

ECONOMETRIC POLICY EVALUATION AND OPTIMIZATION

UNDER RATIONAL EXPECTATIONS

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1. Introduction

This paper is concerned with methods for evaluating given economic policies and for finding optimal policies using an econometric model under the assumption of rational expectations. For the purpose of this discussion, the assumption of rational expectations will be granted without implying that the author will necessarily subscribe to it for most macroeconomic applications. This work, as well as several of the papers to be cited below, has been stimulated by the critical comments of Robert Lucas (1976) on econometric policy evaluation which takes inadequate account of the effects of government policy on the expectations formed by the economic agents. One answer to this critique which Lucas himself advocates is the use of rational expectations. These are conditional expectations of the endogenous variables generated by the econometric model itself given the available information inclusive of the government policy. Such an approach to policy evaluation raises theoretical and computational problems which are to be addressed in this paper. Similar problems will also arise when one wishes to formulate optimal policy under the assumption of rational expectations. Some authors, including Kydland and Prescott (1977) in particular, feel strongly that these problems are insoluble and therefore optimal control theory should not be applied to economic planning. We will attempt to solve these problems and to show that optimal control is applicable in an environment of rational expectations.

There appears to be a consensus among several authors, including R. J. Shiller (1977), K. F. Wallis (1977), P. A. Anderson (1977) and J. B. Taylor (1978), that policy evaluation using an econometric model under rational expectations is fairly straightforward, provided that (1) the model is linear and (2) expectations of future endogenous variables do not appear in the model. We will therefore use this simple set-up as the starting point of our discussion of econometric policy evaluation in section 2. The complications due to the presence of expectations of future endogenous variables and of nonlinear relationships in the model will be introduced in turn. In section 3, we consider the formulation of optimal control policies. Section 4 includes some concluding remarks.

2. Econometric Policy Evaluation

2.1 Linear Model Without Expectations of Future Variables

In the simple situation where the model is linear and expectations of future endogenous variables are absent, we can write the model in its reduced form as

$$(1) \quad y_t = B y_t^* |_{t-1} + A y_{t-1} + C x_t + b_t + v_t .$$

Here y_t denotes a vector of endogenous variables, x_t denotes a vector of policy instruments or control variables, b_t is a vector summarizing the combined effects of the exogenous variables not subject to control, and v_t is a vector of serially uncorrelated, identically distributed disturbances.

$y_t^* |_s$ is the conditional expectation $E(y_t | I_s)$ of y_t given the information

I_s as of the end of period s . As is well-known, one can eliminate the variables y_{t-k} for $k > 2$ and x_{t-k} for $k > 1$ if they exist in the original model and rewrite it as equation (1) by suitable definitions of new variables. Since the expectations of only a small fraction of the endogenous variables may appear in the model, many columns of the matrix B will be zero.

One well-known method to eliminate the expectations $y_{t|t-1}^*$ from the model (1) and to derive a model for only the directly observables is to take conditional expectations of both sides of (1) given I_{t-1} and solve for $y_{t|t-1}^*$:

$$(2) \quad y_{t|t-1}^* = (I-B)^{-1} [Ay_{t-1} + Cx_{t|t-1}^* + b_{t|t-1}^*]$$

Substituting the right-hand side of (2) for $y_{t|t-1}^*$ in (1) and simplifying give

$$(3) \quad y_t = (I-B)^{-1} [Ay_{t-1} + Cx_{t|t-1}^* + b_{t|t-1}^*] + C(x_t - x_{t|t-1}^*) + b_t - b_{t|t-1}^* + v_t.$$

The model (3) can be used for policy evaluation as rational expectations are already incorporated by using (2) to represent the expectations $y_{t|t-1}^*$ in the model (1). To specify completely the time path for y_t from model (3), it is sufficient to assume that both the public and the government policy maker share the same expectations $x_{t|t-1}^*$ and $b_{t|t-1}^*$ for the control variables and the combined effects of the other exogenous variables, and, secondly, that the deviations $(x_t - x_{t|t-1}^*)$ and $(b_t - b_{t|t-1}^*)$ are serially uncorrelated and have zero mean. The terms $C(x_t - x_{t|t-1}^*) + (b_t - b_{t|t-1}^*)$ can be combined with v_t in (3) to form a new residual.

