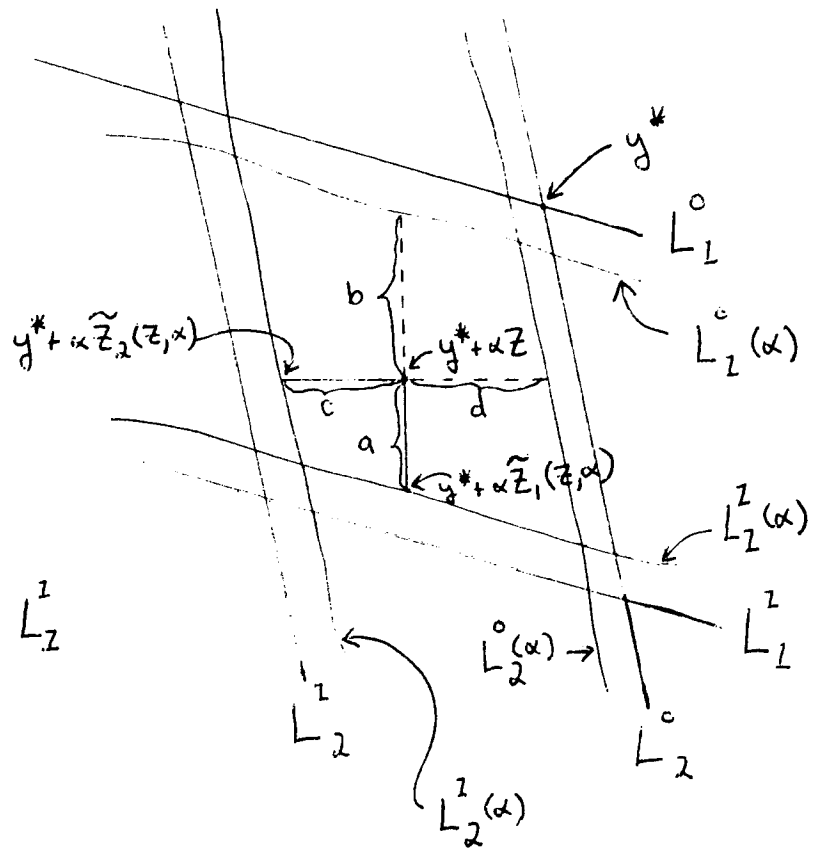


Figure 4

$$\begin{aligned}
 L_1^0 &= \{y^* + \alpha Z \mid y_1^{*'} (B_1 Z_1 + B_2 Z_2) = 0\} \\
 L_1^1 &= \{y^* + \alpha Z \mid y_1^{*'} (B_1 Z_1 + B_2 Z_2) = -\delta_1 y_1^{*'} B_1 y_1^*\} = \{y^* + \alpha Z \mid Z \in \text{EIS}_1(K)\} \\
 L_2^0 &= \{y^* + \alpha Z \mid y_2^{*'} (B_1 Z_1 + B_2 Z_2) = 0\} \\
 L_2^1 &= \{y^* + \alpha Z \mid y_2^{*'} (B_1 Z_1 + B_2 Z_2) = -\delta_2 y_2^{*'} B_2 y_2^*\} = \{y^* + \alpha Z \mid Z \in \text{EIS}_2(K)\} \\
 V(\alpha) &= \{y^* + \alpha Z(v, \alpha) \mid v \in \overline{N(v^*, \epsilon^*)}\} \\
 L_1^1(\alpha) &= \{y^* + \alpha Z \mid Z \in \text{EIS}_1(K, \alpha)\} \\
 L_1^0(\alpha) &= \{y^* + \alpha Z + z_1^*(Z, \alpha) \mid Z \in \text{EIS}_1(K, \alpha)\} \\
 L_2^1(\alpha) &= \{y^* + \alpha Z \mid Z \in \text{EIS}_2(K, \alpha)\} \\
 L_2^0(\alpha) &= \{y^* + \alpha Z + z_2^*(Z, \alpha) \mid Z \in \text{EIS}_2(K, \alpha)\} \\
 \frac{a}{a+b} &= q_1(Z, \alpha) \\
 \frac{c}{c+d} &= q_2(Z, \alpha)
 \end{aligned}$$

sufficiently large (to include the entire area of interest) there exists $\alpha_1 > 0$ such that for all $\alpha \in (0, \alpha_1]$,

$$\min_j \min \{ \|x-y\| \mid x \in \text{EIS}_j(K, \alpha), y \in Q(\alpha) \} \\ > \frac{1}{2} \min_j \min \{ \|x-y\| \mid x \in \text{EIS}_j(K), y \in Q(0) \} > 0$$

where the last inequality follows from Lemma 7 and the definitions of $Q(0)$ and $\text{EIS}_j(K)$.

For each j , for all $Z \in Q(\alpha)$, $\alpha \in [0, \alpha_1]$, the backward map for marginal firms is defined by

$$\psi_j^m(Z, \alpha) = (\tilde{z}_j(Z, \alpha), q_j(Z, \alpha)) \in \text{EIS}_j(K, \alpha) \times (0, 1) \text{ such that} \\ \tilde{z}_j(Z, \alpha) + \frac{1}{\alpha} q_j(Z, \alpha) z_j^*(\tilde{z}_j(Z, \alpha), \alpha) = Z \text{ (where } K \text{ is} \\ \text{sufficiently large to include all relevant points).}$$

Lemma 8: Assume the hypotheses of Theorem 1 are satisfied and v^*, ϵ^* are as in the statement of Lemma 7. Then there exists $\bar{\alpha} > 0$ such that

$$\psi_j^m(Z(v, \alpha), \alpha) = (Z - q_j(Z(v, 0), 0) \delta_j y_j^*, q_j(Z(v, 0), 0) + O(\alpha)) \\ \text{for all } (v, \alpha) \in \overline{N(v^*, \epsilon^*)} \times [0, \bar{\alpha}], \text{ where}$$

$$q_j(Z(v, 0), 0) = \frac{y_j^{*t} \sum_t B_t Z_t(v, 0)}{\delta_j y_j^{*t} B_j y_j^*} .$$

Proof of Lemma 8:

Existence and uniqueness of $\psi_j^m(Z(v, \alpha), \alpha)$ for small α follows from the existence and uniqueness of $\psi_j(Z, \alpha)$ and the fact that $\min_j \min \{ \|x-y\| \mid x \in \text{EIS}_j(K, \alpha), y \in Q(\alpha) \} > 0$ for small α . This also guarantees $q_j(Z(v, \alpha), \alpha)$ is bounded away from 0 and away from 1 for α small and $v \in \overline{N(v^*, \epsilon^*)}$. The expression for $\psi_j^m(Z(v, \alpha), \alpha)$ follows from the properties of $z_j^*(Z, \alpha)$ and the fact that $d(\text{EIS}_j(K, \alpha), \text{EIS}_j(K)) \leq \alpha M_1$ for small α . The expression for $q_j(Z(v, 0), 0)$ follows from the form of $\text{EIS}_j(K)$ and the fact that $\frac{1}{\alpha} z_j^*(Z, \alpha)$ converges to $\delta_j y_j^*$ as α converges to 0. ||

