IDENTIFICATION OF PARAMETRIC FUNCTIONS IN LINEAR SIMULTANEOUS EQUATION MODELS

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1. INTRODUCTION

The identification problem in the linear simultaneous equation model is concerned with the possibility of uniquely determining certain characteristics of the model from prior restrictions and the distribution of the endogenous variables. In their seminal work on the subject, Koopmans and Rubin [6] and later Fisher [2], examined the identifiability of the coefficients of a single equation, subject to linear constraints on these same coefficients. More recently Wegge [13] and Rothenberg [10] have extended the analysis to the identifiability of the complete system subject to linear constraints across equations.

If the complete model or at least some equation of the model is not identifiable, one might be tempted to discard the model. As Hurwicz [3] recognized at an early point, however, certain interesting characteristics of the model may still be identifiable. This fact was emphasized by Wald [12], who examined conditions under which a single coefficient of the linear simultaneous equation model is locally identifiable. Rothenberg [10] has developed conditions under which a single parameter is (globally) identifiable both when there are linear restrictions only on a single equation and also across equations.

Of course, we are often interested in characteristics of linear models other than specific parameters or vectors of parameters, and would like to know whether or not that characteristic is identifiable. Kadane [4]

has suggested that in most cases such characteristics may be conveniently represented as a function of the model parameters. The concept of identifiability of parametric functions is a generalization of the identifiability of a specific parameter, the coefficients of a single equation, or the complete model, since in each case we have a function (possibly vector-valued) of the model parameters. Other parametric functions that have been studied, in the context of the linear model, are linear combinations of coefficients by Richmond [8] and in a limited fashion ratios of coefficients by Kelly [5].

The purpose of this paper is to systematically examine the identifiability of parametric functions in the linear simultaneous equation model when the coefficients are subject to linear restrictions. After presenting the model and the basic concepts of identifiability in the next section, I will develop conditions under which a general parametric function is identifiable in the third section. These conditions will serve, in the fourth section, as a convenient basis for deriving certain previously known results on the identifiability of linear combinations of coefficients. These conditions will be further employed, in the fifth and sixth sections, to study the role of alternative normalization rules when the linear restrictions are homogeneous and apply only to single equations. In the final section, the conditions will be applied to examine the identifiability of ratios of coefficients when the linear restrictions are homogeneous.

2. BASIC CONCEPTS

Consider the system of linear structural relations

$$By_t + \Gamma x_t = u_t \qquad t = 1, 2, \ldots, T$$

where y_t is an M x 1 vector of endogenous variables, x_t is a K x 1 vector of exogenous variables, u_t is an M x 1 vector of structural disturbances, B is an M x M nonsingular matrix of coefficients, and Γ is an M x K coefficient matrix.[1] The disturbance vectors u_t (t = 1, 2, ..., T) are assumed to be identically and independently distributed multivariate normal with mean vector 0 and positive definite covariance Σ.[2] The parameter space of the model consists of the (2M + K)M elements of (B, Γ, Σ) such that B is nonsigular and Σ is positive definite.

Nonsingularity of B implies the existence of the reduced form relations

$$y_t = -B^{-1}\Gamma x_t + B^{-1}u_t$$

$$= \Pi x_t + v_t$$

where $\Pi = -B^{-1}\Gamma$ is an M x K matrix of reduced form coefficients and v_t is an M x 1 vector of reduced form disturbances. The structural disturbances u_t (t = 1, 2, ..., T) are assumed to be independent of the predetermined variables x_t (t = 1, 2, ..., T). Thus y_t given x_t, B, Γ, and Σ (t = 1, 2, ..., T) are distributed multivariate normal with mean vector $\Pi x_t = -B^{-1}\Gamma x_t$ and positive definite covariance $\Omega = B^{-1}\Sigma B'^{-1}$.

Let A = (B:Γ) denote the M x (M+K) matrix whose typical row α_i represents the coefficients of the i-th equation. Thus we can represent all of coefficients in the system by the 1 x M(M+K) vector $\alpha' = (\alpha_1' \; \alpha_2' \; \ldots \; \alpha_M')$. The coefficients are known to satisfy R linear restrictions which

may be written

$$\alpha'\Psi = \lambda'$$

where Ψ is a $M(M+K) \times R$ matrix of known numbers, and λ' is $1 \times R$ vector of known values. Aside from the requirements that B be nonsingular and Σ positive definite these are the only prior restrictions on the parameter space.