2.2 Linear Model with Expectations of Future Variables

A complication arises if $y_{t+k|t-k}^*$ ($k > 1$) appears in the model. We need only to consider the presence of $y_{t+1|t-1}^*$ since the expectations of the other future variables can be eliminated by suitable definition of new variables, as is done in Wallis (1977) for example. The model becomes

$$(4) \quad y_t = B y_{t|t-1}^* + B_1 y_{t+1|t-1}^* + A y_{t-1} + C x_t + b_t + v_t$$

Taking conditional expectations of both sides of (4) given I_{t-1} , solving for $y_{t|t-1}^*$, and substituting the resulting expression for $y_{t|t-1}^*$ back into (4), one obtains a generalization of (3),

$$(5) \quad \begin{aligned} y_t &= (I-B)^{-1} [B_1 y_{t+1|t-1}^* + A y_{t-1} + C x_{t|t-1}^* + b_{t|t-1}^*] + \eta_t \\ &= \tilde{B}_1 y_{t+1|t-1}^* + \tilde{A} y_{t-1} + \tilde{C} x_{t|t-1}^* + \tilde{b}_{t|t-1}^* + \eta_t \end{aligned}$$

where we have let η_t denote $C(x_t - x_{t|t-1}^*) + (b_t - b_{t|t-1}^*) + v_t$, and have defined \tilde{B}_1 as $(I-B)^{-1} B_1$, etc. Unlike (3), the model (5) retains an expectation vector $y_{t+1|t-1}^*$ which is not directly observable.

We shall distinguish between two types of policy, an open-loop policy where a vector (x_1, \dots, x_T) of values for the policy instruments is announced for T periods, T being the planning horizon, and a linear feedback policy which takes the form $x_t = G_t y_{t-1} + g_t$. In either case, the policy is announced at the beginning of the planning period and the economic agents are assumed to form their expectations according to the announcements. In other words, $x_{t|t-1}^*$ is known; it equals the announced policy in the case of an open-loop policy and equals $G_t y_{t-1} + g_t$ in the case of a feedback policy,

G_t and g_t being given constants. Furthermore, it will be assumed that both the policy maker and the public share identical expectations $b_t^*|_{t-1}$ for the combined effects of the exogenous variables; otherwise policy evaluation would be impossible under rational expectations.

For the purpose of policy evaluation, consider equation (4) for period T . It explains y_T using $y_{T+1}^*|_{T-1}$. To obtain a unique (stochastic) model for y_T , and in fact for all y_t ($t=1, \dots, T$), we will assume that $y_{T+1}^*|_{T-1}$ is a given linear function of $y_T^*|_{T-1}$ (and of y_{T-1} if necessary). Each linear function assumed will yield a model for y_T and thus for y_t ($t=1, \dots, T$). This is not to provide a general answer to the multiple-solution problem arising from models under rational expectations as discussed in Taylor (1977) and Shiller (1977) for example. We are merely suggesting that in order to arrive at a unique sequence of predictions, an advocate of rational expectations needs to supply an additional condition, and that equation (4) for the terminal period T is a convenient place to state and examine such a condition. When T is sufficiently large, it is reasonable to assume that selected elements of $y_{T+1}^*|_{T-1}$ are equal or proportional to the corresponding elements of $y_T^*|_{T-1}$ for making a policy evaluation in period one. This assumption can be replaced by the assumption of equality or proportionality between $y_{T+1}^*|_0$ and $y_T^*|_0$. Since the expectations of y_t ($t=1, \dots, T$) to be evaluated are all conditioned on information at time zero, variables in the following procedure can be replaced by their expectations given I_0 without affecting the calculations. Having made an assumption for $y_{T+1}^*|_{T-1}$, we can eliminate it from (4), take expectations given I_{T-1} , and solve out $y_T^*|_{T-1}$ to obtain

$$(6) \quad y_T = \tilde{A}_T y_{T-1} + \tilde{C}_T x_T^*|_{T-1} + \tilde{b}_T|_{T-1} + \eta_T$$

Given the terminal condition (6), and given an open-loop policy, the set of difference equations (4) can be solved backward in time. Taking expectations of both sides of (6) given I_{T-2} and substituting the result for y_T^* in the equation for y_{T-1} , we have

$$(7) \quad y_{T-1} = B y_{T-1}^* |_{T-2} + B_1 [\tilde{A}_T y_{T-1}^* |_{T-2} + \tilde{C}_T x_T^* |_{T-2} + \tilde{b}_T^* |_{T-2}] \\ + A y_{T-2} + C x_{T-1} + b_{T-1} + v_{T-1}$$