Let $q(Z, \alpha) = \begin{pmatrix} q_1(Z, \alpha) \\ \vdots \\ q_n(Z, \alpha) \end{pmatrix}$. By j3 and Lemma 8

$\{q(Z(v, \alpha), \alpha) \mid v \in \overline{N(v^*, \varepsilon^*)}\}$ is the closure of a neighborhood of $q(Z(v^*, 0), 0)$ for α sufficiently small. Thus, there exists an integer $m > 0$ and an $\bar{\alpha} > 0$ such that for all $\alpha \in (0, \bar{\alpha}]$,

$$\{mq(Z(v, \alpha), \alpha) - v \mid v \in N(v^*, \varepsilon^*)\} \supseteq ([-1, 1]^n + \{mq(Z(v^*, 0), 0) - v^*\}).$$

We could use a different integer m_i for each component of q , which corresponds to having a different number of marginal firms in each industry.

Proof of Theorem 1

Let v^*, ε^* be as in Lemma 7, and m as in the remarks after Lemma 8. For all $\alpha > 0$, there exists a vector $\eta^*(\alpha)$ with integral components such that

$$\left| \eta_j^*(\alpha) - \left(\frac{1}{\alpha x_j} + v_j^* \right) + mq_j(Z(v^*, 0), 0) \right| \leq \frac{1}{2} \text{ for all } j, \text{ so}$$

there exists $\alpha^* > 0$ such that

$$\eta_j^*(\alpha) - \left(\frac{1}{\alpha x_j} + \bar{v}_j(\alpha) \right) + mq_j(Z(v(\alpha), \alpha), \alpha) = 0 \text{ for all } j,$$

for some $\bar{v}_j(\alpha) \in N(v^*, \varepsilon^*)$, for all $\alpha \in (0, \alpha^*]$.

$$\left(\text{Note } \left| \eta_j^*(\alpha) - \frac{1}{\alpha x_j} \right| \leq |v_j^*| + m + \frac{1}{2} . \right)$$

Now consider a backward mapping from Z with m marginal firms of each type, and $\eta_j^*(\alpha)$ pure strategy firms of type j , $j = 1, 2, \dots, n$.

If $F(y)$ is not linear in y near y^* then an error is introduced because expected price is not equal to price of

expected output, so the backward maps, and the entry indifference surfaces etc. must be redefined to take account of the effect of the mn firms using probability distributions. However, the effect of all these changes is only $O(\alpha)$ since the support of the aggregate distribution is contained in some ball with radius αK for some $K < \infty$, and for bounded Z , $F(y^* + \alpha Z)$ is essentially linear in Z for small α .

Thus the contribution of the pure strategy firms is $\Psi(Z, \eta^*(\alpha) - \frac{1}{\alpha}\Gamma, \alpha) + O(\alpha)$ where $\Gamma' = (\frac{1}{\delta_1}, \frac{1}{\delta_2} \dots \frac{1}{\delta_n})$, while the contribution of the marginal firms is

$$-m \begin{pmatrix} q_1 & \delta_1 & y_1^* \\ \vdots & \vdots & \vdots \\ q_n & \delta_n & y_n^* \end{pmatrix} + O(\alpha). \quad \text{The backward mapping from } Z(\bar{v}(\alpha), \alpha)$$

with $\eta^*(\alpha) = \frac{1}{\alpha}\Gamma + \bar{v}(\alpha) - m q(\bar{v}(\alpha), \alpha)$ is therefore

$$\Psi(Z(\bar{v}(\alpha), \alpha), \bar{v}(\alpha) - m q(\bar{v}(\alpha), \alpha), \alpha) - m \begin{pmatrix} q_1(\bar{v}(\alpha), \alpha) & \delta_1 y_1^* \\ \vdots & \vdots \\ q_n(\bar{v}(\alpha), \alpha) & \delta_n y_n^* \end{pmatrix} + O(\alpha)$$

(where $q_i(v, \alpha) = q_i(Z(v, \alpha), \alpha)$) which is equal to

$$\begin{aligned} & Z(\bar{v}(\alpha), \alpha) + \begin{pmatrix} -(\bar{v}_j(\alpha) - m q_j(\bar{v}(\alpha), \alpha)) \delta_j y_j^* \\ \vdots \\ -(\bar{v}_j(\alpha) - m q_j(\bar{v}(\alpha), \alpha)) \delta_j y_j^* \end{pmatrix} + \frac{1}{\delta_j} \begin{bmatrix} 0 & -\bar{p}' \\ 0 & I \end{bmatrix} G_j \begin{bmatrix} 0 & 0 \\ \bar{p} & I \end{bmatrix} \begin{pmatrix} \sum_t B_t Z_t + \sum_j B_j y_j^* \\ \vdots \\ \sum_t B_t Z_t + \sum_j B_j y_j^* \end{pmatrix} \\ & - m \begin{pmatrix} q_j(\bar{v}(\alpha), \alpha) \delta_j y_j^* \\ \vdots \\ q_j(\bar{v}(\alpha), \alpha) \delta_j y_j^* \end{pmatrix} + O(\alpha) \\ & = \{I + H_j\} Z(\bar{v}(\alpha), \alpha) + \begin{pmatrix} -\bar{v}_j(\alpha) \delta_j y_j^* \\ \vdots \\ -\bar{v}_j(\alpha) \delta_j y_j^* \end{pmatrix} + \begin{bmatrix} 0 & -\bar{p}' \\ 0 & I \end{bmatrix} G_j \begin{bmatrix} 0 & 0 \\ -\bar{p} & I \end{bmatrix} B_j y_j^* \Bigg|_j + O(\alpha) \\ & = O(\alpha) \text{ by Lemma 6.} \end{aligned}$$

If we call this backward map $\Psi_F(Z, \eta^*, m, \alpha)$, then

$$\frac{\partial \Psi_F}{\partial Z'} = I + H_2 + O(\alpha) \text{ where } H_2 \text{ is the matrix in } j^2 \text{ (we choose } m \text{ to be one of the } m_s \text{ values).}$$