By definition, there is a unique distribution of $(y_1, y_2, \ldots, y_T)$ for each point (B, Γ, Σ) in the restricted parameter space. A point $(B^1, \Gamma^1, \Sigma^1)$ is said to be observationally equivalent to another point $(B^2, \Gamma^2, \Sigma^2)$ if both points yield the same distribution of $(y_1, y_2, \ldots, y_T)$. Now the multivariate normal distribution is completely characterized by its mean vector and variance-covariance matrix. So $(B^1, \Gamma^1, \Sigma^1)$ and $(B^2, \Gamma^2, \Sigma^2)$ yield the same distribution of $(y_1, y_2, \ldots, y_T)$ if and only if $\Pi^1 x_t = \Pi^2 x_t$ for $t = 1, 2, \ldots, T$ and $\Omega^1 = \Omega^2$. Suppose that $X' = (x_1, x_2, \ldots, x_T)$ has full row rank K, then we have the following result:

LEMMA 2.1: The point $(B^1, \Gamma^1, \Sigma^1)$ is observationally equivalent to $(B^2, \Gamma^2, \Sigma^2)$, write $(B^1, \Gamma^1, \Sigma^1) \sim (B^2, \Gamma^2, \Sigma^2)$, if and only if $\Pi^1 = \Pi^2$ and $\Omega^1 = \Omega^2$.

At some point $(B^\circ, \Gamma^\circ, \Sigma^\circ)$ a characteristic of the model is said to be identifiable if the prior restrictions and the distribution of $(y_1, y_2, \ldots, y_T)$ at $(B^\circ, \Gamma^\circ, \Sigma^\circ)$ imply that the model exhibits that property. Usually the characteristics of interest can be represented as values of functions defined on the structural parameters. Let $g(\cdot)$ be a vector-valued function defined on the restricted parameter space, then at $(B^\circ, \Gamma^\circ, \Sigma^\circ)$ the function $g(\cdot)$ is identifiable if and only if $g(B, \Gamma, \Sigma)$ takes on the same value for all $(B, \Gamma, \Sigma) \sim (B^\circ, \Gamma^\circ, \Sigma^\circ)$ that satisfy the prior restrictions.

Now the prior restriction on (B, Γ, Σ) are $|B| \neq 0$, $\alpha'\Psi = \lambda'$, and Σ positive definite. While $(B, \Gamma, \Sigma) \sim (B^\circ, \Gamma^\circ, \Sigma^\circ)$ if $-B^{-1}\Gamma = \Pi^\circ$ and $B^{-1}\Sigma B'^{-1} = \Omega^\circ$ which means $B\Pi^\circ - \Gamma = AW^\circ = 0$ and $\Sigma = B\Omega^\circ B'$ where $W^{\circ\prime} = (\Pi^{\circ\prime}:I_k)$. But Ω° is positive definite so Σ will be positive definite when $|B| \neq 0$, and $AW^\circ = 0$ can be written in stacked form $\alpha'(I_M \otimes W^\circ) = 0$. Combining the restrictions we see that $(B, \Gamma, \Sigma) \sim (B^\circ, \Gamma^\circ, \Sigma^\circ)$ and satisfies the prior restriction if and only if $|B| \neq 0$, $\Sigma = B\Omega^\circ B'$, and $\alpha'(I_M \otimes W^\circ:\Psi) = (0, \lambda')$, whereupon we have

LEMMA 2.2: The parametric function $g(\cdot)$ is identifiable at $(B^\circ, \Gamma^\circ, \Sigma^\circ)$ if and only if $g(B, \Gamma, B\Omega^\circ B') = g(B^\circ, \Gamma^\circ, \Sigma^\circ)$ when $|B| \neq 0$ and $\alpha'(I_M \otimes W^\circ:\Psi) = (0, \lambda')$.

The above definition and lemma assume that $g(\cdot)$ is defined for all $(B, \Gamma, \Sigma) \sim (B^\circ, \Gamma^\circ, \Sigma^\circ)$ and satisfying the prior restrictions. It is possible, however, that certain functions, such as ratios of parameters, may not be defined for all such points. Accordingly, we say that $g(\cdot)$ is <u>domain identifiable</u> at $(B^\circ, \Gamma^\circ, \Sigma^\circ)$ if and only if $g(B, \Gamma, \Sigma)$ takes on the same value <u>where defined</u> for all $(B, \Gamma, \Sigma) \sim (B^\circ, \Gamma^\circ, \Sigma^\circ)$ that satisfy the prior restrictions. The distinction between the two definitions is non-trivial and has been largely overlooked in the limited literature on the identification of parametric functions.