Since (7) no longer contains the expectation $y_T^* |_{T-2}$ of future endogenous variables, we can take conditional expectations given I_{T-2} to obtain an equation for $y_{T-1}^* |_{T-2}$ and substitute the result back into (7) to yield an equation, analogous to (3),

$$(8) \quad y_{T-1} = (I - B - B_1 \tilde{A}_T)^{-1} [A y_{T-2} + B_1 \tilde{C}_T x_T^* |_{T-2} + B_1 \tilde{b}_T^* |_{T-2} + C x_{T-1}^* |_{T-2} + b_{T-1}^* |_{T-2}] \\ + C (x_{T-1} - x_{T-1}^* |_{T-2}) + b_{T-1} - b_{T-1}^* |_{T-2} + v_{T-1} \\ \equiv \tilde{A}_{T-1} y_{T-2} + \tilde{C}_{T-1, T} x_T^* |_{T-2} + \tilde{C}_{T-1, T-1} x_{T-1}^* |_{T-2} + D_{T-1, T} \tilde{b}_T^* |_{T-2} \\ + \tilde{b}_{T-1}^* |_{T-2} + \eta_{T-1}$$

The process continues by taking expectations of both sides of (8) given I_{T-3} and substituting the result for $y_{T-1}^* |_{T-3}$ in the equation for y_{T-2} , yielding an equation for y_{T-2} corresponding to (7). This equation no longer contains the expectation $y_{T-1}^* |_{T-3}$ of future endogenous variables, and can be converted into an equation like (8). In general, the resulting equation for y_t is

$$(9) \quad y_t = \tilde{A}_t y_{t-1} + \sum_{i=t}^T \tilde{C}_{t,i} x_i^* + \sum_{i=t}^T D_{t,i} \tilde{b}_i^* + \eta_t$$

where $D_{t,t} = I$. Given an open-loop policy which specifies the expectations of all future x_t , and given the initial condition y_0 , (9) can be used to generate predictions for y_t ($t=1, \dots, T$) for the purpose of policy evaluation.

If a linear feedback policy $x_t = G_t y_{t-1} + g_t$ is to be evaluated, we can substitute this rule for x_t in (4) to obtain

$$(10) \quad y_t = B y_{t|t-1}^* + B_1 y_{t+1|t-1}^* + R_t y_{t-1} + \bar{b}_t + v_t$$

where $R_t = A + C G_t$ and $\bar{b}_t = b_t + C g_t$. (10) can be solved backward in time once a terminal condition for $y_{T+1|T-1}^*$ is specified. Using the feedback rule, we can replace $x_{T|T-1}^*$ in (6) by $G_T y_{T-1} + g_T$ to yield the terminal condition

$$(11) \quad y_T = \tilde{R}_T y_{T-1} + \bar{b}_T^* + \eta_T$$

where $\tilde{R}_T = \tilde{A}_T + \tilde{C}_T G_T$ and $\bar{b}_T^* = \tilde{b}_T^* + \tilde{C}_T g_T$. One can follow the derivation from (6) to (9) and obtain an equation analogous to (9),

$$(12) \quad y_t = \tilde{R}_t y_{t-1} + \sum_{i=t}^T D_{t,i} \bar{b}_i^* + \eta_t$$

where the expectations of the policy variables have disappeared as they are incorporated in \tilde{R}_t , $D_{t|i}$ and \bar{b}_i^* . In obtaining the matrix coefficients in (9) and (12), one should be aware of the problems of computational errors due to repeated matrix-multiplications in the backward solutions.

However, the author's experience with the similar problem in solving matrix Riccati equations backward in time for systems of about 100 equations for 20 periods indicates that the problem can usually be solved using double-precision arithmetics with a modern computer.

2.3 Nonlinear Model with Expectations of Future Variables

If the model is nonlinear, but expectations of future endogenous variables are absent and the random disturbances are ignored, then the expectations of the endogenous variables generated by the model under rational expectations are equal to the values of the endogenous variables themselves. Therefore, to find the solutions for the endogenous variables (or equivalently their expectations) one can replace all expectations $y_t^*|_{t-1}$ by the variables y_t themselves in the model and proceed to perform nonstochastic simulations given any policy to be evaluated. This approach was taken by Paul Anderson (1977) for the St. Louis model and the FRB-MIT-Penn model. Anderson's main finding is that the short-run Phillips curve is much more nearly vertical according to both models under rational expectations than under the distributed lag formulations of expectations with the lag structure assumed to be constant over alternative policies.