For α sufficiently small $\frac{\partial \psi_F}{\partial Z^T}$ is invertible by j2. Since we are interested in a compact set of v values, and therefore a compact set of Z values, and the functions we consider converge uniformly on compact sets, there exists $\alpha_1 > 0$ such that the $\eta^*(\alpha)$ pure strategy and m marginal firms are all acting optimally and generate aggregate expected output $y^* + \alpha Z(\bar{v}(\alpha), \alpha) + O(\alpha^2)$, and since $\bar{v}(\alpha) \in N(v^*, \xi^*)$, by Lemma 7 and the uniform convergence, there exists $\bar{\alpha} > 0$ such that for all $\alpha \in (0, \bar{\alpha}]$, there exists a true equilibrium with m marginal firms of each type and $\eta_j^*(\alpha)$ pure strategy firms of type j , $j=1, 2, \dots, n$. In this equilibrium

- a) there is an integral number of firms in each industry, $m + \eta_j^*(\alpha)$,
- b) all firms, including marginal firms, are maximizing profit over all mixed strategies on their entire production set,
- c) all pure strategy firms earn strictly positive expected profit, while marginal firms earn zero expected profit,
- d) there is no incentive for entry (entrants earn strictly negative expected profit if existing firms maintain their actions),
- e) there exists $S < \infty$ such that the support of $\rho(\alpha)$ is contained in $N(y^*, \alpha S)$ where $\rho(\alpha)$ is the Cournot production associated with the constructed Cournot equilibrium of $\mathcal{E}(\alpha)$. //

Proof of Theorem 2

Let $\bar{\alpha}_0$ be the α value guaranteed by Theorem 1, and let $\rho: (0, \bar{\alpha}_0] \rightarrow$ probability measures on R^{1n} be defined by $\rho(\alpha) =$ the Cournot production associated with the Cournot equilibrium of $\mathcal{E}(\alpha)$ constructed in the proof of Theorem 1. (This satisfies 2.1.)

As noted in the proof of Theorem 1, there exists $S_1 < \infty$ such that $\text{supp } \rho(\alpha) \subset N(y^*, \alpha S_1)$ for all $\alpha \in (0, \bar{\alpha}_0]$. Since F is C^2 in a neighborhood of y^* , there exists $S_2 < \infty$ and $\bar{\alpha} \in (0, \bar{\alpha}_0]$ such that $\text{supp } \xi(\alpha) \subset N(\bar{p}, \alpha S_2)$ for all $\alpha \in (0, \bar{\alpha}]$, where $\xi(\alpha)$ is the Cournot price distribution corresponding to $(F(\alpha), \rho(\alpha))$ [$F(\alpha) \equiv F$]. \parallel

Proof of Theorem 3

The set of feasible states is a subset of

$$\mathcal{F} = \{y \in R^{1n} \mid y_j \in \hat{Y}_j, \sum_j y_j + \sum_i \omega_i \geq 0\} \text{ for each economy } \mathcal{E}(\alpha).$$

By e3', there exists $\bar{K} < \infty$ such that $\mathcal{F} \subset N(0, \bar{K})$. Also by e3', 0 is an exposed point of \hat{Y}_j so there exists x_j such that $y'x_j > 0$ for all $y \in \hat{Y}_j \setminus \{0\}$.

By decomposing each $y \in \hat{Y}_j \setminus \{0\}$ into a multiple of x_j and a vector orthogonal to x_j , noting that

$\min \left\{ \frac{y'x_j}{\|y\|} \mid y \in \hat{Y}_j \setminus \{0\} \right\} > 0$ and using $\mathcal{F} \subset N(0, \bar{K})$ we see that

there exists $K < \infty$ such that for any countable set A , and $\{y_a \mid a \in A\}$ where $y_a \in \bigcup_{j=1}^n \hat{Y}_j$ for each $a \in A$,

$\sum_{a \in A} \|y_a\| \geq K$ implies that the corresponding economy output $y \in R^{ln}$ is not feasible.

\mathcal{F} is bounded so by desirability and the upper hemi continuity of the Walras correspondence, the set of feasible prices is a subset of a closed cone contained in the interior of Ω , and, as in Lemma 1, there exists $M < \infty$ such that $(*) \text{ supp } \mu(\alpha_k)(j, v) \subset N(0, M\alpha_k)$ for all $0 < v \leq \eta_j(\alpha_k)$, $0 < j \leq n$, $k \geq 1$ (0 dominates any action outside $N(0, M\alpha_k)$, and the mixed strategies are noncooperative).

We now consider the actions of operating firms as independent (for each given k) random vectors with the same underlying probability space (with expectation operator E). Let \bar{X}_{jv}^k be the random vector in R^l corresponding to the j, v firm in $\xi(\alpha_k)$, and let $X_{jv}^k = \bar{X}_{jv}^k - E\bar{X}_{jv}^k$. Since $y^{*k} = (y_1^{*k}, y_2^{*k}, \dots, y_n^{*k})$ and

$$E \sum_{v=1}^{\eta_j(\alpha_k)} \bar{X}_{jv}^k = \sum_{v=1}^{\eta_j(\alpha_k)} E \bar{X}_{jv}^k \rightarrow y_j^{*k} \text{ as } k \rightarrow \infty, \text{ in order to prove}$$

3.1 it is sufficient to show that for all $j = 1, 2, \dots, n$, $i = 1, 2, \dots, l$, and $\delta > 0$,

$$\lim_{k \rightarrow \infty} \text{prob} \left\{ \left| \sum_{v=1}^{\eta_j(\alpha_k)} X_{jvi}^k \right| < \delta \right\} = 1 \text{ where } X_{jvi}^k \text{ is the } i^{\text{th}} \text{ component of } X_{jv}^k.$$

By definition, \bar{X}_{jv}^k induces the measure $\mu(\alpha_k)(j,v)$ on R^ℓ , so by (*) $\text{prob} \{ \bar{X}_{jvi}^k \in [-M\alpha_k, M\alpha_k] \} = 1$ and therefore $\text{prob} \{ |X_{jvi}^k| \leq 2M\alpha_k \} = 1$.

Fix $j \in \{1, 2, \dots, n\}$, $i \in \{1, 2, \dots, \ell\}$, and $\delta > 0$, and for notational convenience let X_v^k denote X_{jvi}^k . Split the sequence

$(|\sum_{v=1}^{\eta_j(\alpha_k)} X_v^k|)_k$ into two parts, depending on k :

$\eta_j(\alpha_k) < (\alpha_k)^{-\frac{1}{2}}$; and, $\eta_j(\alpha_k) \geq (\alpha_k)^{-\frac{1}{2}}$. For the first part,

$\text{prob} \{ |\sum_{v=1}^{\eta_j(\alpha_k)} X_v^k| \leq \eta_j(\alpha_k) 2M\alpha_k < 2M(\alpha_k)^{\frac{1}{2}} \} = 1$.

For k sufficiently large, $2M(\alpha_k)^{\frac{1}{2}} < \delta$, and

$\text{prob} \{ |\sum_{v=1}^{\eta_j(\alpha_k)} X_v^k| < \delta \} = 1$.