3. IDENTIFIABILITY OF PARAMETRIC FUNCTIONS

Assume that the S x 1 vector-valued parametric function $g(\cdot)$ is defined and continuously differentiable on the subspace defined by $(B, \Gamma, \Sigma) \sim (B^\circ, \Gamma^\circ, \Sigma^\circ)$ satisfying the prior restrictions. By Lemma 2.2, we may consider $g(\cdot)$ to be a function only of the coefficients, $A = (B:\Gamma)$ or α, on this subspace. Correspondingly, we may define

$$G(\alpha) = \frac{\partial g(\alpha)}{\partial \alpha'}$$

as the $S \times M(M + K)$ Jacobian matrix of $g(\cdot)$.

In order to simplify notation let

$$Q^\circ = (I_M \otimes W^\circ : \Psi)$$

and

$$r' = (0, \lambda').$$

Then α satisfies the prior restrictions and observational equivalence if and only if $|B(\alpha)| \neq 0$ and $\alpha' Q^\circ = r'$. Without loss of generality, we may partition $\alpha' Q^\circ = r'$ as

$$(\alpha_1', \alpha_2') \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} = (r_1', r_2')$$

where Q_{11} is a square nonsingular matrix with $\rho(Q_{11}) = \rho(Q) = q$. Solving for α_1 in terms of α_2, we find $\alpha' Q^\circ = r'$ if and only if

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{bmatrix} Q_{11}'^{-1}(r_1 - Q_{21}'\alpha_2) \\ \alpha_2 \end{bmatrix}.$$

Define, accordingly, the M(M+K) x 1 vector of affine functions

$$p(\alpha_2) = \begin{bmatrix} Q_{11}'^{-1}(r_1 - Q_{21}'\alpha_2) \\ \alpha_2 \end{bmatrix}$$

which has the constant Jacobian matrix

$$P = \frac{\partial p(\alpha_2)}{\partial \alpha_2'} = \begin{bmatrix} -Q_{11}'^{-1}Q_{21}' \\ I_{M(M+K)-q} \end{bmatrix} .$$

Now $Q^{\circ\prime}P = 0$ since $(Q_{11}' \; Q_{21}')P = 0$ by definition and the rows of $(Q_{12}' \; Q_{22}')$ are spanned by the rows of $(Q_{11}' \; Q_{21}')$. And $\rho(Q^\circ) + \rho(P) = M(M+K)$ since $\rho(P) = M(M+K)-q$ and $\rho(Q^\circ) = q$. Thus $Q^{\circ\prime}$ spans the null space or column kernel of P.

With these assumptions and definitions we can now derive our basic theorem.

THEOREM 3.1: The parametric function $g(\cdot)$ is identifiable at $(B^\circ, \Gamma^\circ, \Sigma^\circ)$ if and only if

$$(3.1) \qquad \rho(I_M \otimes W^\circ : \Psi : G'(\alpha)) = \rho(I_M \otimes W^\circ : \Psi)$$

for all α satisfying $\alpha'(I_M \otimes W^\circ : \Psi) = (0, \lambda')$.

PROOF: (Sufficiency). Suppose $\rho(Q^\circ : G'(\alpha)) = \rho(Q^\circ)$ for all $\alpha' Q^\circ = r'$. Then $G(p(\alpha_2))$ lies in the row space of $Q^{\circ\prime}$ for any α_2 and

$$0 = G(p(\alpha_2))P$$

$$= \frac{\partial g(p(\alpha_2))}{\partial \alpha_2'}$$

since $Q^{\circ\prime}$ spans the null space of P. But this means that $g(p(\alpha_2))$ is constant for all α_2 or $g(\alpha)$ is constant for all α satisfying $\alpha'Q^\circ = r'$, whereupon $g(\cdot)$ is identifiable at $(B^\circ, \Gamma^\circ, \Sigma^\circ)$.

(Necessity). Suppose $\rho(Q^\circ:G'(\alpha)) > \rho(Q^\circ)$ for some $\alpha'Q^\circ = r'$ or equivalently $\rho(Q^\circ:G'(p(\alpha_2)) > \rho(Q^\circ)$ for some α_2. Then the rows of $G(p(\alpha_2))$ are not spanned by the rows of $Q^{\circ\prime}$ or the null space of P and $G(p(\alpha_2))P \neq 0$ for some α_2. It follows there exists a point, α^* say, such that $\alpha^{*\prime}Q^\circ = r'$ while $g(p(\alpha_2^*)) \neq g(\alpha^*)$, since $g(p(\alpha_2)) = g(\alpha^\circ)$ for all α_2 would imply