If the model is nonlinear and stochastic, and the expectation $y_{t+1}^*|_{t-1}$ appears, we suggest linearizing the model about a tentative solution path, applying the methods of section 2.2 to obtain the solution for the linearized model, relinearizing about the expectation of the new solution path, and iterating till convergence. Specifically, let the nonlinear model be

$$(13) \quad F(y_t, y_t^*|_{t-1}, y_{t+1}^*|_{t-1}, y_{t-1}, x_t) = u_t$$

where F is a vector function and u_t is a vector of random disturbances. Using some distributed lag or any reasonable estimates for the expectations Y_t^* and Y_{t+1}^* , simulate the model (13) for $t = 1, \dots, T$ with $u_t = 0$, given either an open-loop policy $x = (x_1^0, \dots, x_T^0)$ or a feedback policy $x_t = G_t Y_{t-1} + g_t$ which is to be evaluated. Let the result of this simulation be Y_1^0, \dots, Y_T^0 .

We next linearize the model about $Y_t^0, Y_t^0|_{t-1} = Y_t^0, Y_{t+1}^0|_{t-1} = Y_{t+1}^0$ and Y_{t-1}^0 . Denoting by F_{jt} the partial derivative of F with respect to its j^{th} argument evaluated at the above point, we write

$$(14) \quad F_{1t}(y_t - Y_t^0) + F_{2t}(y_t^*|_{t-1} - Y_t^0) + F_{3t}(y_{t+1}^*|_{t-1} - Y_{t+1}^0) + F_{4t}(y_{t-1} - Y_{t-1}^0) = u_t$$

In evaluating F_{jt} in (14), we let $x_t = x_t^0$ if an open-loop policy is to be evaluated. For a feedback policy, we let $x_t = G_t Y_{t-1} + g_t$ and combine in F_{4t} the derivatives of F with respect to both its fourth and fifth arguments, the latter being replaced by $G_t Y_{t-1} + g_t$. Multiplying (14) by F_{1t}^{-1} which is assumed to exist if the simultaneous equations model (13) provides a unique solution, we obtain a linear model

$$(15) \quad Y_t = B_t Y_t^*|_{t-1} + B_{1t} Y_{t+1}^*|_{t-1} + A_t Y_{t-1} + b_t + v_t$$

where $v_t = F_{1t}^{-1} u_t$. (15) corresponds to (4), except that the coefficients are now time-dependent and that the given policy has already been incorporated, with no need for the term Cx_t in (4). We can therefore apply the methods of section 2.2 to solve for y_t from the linear model (15). The result, with the residuals η_t set equal to zero, will provide a new set of initial values

y_1^0, \dots, y_T^0 about which the model (13) can be relinearized. One can continue to iterate until convergence.

As in the case of policy evaluation for linear models in which expectations of future endogenous variables appear, some terminal condition on $y_{T+1}^*|_{T-1}$ would have to be imposed to obtain a unique solution. Given such a condition, one could start with some initial guess for the expectations, simulate the model given these expectations, use the results of the simulation to provide a new set of expectations, and continue iteratively. Such an iterative procedure was used by Fair (1978). It differs from the method recommended above as it involves no linearizations of the model and does not solve the equations recursively backward in time. Once a condition on $y_{T+1}^*|_{T-1}$ is imposed, and with $u_t = 0$, the set of equations (13) for $t = 1, \dots, T$ becomes a set of nonlinear algebraic equations in $(y_1, \dots, y_T) = (y_1^*, \dots, y_T^*)$. The method suggested in the last paragraph amounts to the Gauss-Newton method for solving a nonlinear system of equations by repeated linearizations, except that the solution in each linearization is obtained recursively backward in time. For many practical problems, the Gauss-Newton method has been found to have reasonably good convergence properties.

3. Econometric Policy Optimization

3.1 Linear Model Without Expectations of Future Variables

For a linear model without expectations of future variables as treated in section 2.1, the standard techniques of optimal control apply since the expectation $y_t^*|_{t-1}$ can be eliminated from model (1) to yield model (3),

the parameters of which are invariant with respect to government policy under rational expectations. An interesting application of this case has been provided by John Taylor (1978). There appears to be general agreement that policy optimization using an econometric model under rational expectations is possible and useful at least when the expectations of future variables are absent from the model.

