

FISCAL POLICY, INFLATION
AND THE ACCUMULATION OF
RISKY CAPITAL

by

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A B S T R A C T

A model is developed to explore the effect of expenditure and finance policies on capital accumulation in a stochastic environment. When wealth-holders maximize the expected discounted utility of consumption over an infinite horizon, the rate of capital accumulation is found to be invariant to the average share of government expenditure in income and to the level of lump-sum taxation. Tax measures that raise the average return on capital and reduce its risk increase the growth rate of a surplus-labor economy and the capital-labor ratio of a neoclassical economy. An increase in the overall level of uncertainty has an ambiguous effect in either context. The effect is, however, more likely to be negative in a full-employment situation.

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1. Introduction

Recent discussions of a possible "capital shortage" in the United States focus attention on the role of government expenditure and finance on capital accumulation. Proposals to raise the level of business investment have included overall tax cuts, tax incentives to spur investment and a reduction in the level of government spending.

This paper presents a theoretical apparatus to explore the relationships between fiscal policy, government debt, inflation and capital accumulation. A one-sector monetary growth model embodying a stochastic production function is developed. Atomistic agents in the economy choose consumption levels and allocate their wealth between risky capital and risky government debt to maximize expected discounted utility over an infinite horizon. Fiscal policy, by affecting asset yields, affects the composition of portfolios, the price level and the accumulation of capital.

The model is used to answer three general questions:

- (1) Does the level of government expenditures affect the rate of capital accumulation and inflation in steady state?
- (2) Does the mode of government finance, when the choice is between an income tax and deficit finance, affect these magnitudes?
- (3) What is the effect of uncertainty in the productivity of capital on growth and inflation in a monetary economy?

The analysis relates to several areas of literature: (1) the analysis of the relationship between capital accumulation and debt, (2) the theory of the demand for money, (3) considerations of the effect of income taxation on risk bearing, and (4) the analysis of capital accumulation under uncertainty.

1. The effects of the government's expenditure decision and its decision to finance its expenditure by taxation or debt issue on output and the capital stock are explored in two major bodies of economic theory. The literature on the "burden of the debt" as developed by Modigliani [1961], Diamond [1965], and Phelps and Shell [1969], for example, implies that policies which increase the value of outstanding government debt will tend to lower the level of the capital stock by displacing capital with public debt in private wealth. The theory of money and growth as represented by Tobin [1965] and Sidrauski [1967] also implies that government debt displaces capital, but specifically recognizes the role of capital gains.

The conclusion that the mode of government finance is not neutral in terms of its effect on the level of the capital stock depends, in these models, on the assumption that government debt is regarded as net wealth by the private sector. Barro [1974] has pointed out that this assumption is closely tied to a life-cycle theory of savings in which intergenerational bequests do not typically occur. Barro concludes that if a bequest motive is operative "fiscal effects involving changes in the relative amounts of tax and debt finance for a given amount of public expenditure would have no effect on aggregate demand, interest rates, and capital formation." (p.1116) That a finite optimization horizon on the part of wealth holders is crucial to the relationship between finance policy and capital accumulation is implied in an earlier model of Sidrauski [1967a] in which the capital-labor ratio in an economy consisting of a single infinitely-lived household is found to be invariant to the rate of monetary growth. A conclusion of the present analysis is that, while the choice between deficit finance and lump-sum taxation is neutral in its effects on capital accumulation for the infinite horizon case, the choice between deficit finance and income taxation is not.

2. Tobin's [1957] classic interpretation of the demand for money as risk-averse behavior assumes money is riskless. The risk associated with money arising from price level variation has become increasingly evident, however. In the present paper, a stochastic inflation rate is derived from an underlying source of uncertainty in the economy. The return on money is thus related to the return on the alternative asset. Optimal behavior is still typically characterized by portfolio diversification.

3. The effect of income taxation on portfolio allocation has been analyzed by Domar and Musgrave [1944] and Stiglitz [1969] in a partial equilibrium context. While these studies demonstrate that increased taxation of income from capital can increase the demand for a risky asset, the consequences of the implied randomness in tax revenues are not taken into account. If the risky undertaking in question is large in relation to the economy as a whole, the taxation of income from this undertaking will generate uncertainty about either the level of government expenditure or the size of the deficit. Risk in the value of government debt is in either case affected. The present study incorporates a general equilibrium analysis of the effect of taxation on the demand for risky capital taking into account its effects on the yield characteristics of government debt. It is found that an increase in the extent to which the government shares in bearing the risk of capital does not necessarily increase the demand for capital relative to debt.

4. The model developed in this paper also extends the literature on growth under uncertainty developed by Mirrlees [1971], Leland [1974], Brock and Mirman [1971, 1973], Mirman [1972], Bourguignon [1974] and Merton [1975] for non-monetary economies. The last two authors demonstrate that introducing uncertainty about the natural rate of growth tends to increase average capital intensity. In

this paper it is shown that the presence of uncertainty in the production relationship has the opposite effect.

To obtain a tractable closed-form characterization of the steady state of an economy of the type described in this paper requires a number of specific assumptions about behavior, technology and the nature of uncertainty. Exploration of the implications of more general assumptions represents an interesting but difficult undertaking. The present analysis is intended to demonstrate a natural means of introducing government debt and policy into a stochastic, dynamic model and to indicate the types and possible directions of effects that policy and uncertainty may have on steady-state growth.

The model incorporates government debt which is monetary in the sense that it does not bear interest. It is shown in Appendix 3 that introducing explicit interest payments on government debt will not affect the propositions derived from the analysis. In the present context, then, the distinction between money and bonds is not essential.