$$0 = \frac{\partial g(p(\alpha_2))}{\partial \alpha_2'}$$

$$= G(p(\alpha_2))P.$$

By continuity of $g(\cdot)$ and $p(\cdot)$ there exists a neighborhood about α_2^* such that $g(p(\alpha_2)) \neq g(\alpha^\circ)$ for all α_2 in the neighborhood. Now $|B(p(\alpha_2))|$ is a polynomial in the elements of α_2 and nonzero for $\alpha_2 = \alpha_2^\circ$, hence $|B(p(\alpha_2))| \neq 0$ for all α_2 except a set of measure zero. Thus we can find α_2 in a neighborhood of α_2^* such that $|B(p(\alpha_2))| \neq 0$ and $g(p(\alpha_2)) \neq g(\alpha^\circ)$. But this means that $g(\alpha) \neq g(\alpha^\circ)$ for some α satisfying $\alpha'Q^\circ = r'$ and $|B(\alpha)| \neq 0$, whereupon $g(\cdot)$ is not identifiable at $(B^\circ, \Gamma^\circ, \Sigma^\circ)$. END OF PROOF.

In applying Theorem 3.1 we must evaluate $\rho(Q^\circ:G'(p(\alpha_2))$ globally in order to verify that (3.1) holds for all $\alpha'Q^\circ = r'$. Suppose $g(\cdot)$ and hence $G(\cdot)$ are analytic in α_1 then the elements of $(Q^\circ:G'(p(\alpha_2))$ are analytic in α_2 and the matrix achieves maximal rank for all α_2 except on a set of measure zero.[3] This means that $(Q^\circ:G'(p(\alpha_2)$ must achieve maximal rank somewhere in any neighborhood of α_2°. And we need

only verify that $\rho(Q^\circ:G'(p(\alpha_2)) = \rho(Q^\circ)$ locally for all α_2 to guarantee that the condition holds globally for all α_2. Specifically, we have:

COROLLARY 3.1: Suppose that $g(\cdot)$ is analytic, then $g(\cdot)$ is identifiable at $(B^\circ, \Gamma^\circ, \Sigma^\circ)$ if and only if (3.1) holds locally for α satisfying $\alpha'(I_M \otimes W^\circ:\Psi) = (0, \lambda')$.

In actual practice, even this local search is probably unneeded. The matrix $(Q^\circ:G'(p(\alpha_2))$ is said to have a regular point at α_2° if its rank is constant in some neighborhood of α_2°. Fisher [2] has shown, when the elements of a matrix are analytic, the set of points that are not regular is of measure zero. Thus for any $(B^\circ, \Gamma^\circ, \Sigma^\circ)$ and hence α_2° we are almost certain that $(Q^\circ:G'(p(\alpha_2))$ has constant rank in some neighborhood of α_2°. And we can be almost certain that $\rho(Q^\circ:G'(\alpha^\circ)) = \rho(Q^\circ)$ is a necessary and sufficient condition for the identifiability of $g(\cdot)$.

An important special case of the model occurs when the linear restrictions apply only to coefficients of the same equation. That is, Ψ is block diagonal and $\alpha'\Psi = \lambda'$ may be written $\alpha_i'\Psi_i = \lambda_i'$ for $i = 1, 2, \ldots, M$. If the parametric function $g(\cdot)$ is a function only of the coefficients of equation i, then for i = 1, say,

$$(I_M \otimes W^\circ:\psi:G'(\alpha)) = \left[\begin{array}{ccc|ccc|c} W^\circ & 0 \ldots 0 & & \Psi_1 & 0 \ldots 0 & & G_1'(\alpha_1) \\ 0 & W^\circ \ldots 0 & & 0 & \Psi_2 \ldots 0 & & 0 \\ \vdots & \vdots \quad \vdots & & \vdots & \vdots \quad \vdots & & \vdots \\ 0 & 0 \ldots W^\circ & & 0 & 0 \quad \Psi_M & & 0 \end{array}\right]$$

where

$$G_i(\alpha_i) = \frac{\partial g(\alpha_i)}{\partial \alpha_i'}$$

We immediately have the following result, since the numbering of equations is arbitrary.

COROLLARY 3.2: With no across equation restrictions $g(\alpha_i)$ is identifiable at $(B^\circ, \Gamma^\circ, \Sigma^\circ)$ if and only if

$$\rho(W^\circ:\Psi_i:G_i'(\alpha_i)) = \rho(W^\circ:\Psi_i)$$

locally for α_i satisfying $\alpha_i'(W^\circ:\Psi_i) = (0, \lambda_i')$.