For the second part of the sequence, by the Markov inequality

([8] p.158), $\text{prob} \{ |\sum_{v=1}^{\eta_j(\alpha_k)} X_v^k| \geq \delta \} \leq \frac{E|\sum_{v=1}^{\eta_j(\alpha_k)} X_v^k|^r}{\delta^r}$ for any

$r > 0$. Let $r = 3$. The X_v^k are independent (for fixed k)

with zero means, so $E|\sum_{v=1}^{\eta_j(\alpha_k)} X_v^k|^3 = \sum_{v=1}^{\eta_j(\alpha_k)} E|X_v^k|^3$.

Also $\text{prob} \{ |X_v^k| \leq 2M\alpha_k \} = 1$ and $\text{prob} \{ \sum_{v=1}^{\eta_j(\alpha_k)} |X_v^k| > 2K \} = 0$

(otherwise a state which is not feasible occurs with positive probability). Thus,

$\sum_{v=1}^{\eta_j(\alpha_k)} E|X_v^k|^3 \leq \sup \{ \sum_{v=1}^{\infty} |c_v|^3 : \sum_{v=1}^{\infty} |c_v| \leq 2K, |c_v| \leq 2M\alpha_k \text{ for all } v \}$

$< (\frac{2K}{2\alpha_k M} + 1) (2M\alpha_k)^3 = \alpha_k^2 [4KM^2 + \alpha_k 8M^3]$ where the second

inequality follows from the fact that the sum $\sum_{v=1}^{\infty} |c_v|^3$ is maximized at a boundary solution with $|c_v| = 2M\alpha_k$ for as many v as possible. Thus, $\text{prob} \left\{ \left| \sum_{v=1}^{\eta_j(\alpha_k)} x_v^k \right| \geq \delta \right\} = O(\alpha_k^2) \rightarrow 0$ as $k \rightarrow \infty$. This completes the proof of 3.1.

Let \bar{p} be defined by $F(y^*) = (1, \bar{p})$. $\xi(\alpha_k)$ is completely determined by $\rho(\alpha_k)$ and F , and 3.2 follows from 3.1 and the continuity of F at y^* .

Let x^* be the allocation in the exchange economy $\mathcal{E}(y^*) = (x_i, \omega_i + \sum_j \theta_{ij} y_j^*, \lambda_i)$ that is an equilibrium allocation at $(1, \bar{p})$ (x^* exists by definition of F and \bar{p}). F is continuous at y^* , so $(1, \bar{p})$ must be supporting prices for each cone \hat{Y}_j (otherwise entry would occur at some α_k , destroying the Cournot equilibrium in mixed strategies). Thus, $((1, \bar{p}), x^*, y^*)$ is a Walras equilibrium of $\hat{\mathcal{E}}$, which establishes 3.3.

To prove 3.4, suppose $y_j^{*'} B_j y_j^* > 0$. By e2', the boundedness of the set of feasible prices, and 3.2, given $\delta > 0$ there exists $k_1 < \infty$ such that $\text{supp } \mu(\alpha_k)(j, v) \subset N(\alpha y_j((1, p^*)), \alpha^\delta) \cup \{0\}$ for all $k \geq k_1$. For all k sufficiently large, some firm of type j must be active (by 3.1 and $y_j^* \neq 0$). For each k , let $x_k \neq 0$ be an action with positive probability for some active firm of type j , and consider an entrant using pure strategy x_k . The active firm only has actions in $N(\alpha y_j((1, p^*)), \alpha^\delta) \cup \{0\}$.

Conditional on the action of the active firm being 0, the entrant has non-negative expected profit (just as the active firm does at action x_k prior to entry). Conditional on the active firm having action in $N(\alpha y_j((1, p^*)), \alpha \delta)$, the entrant's profit increases for all states with aggregate output near y^* ($y_j^{*'} B_j y_j^* > 0$).

From the proof of 3.1, we see that for any $\varepsilon > 0$, aggregate output is in $N(y^*, \varepsilon)$ with probability $1 - O(\alpha_k^2)$, and for ε, α_k sufficiently small, the increase in profit is greater than $\frac{1}{2} \alpha_k \varepsilon y_j^{*'} B_j (\alpha_k \varepsilon y_j^*)$. The set of feasible prices is bounded, so the worst possible profit for the entrant with action $x_k \in N(\alpha y_j((1, \bar{p})), \alpha \delta)$ is greater than $-c\alpha$ for some $c < \infty$, and the probability of a loss is $O(\alpha_k^2)$. For k sufficiently large (α_k sufficiently small), the net change in expected profit has the same sign as $y_j^{*'} B_j y_j^*$, and entry will occur, upsetting equilibrium. This completes the proof of 3.4.

VI. REMARKS

A. The analysis we have presented differs substantially from standard formal treatments of competitive theory: non-convexity is essential, market power is endogenous, downward sloping demand is a requirement for static equilibrium, there is free entry, etc. Yet our complete dependence on the results of modern competitive theory (as surveyed in Debreu [4]) should be apparent to the reader. To underscore this dependence, we list some of the ways in which the paper relates (even ties together) recent developments in competitive theory.

Theorem 1 on the existence of Cournot equilibrium with entry requires the existence of Walras equilibrium in $\hat{\mathcal{E}}$. As such, the result rests on the existence of Walras equilibrium. The Pareto Optimality of Walras equilibrium translates to the approximate Pareto Optimality of Cournot Equilibrium when efficient scale is small (Theorem 3). Theorem 2, together with the result that Pareto Optima are Walras equilibria in $\hat{\mathcal{E}}$, translates to the theorem that provided efficient scale is small, any Pareto Optimum which satisfies DSD is approximately a Cournot equilibrium (relative to the appropriate specification of private ownership). Also, the replication of consumers, the idea of a limit economy, convergence results, etc, borrow from a pattern well developed in the literature on limits of cores. (A natural way to obtain decreasing returns in $\hat{\mathcal{E}}$ is to exploit a measure theoretic specification of the kind used in that literature.) Theorem 2 contains a result on the

"rate of convergence". The entire framework is equilibrium with endogenous uncertainty. A "differentiable" framework is for a great many reasons natural for the analysis; in addition, the possibility of a stability theory in \hat{E} based on the entry and exit of firms is an attractive possibility. Product differentiation with an infinity of conceivable commodities would require extensions to economies with infinite dimensional commodity spaces.

Finally, the notion of a regular economy is intimately related to the existence of a selection from the Walras correspondence, DSD is a requirement imposed on the Walras correspondence, and existence can only be guaranteed as a generic property of a class of economies.