The model developed in this paper is a positive one. It incorporates optimizing behavior on the part of wealth-holders only in order to characterize consumption behavior and asset demands. Optimal government policy can be derived by maximizing a social welfare function specifying the trade-offs among private consumption, government expenditure, inflation and real money balances, subject to the constraints imposed by the model.

Part 2 of the paper is an analysis of the effects of policy on growth in a stochastic, surplus-labor economy. The effects of policy and uncertainty on the steady state of a stochastic, monetary, neoclassical economy are treated in Part 3. Part 4 contains a summary of the major conclusions and discussion of some possibilities for further inquiry.

2. Inflation and Growth in a Monetary, Surplus-Labor Economy

The effect of fiscal policy and debt on inflation and growth are explored. Section a is a presentation of the aggregate model. The optimal behavior of an atomistic investor in such an economy is derived in section b. In section c the two analyses are joined to obtain a characterization of the demand for money and capital in the equilibrium of this economy. The effects of fiscal policy, debt and uncertainty on the rate of growth are discussed in section d and on the rate of inflation in section e. Appendix 2 explores the stability of the steady state. The consequence of introducing interest-bearing debt is discussed in Appendix 3. Appendix 4 derives the implications of lump-sum taxation for the certainty case.

a. The Aggregate Model

Net output at time t , $Q(t)$, is a stochastic function of the capital stock, $K(t)$, and employment, $L(t)$, of the form:

$$(2.1) \quad Q = K^\alpha L^{1-\alpha} dt + \sigma K dz$$

where dz represents a standard normal Wiener process: expected output is a Cobb-Douglas function of capital and labor while the instantaneous variance of output is proportional to the square of the capital stock.^{1/}

Labor is supplied elastically at a real wage w . Given a capital stock $K(t)$ expected-profit-maximizing firms will employ

$$(2.2) \quad L = [(1-\alpha)/w]^{1/\alpha} K$$

They will produce

$$(2.3) \quad Q = uKdt + \sigma Kdz,$$

where

6.

$$u = [1-\alpha]/w]^{(1-\alpha)/\alpha} .$$

The income from capital before tax is output less wages or

$$(2.4) \quad \alpha u K dt + \sigma K dz .$$

Tax revenue T is a fixed proportion t_K of the deterministic part of capital income and a proportion t' of the stochastic part of capital income:

$$(2.5) \quad T = t_K \alpha u K dt + t' \sigma K dz ; \quad t_K, t' \in (0,1) .$$

Similarly, government expenditure G is a proportion g of the deterministic part of income and a proportion g' of the stochastic part of income:

$$(2.6) \quad G = g u K dt + g' \sigma K dz , \quad g, g' \in (0,1) .$$

The model allows the government to tax the random component of capital income at a rate different from the rate at which it taxes the deterministic part. Such a differential tax rate will arise if, for instance, the government explicitly coinsures risky undertakings in a degree beyond that implied by the average tax rate. Furthermore, a non-proportional income tax with a full-loss offset is better approximated by allowing t' to differ from t_K . Progressivity, for instance, would imply $t_K < t'$. Similarly, the level of public spending, while proportional to income on average, may not vary with random movements in income in the same proportion. This may be due to sluggishness in the budgeting process, for example.

The government deficit is financed by monetary issue,

$$(2.7) \quad G - T = \frac{dM}{p} ,$$

where M represents the nominal money supply and p the price level.

Net investment is net output less consumption C and government spending,

$$(2.8) \quad dK = Q - C - G .$$

All labor income is consumed. The growth rate of the capital stock is therefore

$$(2.9) \quad \frac{dK}{K} = [(\alpha-g)u - C_K/K]dt + (1-g')\sigma dz$$

where C_K represents consumption out of capital income.

The term H is defined as the ratio of real money balances demanded to the capital stock,

$$(2.10) \quad \left(\frac{M}{P} \right)^D \equiv HK .$$

Values of H and C_K consistent with optimal behavior of wealth-holders are derived in section b.

The price level p adjusts instantaneously to maintain equilibrium in the money market,

$$(2.11) \quad p = \frac{M}{HK} .$$

The after tax rate of return on capital r_K is thus given by

$$(2.12) \quad r_K = \alpha u(1-t_K)dt + (1-t')\sigma dz .$$

The real return on money derives from changes in the price level and is equal to the increase in the real value of money less real monetary issue. The rate of return on money r_M is thus

$$(2.13) \quad r_M = \left[d \left(\frac{M}{P} \right) - \frac{dM}{P} \right] / \left(\frac{M}{P} \right) .$$

If H is constant, expression (2.13) is equal to the growth rate of capital and output (2.9) less the deficit per unit of real money balances,

8.

$$(2.14) \quad r_M = [\alpha u(1-t_K) - C_K/K - (g - \alpha t_K)u/m]dt + [(1-t') - (g' - t')/m] \sigma dz$$

where $\frac{m}{1-m} \equiv H$.

Expression (2.9) implies that for a given ratio of consumption to capital, tax parameters have no effect on the rate of capital accumulation. From expressions (2.12) and (2.14) it does appear, however, that tax parameters do affect the rates of return on capital and money. Only to the extent that consumption behavior responds to changes in asset yields will the method of finance of government spending affect the rate of capital accumulation. In the next section, optimal consumption and portfolio plans for atomistic wealth-holders, who take asset returns as given, are derived.

b. Optimal Savings and Portfolio Choice

For analytic convenience, a class distinction of the type associated with Pasinetti [1961] is made: capitalists do not work and workers do not save.

The set of capitalists I is a continuum indexed by i .