4. IDENTIFICATION OF LINEAR COMBINATIONS OF COEFFICIENTS

Sometimes economic meaning can be attached to certain linear combinations of coefficients. It follows that the identifiability of such functions can become important when some or all of the coefficients are unidentifiable. Moreover, any results on the identifiability of linear functions of coefficients include the identifiability of single coefficients and all coefficients as special cases. Consequently, the identifiability of linear combinations of coefficients has been well-studied in the past. While none of the results of this section are new, they are presented nonetheless for the sake of completeness, since Theorem 3.1 has such a simple form in this case.

Let $g(\alpha) = \alpha' t$ where t is an M(M+K) x 1 vector of known weights, then $G'(\alpha) = t$ is constant and we have the following condition first found by Richmond [8].

THEOREM 4.1: The linear function $\alpha' t$ is identifiable at $(B^\circ, \Gamma^\circ, \Sigma^\circ)$ if and only if

$$\rho(I_M \otimes W^\circ : \Psi : t) = \rho(I_M \otimes W^\circ : \Psi).$$

Define d_i as the M x 1 vector with unity in position i and zero elsewhere and e_j as the (M+K) x 1 vector that is zero everywhere but position j, where it is one. Then $\alpha_{ij} = \alpha'(d_i \otimes e_j)$ is recognized as linear and we immediately have Richmond's [8] result on the identification of a single coefficient.

CORROLLARY 4.1: The j^{th} coefficient of the i^{th} equation, α_{ij}, is identifiable at $(B^\circ, \Gamma^\circ, \Sigma^\circ)$ if and only if

$$\rho(I_M \otimes W^\circ : \Psi : d_i \otimes e_j) = \rho(I_M \otimes W^\circ : \Psi).$$

The complete coefficient vector is indentifiable if and only if this condition holds for all i = 1, 2, ..., M and j = 1, 2, ...,M+K. But this occurs if and only if $\rho(Q^\circ:I_{M(M+K)}) = \rho(Q^\circ)$ and we have the condition first derived by Rothenberg [10].

COROLLARY 4.2: All the coefficients, α, are identifiable at $(B^\circ, \Gamma^\circ, \Sigma^\circ)$ if and only if $(I_M \otimes W^\circ:\Psi)$ has full row rank M(M+K).

The corrsponding results for the case when the linear restrictions apply only to coefficients of single equations are obvious. Such results will be needed for later reference, however, and this case is the most frequently occurring form of the model. Thus, the following conditions involving linear combinations of the coefficients of the i^{th} equation are presented despite their redundancy. The notation is the same as before.

THEOREM 4.1: With no across equation restrictions, the linear function $\alpha_i't_i$ is identifiable at $(B^\circ, \Gamma^\circ, \Sigma^\circ)$ if and only if

$$\rho(W^\circ:\Psi_i:t_i) = \rho(W^\circ:\Psi_i).$$

COROLLARY 4.3: With no across equation restrictions, the j^{th} coefficient of the i^{th} equation is identifiable at $(B^\circ, \Gamma^\circ, \Sigma^\circ)$ if and only if

$$\rho(W^\circ:\Psi_i:e_j) = \rho(W^\circ:\Psi_i).$$

COROLLARY 4.4: With no across equation restriction, the coefficients of the i^{th} equation, α_i, are identifiable at $(B^\circ, \Gamma^\circ, \Sigma^\circ)$ if and only if $(W^\circ:\Psi_i)$ has full row rank M+K.

The above conditions, which require the construction of the Π matrix, are needlessly complex, since they can be recast so as to display the structural coefficients directly. Now $(I_M \otimes A^\circ)(I_M \otimes W^\circ) = 0$ and $\rho(I_M \otimes A^\circ) + \rho(I_M \otimes W^\circ) = M(M+K)$, thus $(I_M \otimes W^\circ)$ spans the null space of $(I_M \otimes W^\circ)$. Let D by any M(M+K) row matrix, then $(I_M \otimes A^\circ)(I_M \otimes W^\circ : D)$ has rank equal to the number of columns of D that are independent and also independent of $(I_M \otimes W^\bullet)$, or

$$\rho((I_M \otimes A^\circ)D) = \rho(I_M \otimes W^\circ : D) - \rho(I_M \otimes W^\circ)$$

$$= \rho(I_M \otimes W^\circ : D) - MK.$$

Applying this result to Theorem 4.1 we obtain an equivalent condition in terms of structural coefficients.

