ASYMPTOTIC TESTS FOR THE CONSTANCY OF REGRESSIONS IN THE HETEROSCEDASTIC CASE*

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1. Introduction

A frequently encountered problem is to test the hypothesis that the coefficient vectors in two regression equations are the same. Formally, let

$$Y_1 = X_1\beta_1 + u_1 \tag{1-1}$$

$$Y_2 = X_2\beta_2 + u_2 \tag{1-2}$$

where all symbols have the usual interpretation and Y_i is $m_i x1$, X_i is $m_i xk$, u_i is $m_i x1$, and β_i is $kx1$. We assume $m_i > k$. Customarily it is also assumed that $u_1 \sim N(0,\sigma^2 I_{m_1})$, $u_2 \sim N(0,\sigma^2 I_{m_2})$. It is then required to test $H_0:\beta_1=\beta_2$. Define $Y' = (Y_1' \;\; Y_2')$, $u' = (u_1' \;\; u_2')$, $X' = (X_1' \;\; X_2')$ and

$$X^* = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}$$

Under H_0 (1-1) and (1-2) may be written as

$$Y = X\beta + u \tag{1-3}$$

and under the alternative

$$Y = X^* \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} + u \tag{1-4}$$

Denoting the OLS residuals from (1-3) by $\hat{u}$ and those from (1-4) by $\hat{u}^*$, the

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test, referred to as the Chow-test, is based on the statistic

$$C = \frac{(\hat{u}'\hat{u}-\hat{u}^{*\prime}\hat{u}^{*})/k}{\hat{u}^{*\prime}\hat{u}^{*}/(m_1+m_2-2k)} \tag{1-5}$$

which is distributed as $F(k,m_1+m_2-2k)$ (see [1] and [2] for example).

Crucial for the validity of this test is the assumption that the variance of u_1 is the same as that of u_2. Sometimes, however, it is known that the model is heteroscedastic, e.g., that $u_1 \sim N(0,\sigma_1^2 I_{m_1})$, $u_2 \sim N(0,\sigma_2^2 I_{m_2})$, with $\sigma_1^2 \neq \sigma_2^2$. This case has recently been considered by Toyoda [6], Jayatissa [3], and Schmidt and Sickles [5]. Jayatissa and Schmidt and Sickles are principally concerned with exact small sample tests. Both of these are difficult to apply in the sense that they either require eigenvectors of matrices of order m_1 for the construction of the test or require eigenvalues of matrices of m_1+m_2 for determining the significance level of the test statistic.

In this paper we consider two asymptotic tests: the obvious likelihood ratio test and a variant of the Chow-test which is related to an asymptotic test suggested by Jayatissa. In Section 2 we state the tests. In Section 3 we examine their behavior in small samples by means of sampling experiments.

2. Two Asymptotic Tests

The Likelihood Ratio Test. The maintained hypothesis is $\sigma_1^2 \neq \sigma_2^2$. Then the likelihood function under the alternative hypothesis is

$$L(\Omega) = \prod_{i=1}^{2} \left(\frac{1}{\sqrt{2\pi}\sigma_i}\right)^{m_i} \exp\{-\frac{1}{2\sigma_i^2}(Y_i-X_i\beta_i)'(Y_i-X_i\beta_i)\} \tag{2-1}$$

and, under H_0, it is

$$L(\omega) = \prod_{i=1}^{2} \left(\frac{1}{\sqrt{2\pi}\sigma_i}\right)^{m_i} \exp\{-\frac{1}{2\sigma_i^2}(Y_i-X_i\beta)'(Y_i-X_i\beta)\} \tag{2-2}$$

where Ω and ω represent the unrestricted and restricted parameter spaces respectively. Let $L(\hat{\omega}) = \sup_{\omega} L(\omega)$, $L(\hat{\Omega}) = \sup_{\Omega} L(\Omega)$. Let $\hat{\beta}_1$, $\hat{\beta}_2$ be the OLS estimates from (1-1), (1-2); $\hat{u}_1$, $\hat{u}_2$ the corresponding vectors of residuals; $\hat{\beta}$ the ML estimate from (2-2) and $\tilde{u}' = (\tilde{u}_1' \ \tilde{u}_2')$ the corresponding residuals. It is straightforward to show that if λ denotes the likelihood ratio $L(\hat{\omega})/L(\hat{\Omega})$, then

$$-2\log \lambda = m_1 \log \tilde{u}_1'\tilde{u}_1 + m_2 \log \tilde{u}_2'\tilde{u}_2 - m_1 \log \hat{u}_1'\hat{u}_1 - m_2 \log \hat{u}_2'\hat{u}_2 \qquad (2\text{-}3)$$