Each capitalist i has an objective functional of the form ^{2/}

$$(2.15) \quad E \left[\int_0^{\infty} \frac{1}{\gamma} C_i(t) \gamma e^{-\delta t} dt \right] \quad \delta \in (0, \infty) \\ \gamma \in (-\infty, 1)$$

where $C_i(t)$ represents consumption in period t by capitalist i and δ is the discount rate. The change in the i th capitalist's wealth in period t , $dW_i(t)$, is the flow of income from wealth less consumption:

$$(2.16) \quad dW_i(t) = [r_M m_i(t) + r_K (1 - m_i(t))] W_i(t) - C_i(t)$$

where $m_i(t)$ represents the share of financial wealth held in the form of real money balances in period t .

The set of available assets may equivalently be characterized as one risky asset, capital, with return r_K given by expression (2.12), and one safe asset consisting of a proportion \hat{m} of money and $1-\hat{m}$ of capital, where

$$(2.17) \quad \hat{m} = \frac{m(1-t')}{(g'-t')} .$$

When capital and money are held together in these proportions, the stochastic components of their returns exactly offset each other. The rate on the composite safe asset is

$$(2.18) \quad r_s = \left\{ \alpha u(1-t_K) + \frac{(1-t')}{(g'-t')} [(\alpha t_K - g)u - mC_K/K] \right\} .$$

Expression (2.16) may be written as:

$$(2.19) \quad \begin{aligned} dW_i(t) = & \{ [r_s + (1-\omega_i(t))(\alpha u(1-t_K) - r_s)] W_i(t) - C_i(t) \} dt \\ & + (1-\omega_i(t))(1-t') \sigma W_i(t) dz \end{aligned}$$

where $\omega_i(t)$ is the share of wealth held in the risky asset in period t .

The maximization of (2.15) subject to the stochastic differential equation (2.19) represents a continuous-time stochastic dynamic programming problem of the type discussed by Merton [1969, 1971]. If (2.15) is bounded optimal consumption is given by

$$(2.20) \quad C_i^*(t) = c W_i(t)$$

where

$$c = \frac{\delta - \gamma [r_s + (\alpha u(1-t_K) - r_s)^2 / 2(1-\gamma)(1-t')^2 \sigma^2]}{1 - \gamma}$$

and the optimal share of the risky asset in wealth is

$$(2.21) \quad \omega_i^* = \frac{\alpha u(1-t_K) - r_s}{(1-\gamma)(1-t')^2 \sigma^2} .$$

10.

The optimal share of money in wealth is therefore

$$(2.22) \quad m_1^* = \hat{m}(1-\omega_1^*)$$

c. Portfolio Equilibrium

If all capitalists have identical tastes then

$$(2.23) \quad m_1^* = m$$

and

$$(2.24) \quad C_K = cK/(1-m)$$

The solutions to the individual investors' dynamic programming problem, expressions (2.20) and (2.22), may be substituted into expression (2.10), the aggregate demand for money, to obtain an expression for the ratio of money demanded to capital

$$(2.25) \quad H = \frac{\lambda}{c}$$

where

$$\lambda = u(\alpha t_K - g) + (1-\gamma)(1-g')(g'-t')\sigma^2$$

The following propositions concerning the effect of uncertainty and policy follow from expression (2.25)

Proposition 2.1: For a monetary economy to exist the average deficit per unit of capital must be strictly less than $(1-\gamma)(1-g')(g'-t')\sigma^2$.

This condition guarantees that the demand for money is strictly positive. Introducing a transactions demand for money would imply a less stringent condition for the existence of a monetary economy.

Proposition 2.2: In the case of a Bernoulli utility function ($\gamma=0$) an increase in the variance of the production process increases or decreases the demand for money relative to wealth as $g' \gtrless t'$.

If $g' > t'$ then increased uncertainty in the production relationship increases the variance of the return on money more than the variance of the return on capital. Conversely if $g' < t'$, when $\gamma=0$ uncertainty has no effect on the consumption-wealth ratio and increased uncertainty in the underlying production relationships raises the demand for the safer asset.

When $\gamma \neq 0$, however, increased uncertainty does affect the ratio of consumption to wealth. Higher consumption implies greater inflation and a lower demand for money. One may infer:

Proposition 2.3: When money and capital are equally risky ($g'=t'$) increased riskiness of production will increase or decrease the demand for money relative to capital as $\gamma \gtrless 0$.

When $\gamma < 0$ increased uncertainty lowers consumption and therefore inflation. Conversely when $\gamma > 0$.

Proposition 2.4: If $\gamma < (1-m)$ (a) an increase in the tax rate on the expected return on capital, t_K , lowers the demand for capital; (b) an increase in the share of government's risk bearing, t' , raises the demand for capital; (c) when $t_K = t'$ an increase in the proportional tax rate raises or lowers the demand for capital as

$$\alpha_u \gtrless (1-\gamma)(1-g')\sigma^2.$$

Conversely if $\lambda > (k-m)$.

Increased taxation of the random component of the return on capital reduces its riskiness, making it a more attractive asset. The results of Domar

and Musgrave [1944] and Stiglitz [1969] on the effects of taxation on risk bearing rely on this partial equilibrium effect. If the degree of relative risk aversion is very low, however, a reduction in the variability of the return on capital lowers the ratio of consumption to wealth, reducing inflation and making debt a more attractive asset. A tax-induced reduction in the private riskiness of capital can thus lower the demand for capital relative to debt. The finding that reduced taxation of the average return on capital will lower the demand for capital for low values of risk aversion also derives from reduced consumption lowering inflation.

d. Equilibrium Growth

An expression for the rate of growth of the capital stock is obtained by substituting expressions for optimal consumption (2.20) and demand for money (2.22) into the differential equation (2.9) to obtain

$$(2.26) \quad \frac{dK}{K} = \left\{ \frac{\alpha u(1-t_K) - \delta}{1-\gamma} - (1-g')\sigma^2 [g'-t'+\gamma(1-g')/2] \right\} dt \\ + (1-g')\sigma dz .$$

Two propositions concerning the effects of average government expenditure and taxation policy on the average growth rate follow from expression (2.26):

Proposition 2.5: The average rate of government expenditure does not affect the expected rate of capital accumulation.