THEOREM 4.3: The linear function $\alpha't$ is identifiable at $(B^\circ, \Gamma^\circ, \Sigma^\circ)$ if and only if

$$\rho((I_M \otimes A^\circ)(\Psi : t)) = \rho((I_M \otimes A^\circ)\Psi).$$

And by the same argument we have the corresponding corollaries first given by Rothenberg [10] and Wegge [12], respectively.

COROLLARY 4.5: The j^{th} coefficient of the i^{th} equation is identifiable at $(B^\circ, \Gamma^\circ, \Sigma^\circ)$ if and only if

$$\rho((I_M \otimes A^\circ)\Psi : d_i \otimes A^\circ e_j) = \rho((I_M \otimes A^\circ)\Psi).$$

COROLLARY 4.6: The entire coefficient vector, α, is identifiable at $(B^\circ, \Gamma^\circ, \Sigma^\circ)$ if and only if $((I_M \otimes A^\circ)\Psi)$ has full row rank M^2.

5. HOMOGENEOUS RESTRICTIONS AND LINEAR NORMALIZATIONS

A well-studied case of the model occurs when the linear restrictions on the coefficients are homogeneous and apply only to the coefficients of single equations. The restrictions on the coefficients of the typical equation i may now be written

$$\alpha_i' \Phi_i = 0$$

where Φ_i is an (M+K) x R_i matrix of known weights. In this case α_i will satisfy the prior restrictions and observational equivalence at some point (B°, Γ°, Σ°) if and only if $\alpha_i' \Phi_i = 0$, $|B(\alpha)| \neq 0$ and $\alpha_i' W^\circ = 0$. If α_i meets these conditions, however, so will any scalar multiple and the need arises for a "normalization" rule in order to establish a scale for each such vector. It is of interest to study the role of alternative normalization procedures in the identification problem.

The most common approach to the choice of scale problem is to apply a linear normalization rule by requiring α_i to satisfy

$$\alpha_i' \mu_i = \delta$$

where μ_i is an (M+K) x 1 vector of constants and δ is a nonzero scalar.[4] We assume that this normalization rule is consistent with the homogeneous restrictions and observational equivalence, so there is at least one solution to $\alpha_i'(W^\circ : \Phi_i : \mu_i) = (0, 0, \delta)$. In order to study the identification of the parametric function $g(\alpha_i)$ under this normalization rule, we need merely define

$$\Psi_i = (\Phi_i : \mu_i)$$

and

$$\lambda_i' = (0, \delta)$$

and apply Corollary 3.2 to obtain

THEOREM 5.1: Under the normalization $\alpha_i'\mu_i = \delta$, $g(\alpha_i)$ is identifiable at $(B^\circ, \Gamma^\circ, \Sigma^\circ)$ if and only if

$$\rho(W^\circ:\Phi_i:\mu_i:G'(\alpha_i)) = \rho(W^\circ:\Phi_i)$$

locally for all α_i satisfying $\alpha_i'(W^\circ:\Phi_i:\mu_i) = (0, 0, \delta)$.

A problem arises, however, in that this normalization procedure is not necessarily "neutral," since it may restrict the direction as well as the scale of α_i. Specifically, we may sometimes find α which satisfy all the prior restrictions and observational equivalence but $\alpha_i'\mu_i = 0$. In this case the imposition of the restriction $\alpha_i'\mu_i = \delta$ would eliminate such α from consideration, thereby doing more than just establishing a scale for each direction vector.[5] Accordingly, it is important to establish when a linear normalization is neutral and does not restrict the direction of α_i.

The normalization $\alpha_i'\mu_i = \delta$ is neutral if and only if $\alpha_i'\mu_i \neq 0$ for all α such that $|B(\alpha)| \neq 0$ and $\alpha_j'(W^\circ:\Phi_j) = 0$ where $j = 1, 2, \ldots, M$. Let $c_i(\alpha)$ denote the (M+K) x 1 vector whose first M elements are the cofactors of the i^{th} row of $B(\alpha)$ and last K elements are zero. Then $c_i(\alpha)$ is a function only of α_j for $j \neq i$ and $|B(\alpha)| = \alpha_i'c_i(\alpha)$, whereupon we can derive the following result.

THEOREM 5.2: The linear normalization $\alpha_i'\mu_i = \delta$ is neutral at $(B^\circ, \Gamma^\circ, \Sigma^\circ)$ if and only if

$$\rho(W^\circ:\Phi_i:\mu_i:c_i(\alpha)) = \rho(W^\circ:\Phi_i:\mu_i)$$

for all α satisfying $\alpha_j'(W^\circ:\Phi_j) = 0$ for $j \neq i$.