B. The analysis could be enlarged to admit the case where industry production sets in \hat{E} exhibit (strictly) diminishing returns to scale. This could be modelled in one of two natural ways: diseconomies (external to the firm and internal to the industry); or a measure space of available firms in \hat{E} with differing efficiencies. Then, it would be possible for a firm to have a large positive profit in equilibrium (rents).

C. The assumption that consumers exhibit price taking behavior is not necessary; what is required is that every set of quantity actions by firms yield a well defined vector of prices. For suitably smooth "F", results analogous to

Theorem 1 hold. For example, suppose that in each economy $\mathcal{E}(\alpha)$, the quantity actions of firms result in an allocation in the core of the associated pure exchange economy, and let price be the implied efficiency prices. Since the cores of the exchange economies converge to the Walras equilibria of the exchange economy associated with $\hat{\mathcal{E}}$, the modified (non price taking) Cournot equilibria of $\mathcal{E}(\alpha)$ converge to the Walras equilibria of $\hat{\mathcal{E}}$. In connection with this point, it is appropriate to underscore the asymmetry between our treatment of consumers and producers. The present analysis is designed to explain perfectly competitive producer behavior and has no role in the explanation of price taking consumer behavior. The key ingredients in our explanation of perfectly competitive producer behavior are small (per-capita) efficient scale and the free entry of firms. Whatever the explanation of competitive consumer behavior, it is not free entry and small efficient scale! Firms and consumers are created and exist for different purposes; they perform a different function in the allocation of resources. In contrast with standard models of Walras equilibrium, we highlight the creation of firms; this dictates the asymmetric treatment afforded consumers and producers.

D. Since uncertainty disappears for the equilibria exhibited in Theorem 1 as α converges to zero, the results obtained are easily extended to firms that maximize a (smooth) expected utility of income function. This admits risk aversion into the analysis.

E. As long as there are only a finite number of conceivable commodities, our analysis includes the classical case of product differentiation. For any fixed value of α , there may be many commodities which are not produced. As α converges to zero, the number of different commodities actually produced (from among the large but finite number of available commodities) may increase. For situations in which there are an infinite number of available commodities, it is necessary to consider economies $\hat{\mathcal{E}}$ with infinite dimensional commodity spaces (see; eg, Hart [6] and Mas-Colell [9]).

F. With only a countable number of firms, the introduction of mixed strategies does not remove all discontinuities. The only way to remove the discontinuity associated with entry is to have a continuum of firms. We have constructed a simple example in which equilibrium with entry does not exist in mixed strategies for all $\alpha \geq \bar{\alpha} > 0$ (i.e., introduction of mixed strategies does not imply that equilibrium with entry exists for all α).

More importantly, arbitrary mixed strategies are not consistent with observed behavior. Thus, the fact that only pure and marginal firm strategies are profit maximizing in the equilibria of Theorem 1, even though arbitrary mixed strategies are allowed, is significant.

G. The introduction of marginal firms is essential. Restrictive assumptions are necessary to guarantee the

existence of equilibrium in pure strategies for all α sufficiently small. We have substantially explored such assumptions, and existence results will be reported elsewhere.

H. A natural test of the robustness of the equilibria considered here is given by the ability of a single firm with perfect information and perfect knowledge of the behavior of other agents to increase profit by departing from the prescribed behavior. We observe that the maximum profit such a firm could obtain is less than αK , where K is a constant. Thus, the benefit which accrues to strategies which are more sophisticated than those considered here will eventually be swamped by the costs of obtained information (which do not depend on α). Similarly, from the point of view of cooperative action, any cartel must be sensitive to the possibility of entry if it restricts output to increase profit. Threats aside, cartels must "limit price" to prevent entry, and so the profit which accrues to a cartel is less than αL , where L is a constant. (This is discussed by Novshek [10].)^{12/} For this reason, we believe that the results reported here have natural extensions beyond the case of quantity choosing Cournot equilibrium.

I. Firms that recognize their effect on price do not in general maximize a weighted sum of shareholders' utilities by profit maximization. Consider a sequence of Cournot equilibria, one for each economy $\xi(\frac{1}{k})$, $k = \bar{k}, \bar{k} + 1, \bar{k} + 2, \dots$

as constructed in the proof of Theorem 1, where the $\mathcal{E}(\frac{1}{k})$ are viewed as "per capita" economies for k replications of each consumer type. Let $v_i(p, w)$ be an indirect utility function for consumers of type i . For large k , the equilibrium prices and wealth (p^k, w_i^k) are approximately equal to the competitive prices and wealth $(p^*, w_i^*) = (p^*, p^{*'} \omega_i)$. For any single firm of type j for which the t^{th} consumer of type i has ownership share θ_{itj}^k in $\mathcal{E}(\frac{1}{k})$,

$$\frac{\partial v_i}{\partial y^i} \approx \frac{1}{k} \frac{\partial v_i(p^*, w_i^*)}{\partial p^i} B_j + \theta_{itj}^k \frac{\partial v_i(p^*, w_i^*)}{\partial w} p^{*'}.$$

For $\theta_{itj}^k = \frac{1}{k} \theta_{ij}$ both terms are of the same order of magnitude so either may dominate. Whenever $\theta_{itj}^k \rightarrow 0$ as $k \rightarrow \infty$, if the firm's actions are constrained to lie in a bounded set then the effect of a single firm on any single consumer converges to zero.

If for some itj , θ_{itj}^k is bounded away from zero, then for k sufficiently large,

$$\frac{\partial v_i}{\partial y^i} \approx \theta_{itj}^k \frac{\partial v_i(p^*, w_i^*)}{\partial w} p^{*'}, \text{ and the utility maximizing solution}$$

is approximately the competitive solution which is approximately the profit maximizing solution (as seen in the proof of Theorem 1). In either case, any cost (to the firm) of learning the preferences of a consumer/owner eventually outweighs the gain from knowledge of an exact "welfare function" to be maximized by the firm.¹³ The second case is related to Hart [7], which deals with a stock market model.

J. Implicit in the manner in which firms determine how much they will have to pay for an input is the assumption that in equilibria of \hat{E} the consumer sector consumes some of each commodity. Thus, there are neither pure inputs nor pure intermediate products. The producer sector can always obtain, from the price taking consumer sector, an additional amount of a commodity to use as an input by paying the appropriate price. This could be softened by introducing price taking firms, but such a device would be contrary to the spirit of the analysis.

K. Theorem 3 makes precise the sense in which DSD is a necessary condition for Cournot equilibrium. Degenerate cases aside, the condition will be satisfied whenever the consumer sector behaves as a single utility maximizing consumer (eg, if each consumer maximizes a homogeneous utility function and initial endowments are proportional). More generally, DSD will be satisfied (again excluding some degenerate cases), if the Walras correspondence satisfies the weak axiom of revealed preference locally and in certain directions: there exists $\bar{\lambda} > 0$ such that for all $j \in \{1, 2, \dots, n\}$, $y_j^* \cdot F(y) < y_j^* \cdot F(y^*) = 0$ whenever $y = y^* + \lambda y_j^*$, $\lambda \in (0, \bar{\lambda}]$.