An increase in the rate of government expenditure when the tax structure is unchanged initially increases the inflation rate and lowers the demand for money. The consequent rise in the price level reduces real financial wealth which in turn reduces consumption in an amount equal to the increase in

While the level of government expenditure does not constrain the growth rate directly it does limit the extent to which tax cuts may be used to stimulate growth without causing the demand for money to fall to zero. A reduction in government spending will shift the MM curve to the left but not affect the GG curve. If the tax rate is unchanged the growth rate will not be affected while the demand for money will rise. A larger tax cut which does not eliminate money demand but does raise the average growth rate is now feasible.

The effect of uncertainty on the average growth rate is summarized in the following two propositions:

Proposition 2.7: The average growth rate is positively related to the government's share in bearing the risk of capital, t' .

An increase in the marginal tax rate reduces the riskiness of capital and raises that of money. Money demand falls, prices rise, consumption falls and investment rises.

Proposition 2.8: An increase in the uncertainty of the pre-tax rate of return on capital or a reduction in the marginal expenditure rate g' raises or lowers the maximal sustainable monetary growth rate as $\gamma \lesseqgtr 0$ and raises or lowers the average growth rate as

$$\gamma \lesseqgtr 2(t'-g')(1-g') .$$

An increase in σ^2 or a decrease in g' raises the variability of income from wealth. This increases or reduces savings as $\gamma \lesseqgtr 0$, a result obtained by Leland [1973]. Increased uncertainty also increases the demand for the less risky asset, capital if $t' > g'$ and money otherwise. In contrast with Leland's result, if $t' \neq g'$ increased uncertainty will affect the expected growth rate even if $\gamma=0$ because of its effect on portfolio equilibrium. If,

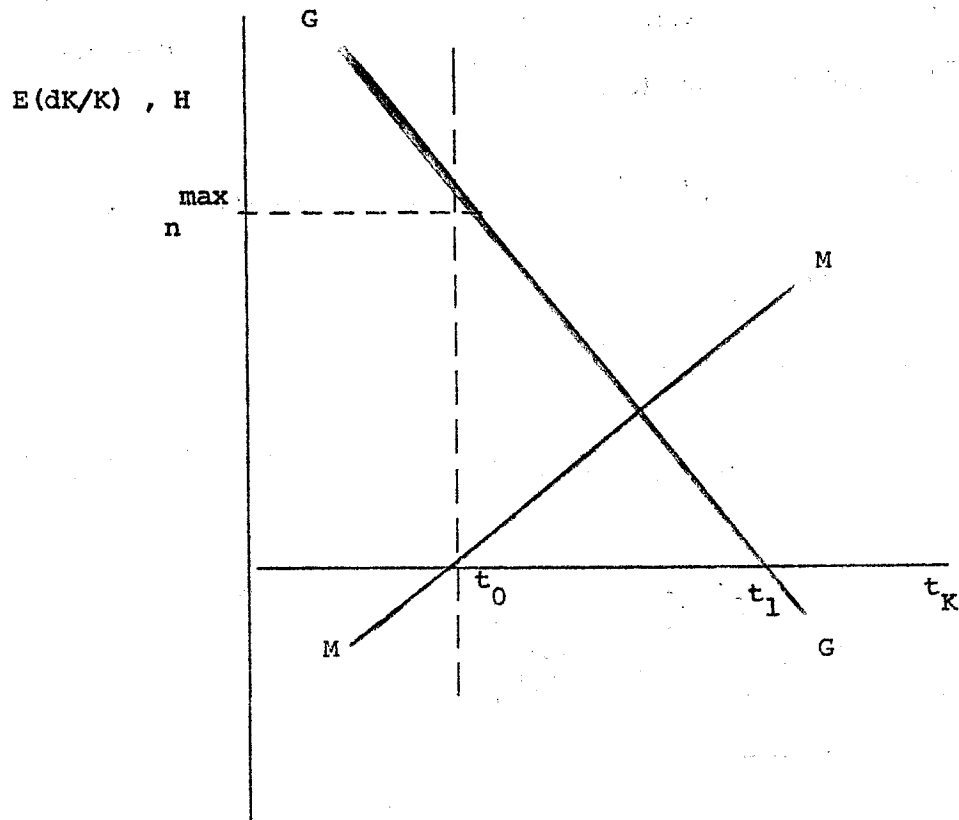


FIGURE 1

for example, $t' > g'$ increased uncertainty may increase the expected growth rate even if the degree of relative risk aversion is less than unity by reducing the demand for money, the riskier asset.

e. The Rate of Inflation

A reduced-form expression for the rate of return on money may be obtained by substituting expressions (2.20) and (2.14) to give

$$(2.30) \quad r_M = [\alpha u(1-t_K) - \lambda'/m]dt + [1-t' - (g'-t')/m]\sigma dz$$

$$\equiv u_M dt + \sigma_M dz \quad . \quad 4/$$

where

$$\lambda' = (1-\gamma)(1-g')(g'-t')\sigma^2 .$$

Differentiating the deterministic part of this expression with respect to t and g yields

$$(2.31a) \quad \frac{du_M}{dt_K} = \alpha u \left[\left(\frac{1-m}{m} \right)^2 \left(\lambda'/c \right) - \gamma \left(\frac{1-m}{m} \right) - 1 \right]$$

and

$$(2.31b) \quad \frac{du_M}{dg} = -u \left(\frac{1-m}{m} \right)^2 (\lambda'/c) .$$

Both are ambiguous in sign. Since, from Proposition 2.5, the growth rate is independent of g and, from Proposition 2.6, it is negatively related to t_K , there is in general no trade-off between inflation and capital growth.^{5/}

3. Neoclassical Growth

This part extends the model developed in Part 2 to the case in which labor is supplied inelastically. The labor force is now assumed to grow at

rate n :

$$(3.1) \quad L(t) = L_0 e^{nt}$$

while output continues to be given by expression (2.1).