PROOF: (Sufficiency). Suppose $\rho(W^\circ:\Phi_i:\mu_i:c_i(\alpha)) = \rho(W^\circ:\Phi_i:\mu_i)$ for all α satisfying $\alpha_j'(W^\circ:\Phi_j) = 0$ where $j \neq i$. Then $\alpha_i'c_i(\alpha) = |B(\alpha)| = 0$ when $\alpha_i'\mu_i = 0$ and $\alpha_i'(W^\circ:\Phi_i) = 0$, since $c_i(\alpha)$ is a linear combination of the columns of $(W^\circ:\Phi_i:\mu_i)$. Thus $\alpha_i'\mu_i \neq 0$ for any α such that $|B(\alpha)| \neq 0$ and $\alpha_j(W^\circ:\Phi_j) = 0$ where $j = 1, 2, \ldots, M$, which implies the linear normalization is neutral.

(Necessity). Assume $\rho(W^\circ:\Phi_i:\mu_i:c_i(\alpha)) > \rho(W^\circ:\Phi_i:\mu_i)$ for some α satisfying $\alpha_j'(W^\circ:\Phi_j)$ for $j \neq 0$. Then $c_i(\alpha)$ is independent of the columns of $(W^\circ:\Phi_i:\mu_i)$ and we can find α_i such that $\alpha_i'c_i(\alpha) = |B(\alpha)| \neq 0$ while $\alpha_i'(W^\circ:\Phi_i:\mu_i) = 0$. Thus $\alpha_i'\mu_i = 0$ for some α satisfying $|B(\alpha)| \neq 0$ and $\alpha_j(W^\circ:\Phi_j) = 0$ $(j = 1, 2, \ldots, M)$, and the linear normalization is not neutral. END OF PROOF.

The above theorem, which requires that we evaluate (5.1) for all α satisfying $\alpha_j'(W^\circ:\Phi_j) = 0$ where $j \neq i$, might prove difficult to verify. By the arguments of section 3., however, we need only show that (5.1) holds locally for all α satisfying the linear restrictions. Specifically we have

COROLLARY 5.1: The linear normalization $\alpha_i'\mu_i = \delta$ is neutral at $(B^\circ, \Gamma^\circ, \Sigma^\circ)$ if and only if (5.1) holds locally for all α satisfying $\alpha_j'(W^\circ:\Phi_j) = 0$ where $j \neq i$.

If, as is usually the case, $\rho(W^\circ:\Phi_i:\mu_i:c_i(\alpha))$ is constant for all α in the restricted neighborhood of α°, then we need only evaluate (5.1) at α° to check neutrality.

The form of Theorem 5.2 suggests that the neutrality of linear normalizations is closely related to the identifiability of $|B(\alpha)|$.

Suppose each equation is subjected to a linear normalization, then all normalizations are neutral if and only if Theorem 5.2 holds for i = 1, 2, ..., M. Define

$$\Psi = \begin{bmatrix} \Psi_1 & 0 \ldots 0 \\ 0 & \Psi_2 \ldots 0 \\ \cdot & \cdot \; \cdot \\ \cdot & \cdot \quad \cdot \\ 0 & 0 \ldots \Psi_M \end{bmatrix}$$

and $\lambda' = (\lambda_1', \lambda_2', \ldots, \lambda_M')$ where as above $\Psi_i = (\Phi_i : \mu_i)$ and $\lambda_i' = (0, 1)$. Let $g(\alpha) = |B(\alpha)|$, then

$$G(\alpha) = \frac{\partial g(\alpha)}{\partial \alpha'} = (c_1'(\alpha), c_2'(\alpha), \ldots, c_M'(\alpha)).$$

and a necessary condition for every normalization to be neutral is that $\rho(I_M \otimes W^\circ : \Psi : G'(\alpha)) = \rho(I_M \otimes W^\circ : \Psi)$ for all α satisfying $\alpha'(I_M \otimes W^\circ : \Psi) = (0, \lambda')$. But, by Theorem 3.1, this condition implies that $g(\alpha) = |B(\alpha)|$ is identifiable at $(B^\circ, \Gamma^\circ, \Sigma^\circ)$ and we have the following remarkable relationship.

COROLLARY 5.2: At $(B^\circ, \Gamma^\circ, \Sigma^\circ)$ the linear normalizations $\alpha_i'\mu_i = \delta$ for $\alpha = 1, 2. \ldots, M$ are all neutral only if $|B(\alpha)|$ is identifiable when these normalizations are imposed.