It is straightforward that $-2\log \lambda$ is distributed asymptotically as $\chi^2(k)$.[1]

<u>An Asymptotic Chow-test</u>. Assume at first that σ_1 and σ_2 are known. Divide (1-1) by σ_1 and (1-2) by σ_2 and write the resulting equations

$$W_1 = Z_1\beta_1 + v_1 \qquad (2\text{-}4)$$

$$W_2 = Z_2\beta_2 + v_2 \qquad (2\text{-}5)$$

where $W_i = Y_i/\sigma_i$, $Z_i = X_i/\sigma_i$ and $v_1 \sim N(0, I_{m_1})$, $v_2 \sim N(0, I_{m_2})$. The Chow-test is now directly applicable to these equations, with (1-5) being the relevant statistic. The test statistic we propose is this same one with σ_1 and σ_2 replaced by any consistent estimates $\hat{\sigma}_1$, $\hat{\sigma}_2$. The resulting statistic is distributed asymptotically as $F(k, m_1+m_2-2k)$.

In order to establish this, assume that symbols $\tilde{W}_i$, $\tilde{Z}_i$, $\tilde{v}_i$ are defined analogously to W_i , Z_i , v_i except that the former are obtained by dividing by $\hat{\sigma}_1$ and $\hat{\sigma}_2$. The statistic in (1-5) can then be written as

$$C = \frac{\tilde{v}'[\tilde{Z}(\tilde{Z}'\tilde{Z})^{-1}\tilde{Z}' - \tilde{Z}^*(\tilde{Z}^{*\prime}\tilde{Z}^*)^{-1}\tilde{Z}^{*\prime}]\tilde{v}}{\tilde{v}'[I - \tilde{Z}^*(\tilde{Z}^{*\prime}\tilde{Z}^*)^{-1}\tilde{Z}^{*\prime}]\tilde{v}} \qquad (2\text{-}6)$$

[1] It is well-known that if $\sigma_1^2 = \sigma_2^2$, the likelihood ratio test and the Chow test are related and $[(\lambda^{-n/2}-1)(m_1+m_2-2k)/k]$ is distributed as $F(k, m_1+m_2-2k)$. It is natural to ask whether in the present case some suitable function of λ might not also have an F-distribution. This does not appear to be the case.

The typical element of $\tilde{v}$, say the jth, is $u_{ij}/\hat{\sigma}_i$, $i=1,2$. Since $u_i \sim N(0,\sigma_i^2)$ and $\text{plim}\ \hat{\sigma}_i = \sigma_i$, $\tilde{v}$ is asymptotically distributed as $N(0,I)$. (We assume that passing to the limit is taken to mean that the number of observations in both regression regimes increases without bound.) Moreover, $\text{plim}\ \tilde{Z} = Z$ and $\text{plim}\ \tilde{Z}^* = Z^*$. Since $\tilde{C}$ is a continuous function of $\tilde{v}$, it follows (Rao [4], p. 124) that $\tilde{C}$ is asymptotically distributed as C, i.e. as $F(k,m_1+m_2-2k)$.

Consistent estimates for σ_1 and σ_2 can be obtained from the OLS regressions (1-1), (1-2). In that event, since $\hat{u}_i^{*\prime}\hat{u}_i^*/(m_i-k) = \hat{\sigma}_i^2$ $(i=1,2)$ by the choice of consistent estimators, the asymptotic Chow statistic can be written as

$$C = [\hat{u}_1'\hat{u}_1/\hat{\sigma}_1^2+\hat{u}_2'\hat{u}_2/\hat{\sigma}_2^2-(m_1+m_2-2k)]/k \qquad (2\text{-}7)$$

It may be noted that the common regression coefficient estimates employed in $\hat{u}_1$ and $\hat{u}_2$ are the OLS estimates obtained by dividing the two subsets of data by $\hat{\sigma}_1$ and $\hat{\sigma}_2$ respectively and pooling the subsets and are not the same as the ML estimates obtained from maximizing (2-2), as is obvious from examining the likelihood equations corresponding to (2-2).

3. Monte Carlo Results

Experiments were performed under H_0. The model was $y_j = \alpha + \beta x_j + u_j$ and the regression coefficients were set $\alpha = \beta = 1.0$. The variance of u for the first m_1 observations, σ_1^2, was 1.0 in all these experiments. The corresponding variance for the last m_2 observations, σ_2^2, varied from 0.1 to 10.0 and the x_j were generated once-and-for all for each sample size either from the uniform distribution from 0 to 10 or from $N(5,8\ 1/3)$. The experiments partly parallel and partly complement the computations in [5]. For each combination of m_1 and m_2 examined, 2000 replications were employed. Table 1 contains the results with KS denoting the Kolmogorov-Smirnov statistic