Tax revenue is a proportion t_L of labor income, t_K of the deterministic part of capital income and t' of the stochastic part of capital income. In per worker terms:

$$(3.2) \quad T/L = [t_L(1-\alpha) + \alpha t_K]k^\alpha dt + t'\sigma k dz$$

where $k \equiv K/L$.

Government expenditure is a proportion g of the deterministic part and g' of the stochastic part of output:

$$(3.3) \quad G/L = gk^\alpha dt + g'\sigma k dz .$$

The deficit is financed by monetary issue.

The rate of change of the capital stock is

$$(3.4) \quad \frac{dK}{K} = [(\alpha - g + (1-\alpha)t_L)k^{\alpha-1} - C_K/K]dt + (1-g')\sigma dz .$$

The after tax return on capital is

$$(3.5) \quad r_K = (1-t_K)\alpha k^{\alpha-1} dt + (1-t')\sigma dz$$

while the return on money is

$$(3.6) \quad r_M = \{ (1-t_K)\alpha k^{\alpha-1} - C_K/K - [g - \alpha t_K - (1-\alpha)t_L]k^{\alpha-1}/m \} dt \\ + [(1-t') - (g'-t')/m]\sigma dz .$$

The optimal portfolio and consumption decisions which the individual capitalist faces are those described in section b of part 2. A complication,

however, is that the capitalist is exposed not only to "asset risk," variation in the instantaneous returns on capital and money, but to "market risk" caused by variation in the capital-labor ratio itself.^{6/} This complicates the solution to the dynamic programming problem for the general isoelastic marginal utility function to the point of intractability.

For the special case of the Bernoulli utility function ($\gamma=0$), however, the presence of market risk is irrelevant to the individual capitalist's dynamic programming problem as demonstrated in Appendix 1.

For this case the rate of change in the capital-labor ratio is given by the stochastic differential equation

$$(3.7) \quad \frac{dk}{k} = [\alpha(1-t_k)k^{\alpha-1} - n - \delta - (1-g')(g'-t')\sigma^2]dt + (1-g')\sigma dz$$

The instantaneous change in k is thus a function of the form

$$(3.8) \quad dk = [a_1 k^\alpha - a_2 k]dt + a_3 k dz \quad a_i \geq 0$$

Bourgignon [1974] has proven that a non-trivial steady-state distribution exists for the continuous-time diffusion process given by expression (3.8).^{7/}

Application of the result presented in Appendix 2 yields as an expression for the marginal product of capital :

$$(3.9) \quad E[\alpha k^{\alpha-1}] = \frac{n + \delta + (1-g')\left[\frac{1}{2}(1+g') - t'\right]\sigma^2}{(1-t_k)}$$

The effect of fiscal policy on the average marginal product of capital is summarized in:

Proposition 3.1: Neither the average share of output devoted to government expenditure nor the rate of taxation of labor income affects average capital intensity. Increased taxation of the average income from capital reduces capital intensity.

The direction of the effects of average expenditure and taxation rates on capital intensity in the neoclassical economy are the same as their effects on the average growth rate of capital in the surplus-labor economy summarized in propositions 2.5 and 2.6. Also parallel is:

Proposition 3.2: An increase in the share of government absorption of the risk of capital t' increases average capital intensity.

The effect of increased uncertainty on capital intensity in the neoclassical context differs, however, from its effect on the growth rate of the surplus labor economy:

Proposition 3.3: An increase in the coefficient of the random component of the production function raises or lowers the expected marginal product capital as

$$(3.10) \quad t' \lesseqgtr (1+g')/2 .$$

In the surplus-labor model the corresponding condition is that increased uncertainty raises or lowers the growth rate as $t' \gtrless g'$. In both cases increased uncertainty has a "portfolio adjustment" effect, shifting demand toward the safer asset. An additional effect is present only in the neoclassical case. Increased uncertainty in output implies increased variability in the rate of capital accumulation and hence in the capital-labor ratio. Since output is a strictly concave function of k and the interest rate a strictly convex function of k increased uncertainty in the distribution of k lowers the expected value of output and hence of savings and raises the expected marginal product of capital. Unless money is much riskier than capital a reduction in overall uncertainty will tend to increase capital intensity.

Bourgignon [1974] and Merton [1975] have shown that increased uncertainty in the natural growth rate raises capital intensity in a non-monetary

neoclassical economy. Their result derives from the convexity of the capital-labor ratio itself in labor. In the non-monetary case of the present model $g'=t'=0$ and proposition 3.3 implies that increased uncertainty in the return on capital will reduce capital intensity, in contrast with their result.