At one time, there might have been some misunderstanding of the conclusion reached by Sargent and Wallace (1975) to the effect that "rational expectations" would rule out any effect of monetary policy on the real economy. These authors stated clearly in their paper (1975, p. 254) that their conclusion depends, in addition, on the aggregate supply hypothesis of Lucas which embodies the natural rate hypothesis. According to this hypothesis, real output can deviate from the natural rate only when there is a deviation of the actual price level from the expected price level. As is well-known and is easily seen from (3) under the assumption of rational expectations, the deviation of any endogenous variable (including the price level) from its expectation in a linear model is determined by the deviations of the policy instruments from their expectations and is thus unaffected by announced policy changes. Hence, announced monetary policy has no effect on real output. Note that the assumption of rational expectations alone does not rule out the effectiveness of monetary policy. Also, the effects of fiscal policy on the real economy and the effect of monetary policy on the price level are interesting questions to pursue even when some form of the natural rate hypothesis is accepted in addition. Therefore, one cannot escape the conclusion that the tools of optimal control remain useful under rational expectations at least for models having no expectations of future endogenous variables.

3.2 Linear Model with Expectations of Future Variables

The situation is more complicated when expectations of future endogenous variables appear in the model, as we will discuss in this subsection. However, for readers of section 2.2, the applicability of optimal control techniques to such a model seems obvious. If the consequences of a given policy, be it open-loop or feedback, can be evaluated, then one can always search for an optimal policy once a welfare or loss function is given. Let us pursue this viewpoint. We assume that a loss function is given and that the proposed policy is to be announced at the beginning of period one and it will be used, together with the econometric model, by the economic agents to form their expectations. The policy maker wishes to find an optimal policy in this setting which will minimize total expected loss for T periods. We will consider in turn open-loop policy and feedback control policy.

Optimal open-loop policy can be obtained by first stacking up the difference equations (6) and (9) as a system, with x_t replacing x_t^* as the policy is announced and accepted, and assuming $\tilde{b}_{t|s}^* = \tilde{b}_{t|0}^* \equiv \tilde{b}_t^*$,

$$(16) \quad \begin{bmatrix} \bar{I} & -\bar{A}_T & 0 & \dots & 0 & \bar{y}_T \\ 0 & \bar{I} & -\bar{A}_{T-1} & \dots & 0 & \bar{y}_{T-1} \\ & & \ddots & & & \vdots \\ 0 & \dots & 0 & \bar{I} & & \bar{y}_1 \end{bmatrix} = \begin{bmatrix} \bar{c}_T & 0 & & & 0 \\ \bar{c}_{T-1,T} & \bar{c}_{T-1,T-1} & \dots & & 0 \\ & \dots & & & \\ \bar{c}_{1,T} & \bar{c}_{1,T-1} & \dots & \bar{c}_{1,1} & & \end{bmatrix} \begin{bmatrix} \bar{x}_T \\ \bar{x}_{T-1} \\ \vdots \\ \bar{x}_1 \end{bmatrix} \\
 + \begin{bmatrix} \bar{I} & 0 & \dots & 0 \\ \bar{D}_{T-1,T} & \bar{I} & \dots & 0 \\ & \dots & & \\ \bar{D}_{1,T} & \bar{D}_{1,T-1} & \dots & \bar{I} \end{bmatrix} \begin{bmatrix} \bar{b}_T^* \\ \bar{b}_{T-1}^* \\ \vdots \\ \bar{b}_1^* + \bar{A}_1 \bar{y}_0 \end{bmatrix} + \begin{bmatrix} \bar{\eta}_T \\ \bar{\eta}_{T-1} \\ \vdots \\ \bar{\eta}_1 \end{bmatrix}$$

or, written more compactly,

$$(17) \quad Ay = Cx + Db + \eta$$

If the loss function is quadratic in y , one can easily solve (17) for y , substitute the result in the loss function, evaluate expected loss given I_0 , and minimize with respect to x . Note, however, that a vector of control variables (x_1, \dots, x_T) determined at the beginning of period one and not to be altered as more observations become available cannot be optimal. In the model (4), the first three terms on the right-hand-side are all functions of y_{t-1} , implying that knowledge of y_{t-1} will be useful in controlling y_t . Hence a feedback policy may be better than a deterministic open-loop policy.