L. When α is small, changes in demand lead to new equilibria in which both firms' actions and the number of active firms have changed. As $\alpha \rightarrow 0$, the change in the number of firms is dominant. This suggests the possibility of a "real time" dynamics in which the adjustment of price is determined by the rate at which firms enter. In addition, since firms recognize their effect on price, there is no contradiction in having them change the amount which they offer to the market (or even enter!) when preferences, technology, or other data of the economy change. It is only in the limit that a tension arises between price taking behavior and the need for prices to adjust in order to clear markets (see Arrow [1]).

M. By definition, every Pareto Optimum of $\hat{\mathcal{E}}$ which does not satisfy DSD has the characteristic that a planner (just as a firm) could buy resources at market prices and sell their product at a price which would more than cover cost. As such, they fail a natural cost benefit test. Theorem 2 then generates the result that provided efficient scale is small, every Pareto Optimum of $\hat{\mathcal{E}}$ which satisfies the appropriate cost benefit test is approximately a Cournot equilibrium (relative to the appropriate specification of private ownership).

N. Under what conditions does the Arrow-Debreu model apply? If one views industry constant returns to scale as being an idealization of small per-capita efficient scale and free entry, as done here, the Arrow-Debreu model with constant returns to

scale is applicable, but only if downward sloping demand is satisfied. If one views constant returns to scale for firms as empirically correct, then the Arrow-Debreu model captures formally the notion of perfect competition only in so far as we acknowledge that the technology is freely available to all users. (Why else will a firm take prices as given?) In this case, our argument for downward sloping demand as a requirement for static equilibrium applies again, and perfectly competitive equilibrium which do not satisfy the condition are simply the result of actions which are irrational (and in no approximate sense rational). If we understand the Arrow-Debreu theory as appropriate for conditions of decreasing returns to scale for firms and no free entry, there is the "embarrassment" of possibly significant profit and possibly significant market power to explain. Free entry is not available to make these go away (and if it was available the model would collapse). Finally, the Arrow-Debreu model is not applicable under conditions of increasing returns to scale, since equilibrium will generally not exist.

VII. CONCLUSION

Theorem 1 proves the existence of Cournot equilibrium with entry. Theorems 2 and 3 unify the equilibrium concepts of Walras and Cournot by establishing that Cournot equilibria are approximated by certain Walras equilibria.

The Arrow-Debreu model is viewed as representing a "frictionless system", whose "frictions" are barriers to entry and non-infinitesimal efficient scale. When efficient scale is small (but significant) and entry is free, certain Walras equilibria serve as good approximations to Cournot equilibria with free entry. However, when there are barriers to entry, or if efficient scale is not attained at a level which is per-capita small, the "frictionless" idealization is no longer appropriate. In the frictionless system, demand determines the measure of firms which are active in each industry. Zero profit becomes a consequence of free entry, and firms take prices as given, not because they want to, but because prices really are beyond their control. However, the analysis shows that the viewpoint of the Arrow-Debreu model as representing a frictionless system is only proper under conditions of downward sloping demand and when the efficient scale of firms is bounded away from zero. (Thus two cornerstones of the classical partial equilibrium diagram are introduced into formal general equilibrium analysis and play a central role.)

We present a simple proof that the perspective we have offered does "make a difference". Recall the standard line: consider a perfectly competitive economy with one firm (two or three firms) and constant returns to scale. From the present viewpoint, it makes no sense: the number of firms is not endogenous, and perfectly competitive behavior is not a primitive solution concept; it applies if conditions are right.

Finally, competitive behavior is unlikely to obtain with only one firm (two or three firms) active in equilibrium. In contrast, the formalism of the present analysis forces you to say this: consider an economy with one industry (two or three industries), free entry, small efficient scale, and downward sloping demand. If efficient scale is small, then the production sets of each industry exhibit approximately constant returns to scale. Small efficient scale with entry dictates the result that firms have almost no effect on price when attention is restricted to the region in which they make positive profit. The addition of a demand sector determines the number of active firms in each industry. In each industry in which there is positive output, there are a large number of firms active, each of which is producing a small amount (at approximately efficient scale).

FOOTNOTES

1 ' denotes transpose.

2 By industry, we mean the set of firms with identical technology. Several industries may use the same inputs to produce the same outputs.

3 $x^{*'} = (x_1^{*'}, x_2^{*'}, \dots, x_m^{*'}) \in R^{\ell m}$, and
 $y^{*'} = (y_1^{*'}, y_2^{*'}, \dots, y_n^{*'}) \in R^{\ell n}$.

4 In an abuse of notation, we use y_j to indicate a vector in R^ℓ and a vector in $R^{\ell n}$ (with all components zero except possibly $\ell(j-1) + 1, \ell(j-1) + 2, \dots, \ell(j-1) + \ell$). For example, in $y + \{y_j + e_1\}$, $e_1 = (1 \ 0 \ 0 \ \dots \ 0)' \in R^\ell$ is added to $y_j \in R^\ell$, which is then imbedded in $R^{\ell n}$, and added to $y \in R^{\ell n}$.

5 Let C be the set of closed convex cones in R^ℓ with vertex the origin and dimension greater than or equal to one, and define the metric d on C by

$d(c_1, c_2) = h(c_1 \cap N_\ell(0,1), c_2 \cap N_\ell(0,1))$, where h is the Hausdorff metric. For each $c \in C$ let $P(c) =$

$\{p \in R^\ell \setminus \{0\} \mid p'y \leq 0 \text{ for all } y \in c \text{ and } p'x = 0 \text{ for some } 0 \neq x \in c\}$.

Let $(c_1, c_2, \dots, c_n) \in C^n$. Then the property that $\bigcap_{i \in I} P(c_i) = \phi$ whenever $\#I > \ell - 1$ is generic $i \in I \subset \{1, 2, \dots, n\}$

(i.e., the property holds on an open dense subset of C^n).

- 6 Peter Hammond pointed this fact out to us. See, for example, Gabszewicz and Vial [5] pp. 398-400. Aside from the requirement that the numeraire is a commodity which is smoothly variable for each firm, the existence theorem remains valid independent of the choice of numeraire. This is because there are only a finite number of commodities and an $\bar{\alpha}$ which will work for each one of them.
- 7 $I(x, d\mu)$ is the Lebesgue-Stieltjes integral.
- 8 For convenience, we write $F(y)$ for $F((\omega_i + \sum_j \theta_{ij} y_j)_i)$ where $y' = (y_1', y_2', \dots, y_n') \in R^{\ell n}$.
- 9 A probing question by Kenneth Arrow led us to consider the possibility of marginal firms.
- 10 The DSD condition only applies to the industries which are "active" in the target Walras equilibrium (p^*, x^*, y^*) , i.e., those j such that there exists $0 \neq y_j \in \hat{Y}_j$ such that $p^* y_j = 0$. As noted in the remarks after e, all other industries are completely inactive, with no incentive for entry, for all prices near p^* , and therefore they can be ignored. (It is possible that there is a $0 \neq y_j \in \hat{Y}_j$ such that $p^* y_j = 0$ but $y_j^* = 0$. In this case, we use y_j rather than y_j^* in the DSD condition.) See remark K for further discussion of DSD.