$\sup|F(x_i)-S(x_i)|$ where $S(x_i)$ is the sample cumulative distribution evaluated at x_i and $F(x_i)$ the cumulative distribution predicted by theory. (In the case of the conventional Chow statistic this is taken to be $F(k, m_1+m_2-2k)$, as if we did not know that this is inappropriate). P denotes the frequency in the sample of values of the various statistics greater than the critical value at the .05 level from the predicted distribution. C, LR, and AC refer to the conventional Chow statistic, the likelihood ratio statistic and the asymptotic Chow statistic respectively. The following conclusions emerge: (1) Wherever comparisons are possible, the results for C are in excellent agreement with those reported for the exact distribution reported in [5] and confirm the finding there that C gives extremely bad results when sample sizes are highly disparate; (2) When $\sigma_2^2 = 1$, C is of course the appropriate statistic and the table verifies this; (3) The distribution of AC is essentially indistinguishable from the appropriate asymptotic F-distribution for all sample sizes; (4) Although convergence in the case of LR is slower than AC, it gives results which would often be acceptable in practice; (5) Increasing sample size improves the closeness of the approximation in all three cases for symmetric sample sizes.

Some experiments were also performed in order to examine the powers of the proposed tests when the alternative hypothesis is true. These experiments are extremely limited and are reported only for illustrative purposes. The true coefficients were $\alpha = 1.0$, $\beta = 1.0$ for the first m_1 observations and $\alpha = 3.0$, $\beta = 0.6$ for the last m_2 observations. Over the range of the x-variable the two regression lines are very close to each other and the regression scatters overlap each other substantially. Obviously the power will be greater the more the two scatters fail to overlap. The power will also depend on the absolute values of the error variances or on R^2. Four specific cases were examined, namely cases in which $R^2 = 0.9$ or 0.3 for the two sub-samples; these R^2's were achieved by letting σ_1^2 alternatively equal .926

and 19.444 and σ_2^2 equal .337 and 7.0. In each such case two subcases were examined, with $m_1 = m_2 = 10$ and $m_1 = m_2 = 50$ respectively. The estimated probabilities of rejecting H_0 are shown in Table 2. For the case in which R^2 is high for both subsamples, the power is excellent. For the other cases it is much lower, but always increases substantially with the sample size. In general the three tests do not show radically different behavior though overall the likelihood ratio test seems to have the best performance.

4. Conclusions

Two asymptotic tests are suggested for testing the constancy of regression coefficients in cases in which heteroscedasticity makes the conventional Chow test inapplicable. Monte Carlo experiments show that in reasonably small samples the limiting distributions fit excellently and that the power of the tests is reasonable.

Table 1

Results when H_0 is True*

m_1	m_2	σ_2^2		x's Uniform C	x's Uniform LR	x's Uniform AC	x's Normal C	x's Normal LR	x's Normal AC
10	10	.1	KS	.069*	.088*	.025	.092*	.087*	.038*
			P	.045	.096	.067	.109	.101	.088
		1	KS	.022	.069*	.024	.013	.088*	.018
			P	.050	.089	.052	.057	.098	.063
		10	KS	.058*	.073*	.019	.093*	.092*	.023
			P	.090	.086	.062	.042	.092	.060
25	25	.1	KS	.053*	.052*	.025	.022	.042*	.021
			P	.034	.068	.059	.052	.065	.056
		1	KS	.018	.037*	.018	.020	.050*	.021
			P	.052	.066	.052	.051	.062	.050
		10	KS	.050*	.026	.019	.035*	.041*	.016
			P	.078	.058	.052	.070	.060	.052
50	50	.1	KS	.015	.041	.021			
			P	.055	.055	.054			
		1	KS	.021	.023	.021			
			P	.045	.050	.045			
		10	KS	.021	.018	.016			
			P	.041	.050	.047			
40	10	.1	KS	.377*	.048*	.018			
			P	.000	.066	.050			
		1	KS	.025	.084*	.038*			
			P	.047	.082	.070			
		10	KS	.423*	.108*	.071*			
			P	.420	.090	.092			

*Kolmogorov-Smirnov statistics significant at the .05 level are indicated by * .

Table 2

Estimated Probabilities of Rejecting H_0

		$\sigma_2^2=.337$			$\sigma_2^2=7.0$		
		C	LR	AC	C	LR	AC
$\sigma_1^2=.926$	$m_1=m_2=10$	.654	.798	.734	.168	.155	.123
	$m_1=m_2=50$	1.000	1.000	1.000	.895	.707	.699
$\sigma_1^2=19.444$	$m_1=m_2=10$	.082	.170	.142	.062	.152	.081
	$m_1=m_2=50$	.321	.333	.323	.245	.262	.249

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