To find an optimal feedback policy, one might be tempted to apply the familiar method of dynamic programming using equations (6) and (9). By this method, one would first minimize with respect to x_T the expected loss attributable to y_T of the last period, given I_{T-1} , and obtain an optimal feedback control equation for x_T . Having obtained the optimal rule for x_T , one would go back in time to find an optimal rule for x_{T-1} , etc. As pointed out by Kydland and Prescott (1977), this method would not be optimal because, by considering equation (6) alone for determining x_T , one ignores the effects of expected x_T on previous y_t ($t < T$) as given by equation (9). Kydland and Prescott (1977, p. 474) write, "current decisions of economic agents depend in part upon their expectations of future policy actions ... only if these expectations were invariant to the future policy plan selected would optimal control theory be appropriate." They claim (pp. 473-474) that optimal control theory "is not the appropriate tool for

economic planning even when there is a well-defined and agreed-upon fixed social objection function Rather, by relying on some policy rules, economic performance can be improved." I agree with these authors that, if expectations of future endogenous variables enter an econometric model, the method of dynamic programming cannot be applied without allowing for the expectational effects of future policy on current actions. However, I cannot concur that optimal control theory is useless and that "by relying on some policy rules, economic performance can be improved." What rules should the policy maker follow? If one considers linear, time-invariant feedback rule of the form $Y_t = GY_{t-1} + g$, one can use equations (11) and (12) for a description of the dynamics of the system under any given rule. The expectations of welfare loss associated with any given parameters G and g can be evaluated. One can then minimize expected loss with respect to G and g by some gradient method.

Furthermore, dynamic programming can be applied to equation (5) to find an optimal feedback rule. Treating $Y_{t+1}^*|_{t-1}$ in (5) as given and minimizing the expectation of a quadratic loss function for T periods, we can apply dynamic programming as in Chow (1975, Chap. 8) to find an optimal feedback control equation

$$(18) \quad \hat{x}_t = G_{1t}Y_{t+1}^*|_{t-1} + G_{2t}Y_{t-1} + g_t$$

Under certain conditions concerning the system parameters as discussed in Chow (1975, pp. 170-172), the coefficients G_{1t} , G_{2t} and g_t may become time-invariant as T increases. It may also happen that G_{1t} and G_{2t} are time-invariant and g_t changes through time to reflect changes in $\tilde{b}_t^*|_{t-1}$, but the system under optimal control will remain covariance stationary. Let us assume that the system (4) or (5) can be made covariance-

stationary by using such a rule, i.e., when (18) is substituted for $x_t^*|_{t-1}$ in (5), we will obtain a covariance-stationary system:

$$(19) \quad y_t = R_1 y_{t+1}^*|_{t-1} + R_2 y_{t-1} + r + \eta_t$$

where $R_1 = \tilde{B}_1 + \tilde{C}G_1$, $R_2 = \tilde{A} + \tilde{C}G_2$ and $r = \tilde{b}_t^*|_{t-1} + \tilde{C}g_t$. If (19) is covariance-stationary under rational expectations, there must exist an observationally equivalent system

$$(20) \quad y_t = Qy_{t-1} + q + \eta_t$$

where the roots of the matrix Q are all smaller than one in absolute value. To find the matrix Q , we use (20) to derive the expectation

$$(21) \quad y_{t+1}^*|_{t-1} = Q^2 y_{t-1} + (Q + I)q$$

and substitute the result into (19),

$$(22) \quad y_t = (R_1 Q^2 + R_2) y_{t-1} + R_1 (Q + I)q + r + \eta_t$$

Equating coefficients in (20) and (22) yields

$$(23) \quad Q = (I - R_1 Q)^{-1} R_2; \quad q = [I - R_1 (Q + I)]^{-1} r$$

Having solved for Q and q , we can substitute (21) into (18) to get an optimal feedback rule as a function of y_{t-1} only. This rule is optimal because it is a time-invariant feedback on $y_{t+1}^*|_{t-1}$ formed by rational expectations, thus allowing for the effect of future policy on current actions. If the matrices G_{1t} and G_{2t} computed from the optimal control algorithm turn out to change appreciably for $t = 1, 2, \dots$, then the system cannot be made stationary and the method of dynamic programming breaks down. Yet, one can still fall back on the gradient method mentioned in the preceding paragraph.