- 11 The definition of B_j implicitly assumes that F is defined for certain points near y^* ; later we will require that F is twice continuously differentiable in a neighborhood of y^* . If in the Walras equilibrium (p^*, x^*, y^*) , all consumers do not consume a bundle in the interior of their consumption set, then the definition of $F(y)$ for some y near y^* may depend on a notion of equilibrium in which some consumers are not in their consumption sets. See remark J for a related discussion.
- 12 Since entrants can produce as efficiently as cartel members, each cartel member faces at least the same loss as an entrant when threats are carried out.
- 13 If $\theta_{itj}^k = \frac{1}{k} \theta_{ij}$, it is worthwhile for consumer/owners to have whole industries act collusively, in which case equilibrium (if it exists) is not in general approximately competitive. However, we are interested in an extension of the model in which $\theta_{itj}^k \neq \frac{1}{k} \theta_{ij}$ in general, and instead of exact replications, "similar" consumers are added as k increases, i.e., for each k there is a finite measure ν_k on the space of consumers (preferences, initial endowments and ownership shares) and $\nu_k \rightarrow \nu$ as $k \rightarrow \infty$.

A P P E N D I X I

PARTIAL EQUILIBRIUM (Novshek, [10])

Assumptions

For the cost function $C(y)$,

$$(C) \quad C(y) = \begin{cases} 0 & y=0 \\ C_0+V(y) & y>0 \end{cases} \quad \text{where } C_0 > 0, \text{ and for all } y \geq 0$$

$v \geq 0, v' > 0, v'' \geq 0$. Average cost is minimized uniquely at $y = 1$.

For the inverse demand function $F(Y)$,

(F) $F \in C^2([0, \infty))$ with $F' < 0$ whenever $F > 0$, and there exists $Y^* > 0$ such that $F(Y^*) = C(1)$ (equals minimum average cost). Y^* is the competitive output.

Definitions

II,1) An α size firm corresponding to C is a firm with cost function $C_\alpha(y) = \alpha C(\frac{y}{\alpha})$. For each α, C , and F , one considers a pool of available firms, each with cost function C_α , facing market inverse demand F .

II,2) Given C, F , and an $\alpha \in (0, \infty)$, an (α, C, F) market equilibrium with free entry is an integer n and a set $\{y_1, \dots, y_n\}$ of positive outputs such that

a) $\{y_1, \dots, y_n\}$ is an n firm Cournot equilibrium (without entry), i.e., for all $i=1, \dots, n$,

$$F\left(\sum_{j \neq i} y_j + y_i\right)y_i - C_\alpha(y_i) \geq F\left(\sum_{j \neq i} y_j + y\right)y - C_\alpha(y) \text{ for all}$$

$y \geq 0$, and

b) there is no profit incentive for additional entry, i.e.,

$$F\left(\sum_{j=1}^n y_j + y\right)y - C_\alpha(y) \leq 0 \text{ for all } y \geq 0.$$

The set of all (α, C, F) market equilibria with free entry is denoted $E(\alpha, C, F)$.

It is easily demonstrated that the Nash-Cournot equilibria exhibited converge to the competitive equilibrium Y^* . I.e., given C satisfying (C), F satisfying (F) and $\alpha \in (0, \infty)$, if n , $\{Y_1, Y_2, \dots, Y_n\} \in E(\alpha, C, F)$. Then $\sum_{j=1}^n y_j \in [Y^* - \alpha, Y^*]$, ([10], p. 8). This observation is made whole by the following theorem.

Theorem ([10], p.8) Given C satisfying (C) and F satisfying (F), there exists $\alpha^* > 0$ such that for all $\alpha \in (0, \alpha^*]$, $E(\alpha, C, F) \neq \emptyset$

A P P E N D I X I I

In order to show the genericity of $j_1, j_2,$ and $j_3,$ we use as basic $G_j, y_j^*, F(y^*), B_j,$ and θ_{ij} subject to

- a) G_j is negative semi definite and $x'G_jx < 0$ for $x \in P_j(R^l) \setminus \{0\}.$
- b) $y_j^{*'} F(y^*) = 0$ for all $j,$
- c) $F_1(y^*) \equiv 1,$
- d) $y_j^{*'} B_j y_j^* < 0$ for all $j,$
- e) $(1 \ 0 \ 0 \ \dots \ 0) B_j = (0 \ 0 \ \dots \ 0)$ for all $j,$ and
- f) $\sum_i \theta_{ij} = 1$ for all $j.$

The ownership shares, $\theta_{ij},$ and the shape of production sets at competitive outputs, $G_j,$ are clearly basic. The $F(y^*), y^*,$ and $\frac{\partial F}{\partial \omega},$ are considered basic in the spirit of the result that given any prices, $p,$ output, $y,$ and Jacobian $[\frac{\partial y}{\partial p}],$ a set of consumers exist whose aggregate demand is y at prices $p,$ and has Jacobian $[\frac{\partial y}{\partial p}]$ at (y,p) (Sonnenschein [15]). The B_j are used as basic rather than the $\frac{\partial F}{\partial \omega},$ since

$$[B_1 \ \dots \ B_n]_{l \times ln} = [\frac{\partial F}{\partial \omega}]_{l \times lm} \begin{bmatrix} \theta_{11} I_{l \times l} & \dots & \theta_{1n} I_{l \times l} \\ \vdots & & \vdots \\ \theta_{m1} I_{l \times l} & \dots & \theta_{mn} I_{l \times l} \end{bmatrix} \quad \text{so if } m \geq n$$

$lm \times ln$

and $\begin{bmatrix} \theta_{11} & \dots & \theta_{1n} \\ \vdots & & \vdots \\ \theta_{m1} & \dots & \theta_{mn} \end{bmatrix}_{m \times n}$ has rank n (which is a generic property

if $m \geq n$ since the θ_{ij} are basic) then if any $\frac{\partial F}{\partial \omega}$, (with $\frac{\partial F}{\partial \omega'} = 0'$) is possible, so is any $[B_1 \dots B_n]$ (with $(1 \ 0 \ \dots \ 0)_{1 \times \ell} [B_1 \dots B_n]_{\ell \times \ell n} = (0 \ \dots \ 0)_{1 \times \ell n}$).