3. Conclusion

The model developed in this paper has explored the effect of uncertainty on capital accumulation in a monetary, general equilibrium context. The findings support the view that if economic agents optimize over an infinite horizon the choice between lump-sum taxation and deficit finance of a given level of government expenditure is irrelevant to capital accumulation. The model demonstrates furthermore that the average level of government expenditure is relevant only indirectly. When the alternative to deficit finance is taxation of income from capital, however, the mode of finance is crucial to the question of capital accumulation. Fiscal policy then changes the average yield and riskiness of capital relative to government debt thus affecting investment and thereby growth or capital intensity. The results do not suggest a trade-off between inflation and capital implied by some previous models of money and growth for the general case. They do suggest, however, that to a limited degree deficit finance will yield greater capital growth than tax finance.

It would be useful to know the extent to which the assumptions embedded in the model may be relaxed without affecting the basic results. Certain restrictions such as the Cambridge-style savings assumption, Cobb-Douglas technology and isoelastic marginal utility of consumption are probably innocuous.^{8/} Others, such as the restriction to a single sector and a single source of uncertainty may be more crucial. The first precludes discrepancies from arising between the replacement cost and market value of capital and no doubt causes the model to understate the efficacy of policy. The model has demonstrated by construction the existence of an equilibrium to a stochastic, monetary economy. A useful

task for future research would be to determine the most general set of assumptions about tastes, technology and policy which ensure the existence of a steady state equilibrium.

APPENDIX I: Optimal Consumption and Portfolio Allocation When Asset Yields Have Variable Means and Variances

Define:

$$(A1.1) \quad J(W(t), x(t), t) = e^{-\delta t} \max_{C(\tau), \omega(\tau)} E \left[\int_t^{\infty} \frac{1}{Y} C(\tau) e^{-\delta \tau} d\tau \right]$$

Subject to:

$$(A1.2) \quad dW(\tau) = [\omega(\tau)(r_K(x) - r_S(x)) + r_S(x)]W(\tau) - C(\tau)$$

Expression (A1.1) may be decomposed to:

$$(A1.3) \quad J(W(t), x(t), t) = \max_{C(\tau), \omega(\tau)} E \int_t^{t+\epsilon} \frac{1}{Y} C(\tau) e^{-\delta \tau} d\tau + J(W(t+\epsilon), x(t+\epsilon), t+\epsilon)$$

Taking the limit of (A1.3) as $\epsilon \rightarrow 0$:

$$\max_{C, \omega} \left[\frac{1}{Y} C(\tau) e^{-\delta \tau} + dJ \right] = 0.$$

dJ is obtained by applying Itô's lemma to differentiate J yielding:

$$(A1.4) \quad dJ = J_W [\omega(u_K(x) - r_S(x)) + r_S(x)W - C] + \frac{1}{2} J_{WW} \omega^2 W^2 \sigma_K^2 \\ + J_X \alpha(x) + \frac{1}{2} J_{XX} \sigma_X^2(x) + J_{XW} \sigma_{XK} \omega W + J_t.$$

Here $x(t)$ is a stochastic process given by

$$(A1.5) \quad dx(t) = \alpha(x(t))dt + \sigma_X(x(t))dz_1$$

such as k given by (5.7) and

$$(A1.6) \quad r_K(x) = u_K(x)dt + \sigma_K(x)dz_2.$$

Necessary conditions for a maximum are

$$(A1.7) \quad C^* = (J_W)^{1/\gamma-1}$$

$$(A1.8) \quad \omega^* = \frac{-J_W(u_K(x) - r_S(x)) - J_{xW}\sigma_{xK}}{J_{ww}W\sigma_K^2}$$

Consider a solution of the form

$$(A1.9) \quad J(W,x,t) = e^{-\delta t} a(x)W^\gamma/\gamma.$$

Substituting expressions (A1.7), (A1.8) and (A1.9) into (A1.4) gives:

$$(A1.10) \quad a^{\frac{1}{\gamma-1}}(1-\gamma) + \gamma[r_S(x) + (u_K(x) - r_S(x))/2(1-\gamma)\sigma_K^2] - \delta \\ + \gamma a' \sigma_{xK} / \sigma_x^2 + a' \alpha(x) / a + a'' \sigma_x^2 / 2a = 0.$$

A simple solution for (A1.10) does not exist for arbitrary values of γ . For $\gamma=0$ $a=1/\delta$ provides a solution. For the case of the Bernoulli utility function, then, the probability distribution of x is not relevant to optimal consumption or portfolio decisions.

APPENDIX 2: Stability of the Steady-State Under Adaptive Expectations

For any level of the capital stock, the state of the economy modeled in sections a and b of Part 2 is fully characterized by the variable m , the share of real money balances in total wealth. When condition (2.25) obtains capitalists are satisfied with existing money balances at the given inflation rate. This inflation rate in turn is sustained at the existing level of money balances. Condition (2.25) thus characterizes an equilibrium value of m and hence the equilibrium of the economy.

The rate of return on capital, expression (2.12), is independent of m and therefore is the same whether or not the economy is in equilibrium. Both

the mean and the variance of the rate of return on money, expression (2.14), however, depend on m .

In steady state the rate of return on debt is given by expression (2.30): the equilibrium rate of return on money exceeds the rate of return on capital by a risk premium of $-\lambda'/m$.

The difference between the actual and the equilibrium return on money for any value of m is:

$$(A2.1) \quad e_1 \equiv (\lambda/m - c/(1-m))$$

while

$$(A2.2) \quad e_2 \equiv (t'-g')/m$$

is the difference between the standard deviations of the returns on money and on capital.

No attempt is made to derive explicit formulations of optimal consumption and portfolio programs in disequilibrium. Instead behavior is characterized by three assumptions:

Assumption 1. The ratio of capitalists' consumption to wealth is given by $c(m, \dot{e}_1^e, \dot{e}_2^e)$ where $c(m, 0, 0)$ is defined in expression (2.20) and where \dot{e}_i^e is the expected change in the variable e_i , $i=1, 2$.