Therefore, contrary to the claim of Kydland and Prescott (1977), optimal control theory is still applicable when "current decisions of economic agents depend in part upon their expectations of future policy actions." Kydland and Prescott have correctly pointed out that optimal control theory cannot be applied to models such as (4) where expectations of future variables enter without appropriately taking these expectations into account. However, if the effects of future expectations are properly incorporated, as in the procedure described above, optimal control theory is applicable. This discussion is parallel to the discussion following the original critique of Lucas (1976) on existing econometric policy evaluation. The consensus, shared by Lucas (1976), Shiller (1977), Wallis (1977), Anderson (1977) and Taylor (1978), appears to be that while one should not apply policy analysis directly to model (1) without adequately allowing for the effects of policy on the expectation $y_t^*|_{t-1}$, one can apply policy analysis and optimal control to model (3) because the effects of policy on $y_t^*|_{t-1}$ are properly incorporated. The comment of Kydland and Prescott appears to be that one should not apply optimal control to model (4) without due allowance for the effects of future policy on $y_{t+1}^*|_{t-1}$. Our response is that, if the model can be made stationary, one can incorporate the effects of future policy on $y_{t+1}^*|_{t-1}$ by using rational expectations to derive optimal control policies. Even if the model cannot be made stationary, one can apply some gradient method to minimize expected loss for T periods once the consequences of any policy rule can be properly evaluated under rational expectations.

A few words should be said concerning the usefulness of the assumption that the model can be made stationary through time. My view is that this assumption is likely to be appropriate if one formulates the econometric model correctly as one should. Variables such as the rate of inflation and the rate of unemployment as generated by an econometric model should eventually approach

a covariance stationary state. For growth variables such as real GNP, if we formulate the model and specify the loss function in terms of the rates of change, the system under control can be covariance-stationary. The level of GNP will not be stationary, but it is determined, through an identity, from a stationary system under control. Even when one fails to obtain a covariance-stationary system from the optimum control calculations, one uncovers important dynamic properties of the model through these calculations. One would learn that no control rules with constant coefficients can make the economic system covariance-stationary under rational expectations.

The methods of this paper assume that the policy makers will follow the policy which they announce and that the public in forming its expectations believes the policy makers to be honest. We have not studied the possibility of government deception by announcing a future policy to influence the public's expectations and then revising it when the time comes. As Lucas (1976, p. 42) remarks, "policy makers, if they wish to forecast the response of citizens, must take the latter into their confidence."

3.3 Nonlinear Model with Expectations of Future Variables

As in section 2.3 dealing with policy evaluation, we propose to linearize a nonlinear model and apply the methods of section 3.2 iteratively to obtain an approximately optimal policy under rational expectations when expectations of future endogenous variables are present. Given any tentative policy, be it open-loop or feedback, one can linearize the nonlinear model as described in section 2.3. The only modification is that, when policy optimization is considered, one needs to treat the policy variables x_t explicitly by adding

the term $F_{5t}(x_t - x_t^0)$ in equation (14) and accordingly the term $C_t x_t$ in equation (15). The modified equation (15) takes the place of equation (4). It can be converted to equation (5) or (9) as needed, with the subscript t added to the coefficient matrices. To obtain an optimal open-loop policy, we use equation (9) which can be rewritten as (16) for the purpose of optimization. The tentatively optimal policy will provide a new initial path for relinearizing the model. To obtain an approximately optimal linear feedback rule, we use equation (5) to obtain the feedback control equations (18). We then apply the control equation for period one as a stationary rule and use the method of equations (19) - (23) to obtain a new feedback rule for the purpose of relinearization. If this iterative process fails to produce a nearly time-invariant feedback rule, we will use equations (11) and (12) to evaluate the expected loss for a given time-invariant feedback rule, and apply a gradient method to minimize the expected loss with respect to the parameters of the rule.

4. Concluding Remarks

In this paper, I have shown that policy evaluation and optimal policy formulation can be carried out under the assumption of rational expectations. The methods proposed amount to modest modifications of the existing methods of policy evaluation and optimal control.

We have assumed throughout that the econometric model to be employed has been properly estimated, and have not treated the problem of estimating an econometric model under rational expectations. The estimation problem has been treated by Wallis (1977) under the assumption that the model is linear and that the control variables themselves follow an autoregressive-moving-average process. If either assumption is relaxed, the latter being replaced by the

assumption that the government acts as if it attempts to maximize the expectation of an objective function with unknown parameters, there appear to be interesting theoretical and applied econometric problems for further research.

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