We only consider economies which satisfy the required differentiability assumptions, and a) - f). Each economy ξ in this space of economies can be represented by a vector $((G_j)_j, (y_j^*)_j, (\theta_{ij})_{i,j}, (B_j)_j, F(y^*))$ where the matrices G_j are written as vectors (row 1, row 2, ..., row ℓ). The distance between any two economies is defined to be 1 if the number of consumers, commodities, or industries is different, or if for any industry j , the set of smoothly variable commodities is different. Otherwise, the distance is defined to be the usual distance in $R^{n\ell^2 + n\ell + mn + n\ell^2 + \ell}$. We show that the nondegeneracy conditions hold in an open dense subset of this metric space, and are therefore generic properties. Since eigenvalues and determinants are continuous functions of the entries of the corresponding matrices, and $j1 - j3$ are open conditions, they hold in a neighborhood of an economy at which they hold. It only remains to show that $j1 - j3$ hold in a dense subset of the space.

j1 Using the form of $\begin{bmatrix} 0 & -\bar{p}' \\ 0 & I \end{bmatrix} G_j$ (the first row is a linear combination of the other rows, and row i is zero if i is not a smoothly variable component), $j1$ can be transformed into an equivalent form (eliminating all components which are identically zero in $P_1(R^\ell) \times P_2(R^\ell) \times \dots \times P_n(R^\ell)$), determinant $\{D + \tilde{G}^{-1}\} \neq 0$

where $\tilde{G} = \begin{bmatrix} \tilde{G}_1 & & 0 \\ & \ddots & \\ 0 & & \tilde{G}_n \end{bmatrix}$, with \tilde{G}_j the negative definite submatrix of G_j corresponding to $P_j(R^\ell)$ (if $G_j = [0]$ then \tilde{G}_j is skipped) and where D is independent of \tilde{G} . Since \tilde{G}^{-1} has full rank, all entries of \tilde{G}^{-1} can be varied, so if $j1$ fails at ξ , it is satisfied at a sequence of economies $\xi_x \rightarrow \xi$.

j2 By $j1$, the existence of $\{I+H_1\}^{-1}$ is generic. Let λ be the smallest strictly positive eigenvalue of $\{I+H_1\}^{-1}\{H_1-H_2(1)\}$. If $H_2(m)x = -x$ then $\{I+H_1\}x = \{H_1-H_2(m)\}x = m\{H_1-H_2(1)\}x$ and $x = m\{I+H_1\}^{-1}\{H_1-H_2(1)\}x$. When $m > \frac{1}{\lambda}$ ($m \geq 1$ if λ does not exist), then $x = m\{I+H_1\}^{-1}\{H_1-H_2(1)\}x$ implies $x = 0$, so $H_2(m)$ does not have eigenvalue -1 .

j3 (a) It is generic that $N = \begin{bmatrix} y_1^* & & 0 \\ & \ddots & \\ 0 & & y_\ell^* \end{bmatrix} \begin{bmatrix} B_1 & \dots & B_n \\ & \ddots & \\ B_1 & \dots & B_n \end{bmatrix} \begin{bmatrix} y_1^* & & 0 \\ & \ddots & \\ 0 & & y_n^* \end{bmatrix}$

has rank n . To see this, let $B_j(x) = B_j + x \begin{bmatrix} 0 & 0 \\ & I_{\ell-1} \end{bmatrix}$ for all j (I is $(\ell-1) \times (\ell-1)$) and let $N(x)$ be the matrix corresponding to N

when B is replaced by $B(x)$. Then $N(x) = N + x \begin{bmatrix} z_1' & z_1 & \dots & z_1' & z_n \\ & \vdots & & & \vdots \\ z_n' & z_1 & \dots & z_n' & z_n \end{bmatrix}$

where $z_j' = (y_{j2}^*, y_{j3}^* \dots y_{j\ell}^*)$. The $z_1, z_2, \dots, z_n \in R^{\ell-1}$ are generically independent since $n \leq \ell-1$ and the y_j^* are basic, so the matrix multiplied by x has rank n generically. Expanding the determinant of $N(x)$, we find

$$\det N(x) = a_n + xa_{n-1} + x^2a_{n-2} + \dots + x^{n-1}a_1 + x^na_0.$$

Let s be the largest integer such that $a_s \neq 0$ (such an s exists since $a_0 = \det ([z_i'z_j]_{ij}) \neq 0$). Then

$$\frac{1}{x^{(n-s)}} \det N(x) = a_s + xa_{s-1} + \dots + x^sa_0 + a_s \neq 0 \text{ as } x \rightarrow 0.$$

For all x sufficiently small, $\det N(x) \neq 0$ so if N does not have rank n for economy ξ , $N(x)$ does for a sequence of economies $\xi_x \rightarrow \xi$.

(b) It is generic that

$$\begin{aligned} M &= \begin{bmatrix} y_1^{*'} \\ \vdots \\ y_n^{*'} \end{bmatrix} [B_1 \dots B_n] \{I+H_1\}^{-1} \begin{bmatrix} y_1^* & 0 \\ \vdots & \vdots \\ 0 & y_n^* \end{bmatrix} \\ &= \begin{bmatrix} y_1^{*'} & 0 \\ \vdots & \vdots \\ 0 & y_n^{*'} \end{bmatrix} \begin{bmatrix} B_1 & \dots & B_n \\ \vdots & \vdots & \vdots \\ B_1 & \dots & B_n \end{bmatrix} \{I+H_1\}^{-1} \begin{bmatrix} y_1^* & 0 \\ \vdots & \vdots \\ 0 & y_n^* \end{bmatrix} \quad \text{has rank } n. \end{aligned}$$

To see this, let $B(x) = B + xB\{I-xH_1H_1\}^{-1} H_1\{I+H_1\}$ where

$$B = \begin{bmatrix} B_1 & \dots & B_n \\ \vdots & \vdots & \vdots \\ B_1 & \dots & B_n \end{bmatrix}_{n \times n} \quad (\text{the inverse exists for all small } x), \text{ and}$$

let $M(x)$ be the matrix corresponding to M when B is replaced by $B(x)$ (note B occurs in H_1 , which becomes $H_1(x)$).

Computation yields

$$B(x)\{I+H_1(x)\}^{-1} = B\{I+H_1\}^{-1} [(1-x)I + x\{I+H_1\}] = (1-x)B\{I+H_1\}^{-1} + xB.$$

Thus $M(x) = (1-x)M + xN$ where N (the matrix from (a)) generically has rank n , and, proceeding as in (a), we see $M(x)$ has rank n for all small nonzero x . If j_3 fails for economy ξ , it is satisfied for a sequence of economies $\xi_x \rightarrow \xi$.

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