Assumption 2. The change in capitalists' share of real money balances in wealth is

$$(A2.3) \quad \dot{m} = h(e_1, \dot{e}_1^e, \dot{e}_2^e) \quad h_1 > 0, h_2 > 0, h_3 < 0; \quad h(0, 0, 0) = 0.$$

Assumption 3. The expected change in the variable e_i is formed according to the rule:

$$(A2.4) \quad \dot{e}_i^e = b_i \dot{e}_i \quad i = 1, 2.$$

Under assumptions 1-3 local stability obtains if and only if $\frac{dm}{dm} < 0$ in the neighborhood of the steady state value of m . Differentiating expression (A2.3) totally with respect to m and evaluating the expression at $\dot{m} = 0$ yields

$$(A2.5) \quad \left. \frac{d\dot{m}}{dm} \right|_{\dot{m}=0} = \frac{h_1 \Delta}{1 - h_2 b_1 \Delta - h_3 b_2 (g' - t') / m^2}$$

where

$$\Delta = -\lambda/m^2 - c/(1-m)^2 - [c_1 + c_2 b_1 \Delta + c_3 b_2 (g' - t') / m^2] / (1-m).$$

Unless the change in the money-wealth ratio affects the consumption-wealth ratio significantly the numerator of expression (A2.5) is negative in a monetary economy.^{9/} For $b_1 > 0$ the first two terms of the denominator are positive if the change consumption-wealth ratio is again small. For $b_2 > 0$ the third term in the denominator is positive or negative as $g' > t'$. These observations may be summarized in

Proposition A2. Under assumptions 1-3 local stability is guaranteed if (1) the consumption-wealth ratio is a non-increasing function of m , (2) money is riskier than capital and (3) expectations are non-regressive.

This result contrasts with the instability often observed in monetary growth models under adaptive expectations and myopic perfect foresight ($b_1 = b_2 = 1$) in particular.^{10/} In the present model the effect of a positive perturbation in real money balances is to increase real wealth and therefore consumption. Increased consumption in turn leads to increased inflation and a lower demand for money. The wealth effect in the consumption functions thus provides a stabilizing element. Also contrasting with other monetary growth models is the finding here that regressive expectations are potentially destabilizing.^{11/}

APPENDIX 3. Bond Finance and Growth

This section modifies the previous analysis by considering government debt which bears interest. Following the development of Blinder and Solow [1973] the

government is assumed to finance its deficit by issuing bonds which continuously yield one unit of output in perpetuity. The number of such bonds is B each of which has a value in terms of commodities of p_B . The government budget constraint becomes

$$(A3.1) \quad B + G - T = p_B dB.$$

The demand for real bonds represents a fraction $D = b/(1-b)$ of the capital stock, i.e.

$$(A3.2) \quad (p_B B)^D \equiv DK$$

and the price of bonds adjusts instantaneously to maintain equilibrium in the bond market.

The capital gain on bonds is given by the change in the real value of bonds less the value of new bonds issued:

$$(A3.3) \quad d(p_B B) - p_B dB.$$

Capital gains are taxed at rate t_{cg} while coupon payments are taxed at rate t_B . Tax revenue is therefore:

$$(A3.4) \quad T = [t_K cuK + t_B]dt + t_{cg} [d(p_B B) - p_B dB] + t'OKdz$$

and the after tax rate of return on bonds is

$$(A3.5) \quad r_B = \frac{(1-t_{cg}) [d(p_B B) - p_B dB] + (1-t_B)Bdt}{p_B B}$$

Solving this, one obtains an expression for r_B identical to (2.14), the return on money; after-tax capital losses exactly offset the after-tax debt service. Subsequent analysis is thus identical to that for the monetary case.

The result for the neoclassical case is equivalent. One may conclude:

Proposition A3: Whether or not an interest coupon is paid on government debt is irrelevant to the rate of return on debt and to the rate of growth of output.

A corollary to this result is that the tax treatment of debt service and capital gains on debt is also irrelevant to the real return on debt and to the rate of capital accumulation. According to this model the taxation of interest payments on debt is not relevant to the bonds as net wealth issue as suggested by Feldstein [1976a].

APPENDIX 4: Lump-Sum Taxation

The effects of tax policy on steady-state growth obtained in part 2 pertain to taxation of income from capital. Taxation therefore places a wedge between the marginal product of capital and the net return on capital. This section considers the effect of lump-sum taxation on the steady-state growth path. To simplify the analysis lump-sum taxation is introduced into a deterministic version of the model developed in section 2.

A lump-sum tax in amount ΘK is imposed on capitalists. The discounted value of future tax obligations at time t is therefore

$$(A4.1) \quad T = \int_t^{\infty} \Theta K(t) e^{(n-r)t} dt$$

$$= K(t)/(r-n)$$

where n is the now nonstochastic growth rate of the capital stock and therefore tax obligations and $r = \alpha(1-t)$ is the after tax rate of return on capital.

In the deterministic model in which both money and capital are held in strictly positive amounts the rate of return on money must equal the interest

rate. The inflation rate is equal to the growth rate of the money supply minus the growth rate of capital:

$$(A4.2) \quad \frac{dp}{p} = [dM/M - n]dt.$$

The government budget constraint implies that

$$(A4.3) \quad \frac{dM}{p} = [g - \alpha t]u - \theta K$$

so that

$$\frac{dp}{p} = \frac{(g - \alpha t)u - \theta}{H} - n.$$

Since

$$r = -dp/p$$

in equilibrium

$$(A4.4) \quad H^* = [(\alpha t - g)u + \theta] / (r - n)$$

is the only value of the money-capital ratio consistent with portfolio diversification. Note that the demand for money relative to capital depends positively on the lump-sum tax rate.

The optimal consumption program of the capitalist described in section b of part 2 is given by

$$(A4.5) \quad C_i^*(t) = cW_i(t)$$

where

$$W_i(t) = K_i(t) + \left(\frac{M}{p}\right)_i(t) - T_i(t)$$

and

$$(A4.6) \quad c = (\delta - \gamma r) / (1 - \gamma).$$

Here T_i represents the discounted present value of the lump-sum taxes imposed on capitalist i .

In the aggregate, therefore,

$$(A4.7) \quad C_K = c[1 + H - \Theta/(r-n)]K$$

and the rate of capital accumulation is

$$(A4.8) \quad \frac{\dot{K}}{K} = n = (\alpha-g)u - c[1 + H - \Theta/(r-n)] .$$

When $H = H^*$ the growth rate is given by

$$(A4.9) \quad n^* = r + (g-\alpha t)u \left[\left(\frac{c}{n^*-r} \right) - 1 \right] - c$$

an expression in which Θ is absent; an increase in lump-sum taxation has the direct effect of lowering capitalists net wealth and therefore consumption. It has the indirect effect, however, of raising the demand for money which increases wealth and consumption. These effects are offsetting.

Proposition A4: An increase in the level of lump-sum taxation raises the demand for money but has no effect on the steady-state growth rate.

Expression (4.9) is quadratic in n^*-r . Its solutions are given by

$$(A4.10) \quad n^* = (r-\delta)/(1-\gamma) \\ = u(\alpha-g) .$$

Only the first is compatible with a positive level of consumption by capitalists.

APPENDIX 5: Properties of Functions of Stationary Diffusion Processes

Expression (3.9) is derived from expression (3.7) by application of the following theorem which is presented in Soong [1973] and Merton [1975].

Theorem

Let $x(t)$ represent a continuous-time diffusion process with equation of motion given by

$$(A5.1) \quad dx(t) = \alpha(x(t))dt + \sigma(x(t))dz(t) .$$

If $x(t)$ has a non-trivial stationary distribution, then for any continuous function $f(x(t))$ which is bounded on any finite interval:

$$(A5.2) \quad E[\alpha(x(t)) \cdot f'(x(t))] + \frac{1}{2}E[\sigma^2(x(t))f''(x(t))] = 0 .$$

Proof:

If $x(t)$ is stationary then so is $f(x(t))$.

Therefore

$$(A5.3) \quad \frac{dE[f(x(t))]}{dt} = 0 .$$

Applying Itô's lemma to evaluate this expression gives expression (A5.2).

Expression (3.9) may be obtained by applying this theorem to the diffusion process for k in (3.7) using $f(k(t)) = \log k(t)$.

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Footnotes

1. Randomness of this form can be generated by a stochastic rate of depreciation of capital, for instance. This specification of uncertainty generates asset returns which are themselves Wiener processes of the form assumed in the capital-asset pricing models of Merton [1969, 1971, 1973]. This is not the case when randomness derives instead from a stochastic rate of labor-force growth as in the models of Bourignon [1974] and Merton [1975].
2. Assuming an infinite horizon of optimization removes problems of non-stationarity and specification of the bequest function. It need not imply intergenerational altruism if, for instance, age at death is characterized by a Poisson process as Cass and Yaari [1967] and Merton [1971] have demonstrated.

The assumption of isoelastic marginal utility of consumption is highly convenient in its implication that optimal consumption and asset demands are proportional to wealth. It is also consonant with empirical analysis of behavior toward risk. On the basis of a cross-sectional estimation of asset demands, Friend and Blume [1975] conclude that their empirical results "indicate that the assumption of constant proportional risk aversion for households is a fairly accurate description of the market place." (p. 919).
3. The transversality condition is satisfied if and only if $c > 0$. See Merton [1969].

4. The average real return on money differs from the negative of the inflation rate by an amount proportional to the variance of the inflation rate, a manifestation of Jensen's inequality. Since the expression for the rate of return on money is slightly less complicated than that for the inflation rate, this analysis addresses itself to the first magnitude.
5. Feldstein [1976b] also finds that increased inflation may be accompanied by reduced capital accumulation when the rate of savings out of disposable income depends positively on the interest rate. His results are not fully comparable since he does not explicitly relate inflation to fiscal policy.
6. Merton [1973] discusses this distinction in detail.
7. Loosely speaking, the Inada conditions guarantee that the drift term in expression (5.7) is positive and large relative to the variance for k near 0, while it is negative and large in absolute value for very large values of k . This guarantees that the distribution of k is compact and does not include the origin.

A non-trivial steady-state distribution would not exist if, say, the tax treatment of capital were highly regressive rather than proportional in the neighborhood of the origin. The convexity of the tax function would then "undo" the concavity of the production function and introduce a positive probability of extinction.
8. Certain, although perhaps pathological, specifications of preferences, technology and policy may threaten the existence of a steady state. See Mirman and Zilcha [1976] for an example of this in a non-monetary stochastic growth model.
9. For the case of the Bernoulli utility function ($\gamma=0$) the optimal consumption-wealth ratio is independent of asset returns as demonstrated in Appendix 1. For this case, Δ is unambiguously positive.
10. See Sidrauski [1976b] and Sidrauski, Shell and Stiglitz [1969] for example. Ross [1975] shows that introducing randomness of bounded variation in the returns on heterogeneous capital goods does not in itself enhance the stability of the heterogeneous capital-good model.
11. In the stationary-state model analyzed by Tobin and Buiter [1976], stability is also enhanced under adaptive expectations. Wealth effects play an important role in this result as well.