6. HOMOGENEOUS RESTRICTIONS AND QUADRATIC NORMALIZATIONS

A less restrictive approach to normalization is to impose the quadratic restriction

$$\alpha_i' D \alpha_i = \delta$$

where D is a known (M+K) x (M+K) positive definite matrix and δ a nonzero scalor constant.[6] Since D is positive definite then $\alpha_i' D \alpha_i \neq 0$ for any nontrivial α_i. Thus setting $\alpha_i' D \alpha_i = \delta$ only restricts the scale of and not the direction of α_i, whereupon the normalization is neutral. It is of interest to examine the identifiability of the parametric function $g(\alpha_i)$ when such a normalization rule is applied in the case of homogeneous restrictions.

Without loss of generality, we can simplify the analysis by setting $\delta = 1$, since we can always rescale D such that the restriction is unchanged. Now $g(\alpha_i)$ for α_i satisfying $\alpha_i' D \alpha_i = 1$ and $g((\alpha_i' D \alpha_i)^{-1/2} \alpha_i)$ for α_i unrestricted except to be nontrivial have the same range. Thus at $(B^\circ, \Gamma^\circ, \Sigma^\circ)$, $g(\alpha_i)$ is constant or identifiable under the normalization $\alpha_i' D \alpha_i = 1$ if and only if $g((\alpha_i' D \alpha_i)^{-1/2}\alpha_i)$ is constant or identifiable without the imposition of any normalization rule. We need only apply Corollary 3.2 to obtain the following result.

THEOREM 6.1: Under the normalization $\alpha_i' D \alpha_i = 1$, $g(\alpha_i)$ is identifiable at $(B^\circ, \Gamma^\circ, \Sigma^\circ)$ if and only if

$$\rho(W^\circ:\Phi_i:G'(\alpha_i) - D \alpha_i \alpha_i' G'(\alpha_i)) = \rho(W^\circ:\Phi_i)$$

locally for all α_i satisfying $\alpha_i'(W^\circ:\Phi_i) = 0$ and $\alpha_i' D \alpha_i = 1$.

PROOF: Define $\alpha_i^* = (\alpha_i' D \alpha_i)^{-1/2} \alpha_i$ as the normalized vector, then we may consider $g^*(\alpha_i) = g(\alpha_i^*)$ to be a composite function in α with

$$G^*(\alpha_i) = \frac{\partial g^*(\alpha_i)}{\partial \alpha_i'}$$

$$= \frac{\partial g(\alpha_i^*)}{\partial \alpha_i^{*\prime}} \quad \frac{\partial \alpha_i^*}{\partial \alpha_i'}$$

$$= (\alpha_i' D \alpha_i)^{-1/2} (G(\alpha_i^*) - G(\alpha_i^*) \alpha_i^* \alpha_i^{*\prime} D).$$

Let $\Psi_i = \Phi_i$, then by Corollary 3.2 $g^*(\alpha_i)$ is identifiable if and only if $\rho(W^\circ:\Phi_i:G^{*\prime}(\alpha_i)) = \rho(W^\circ:\Phi_i)$ for all α_i yielding $\alpha_i'(W^\circ:\Phi) = 0$. Eliminating the scale factor $(\alpha_i' D \alpha_i)^{-1/2}$, we find that $g^*(\alpha_i) = g(\alpha_i^*)$ is identifiable if and only if $\rho(W^\circ:\Phi_i:G'(\alpha_i^*) - D \alpha_i^* \alpha_i^{*\prime} G'(\alpha_i^*)) = \rho(W^\circ:\Phi_i)$ for all α_i such that $\alpha_i'(W^\circ:\Phi_i) = 0$ or equivalently all α_i^* yielding $\alpha_i^{*\prime}(W^\circ:\Phi_i) = 0$ and $\alpha_i^* D \alpha_i^* = 1$. END OF PROOF.

This condition for identifiability can be somewhat simplified. Without loss of generality, we can restrict our attention to the case where $g(\cdot)$ is a scalar function, whereupon $G'(\alpha_i) = \nabla g(\alpha_i)$ and the condition for identifiability becomes $\rho(W^\circ:\Phi_i: \nabla g(\alpha_i) - D \alpha_i \alpha_i' \nabla g(\alpha_i)) = \rho(W^\circ:\Phi_i)$ for all α_i yielding $\alpha_i'(W^\circ:\Phi_i) = 0$ and $\alpha_i' D \alpha_i = 1$. Now this condition is equivalent to the existence of a vector z such that $(W^\circ:\Phi_i)z + (\nabla g(\alpha_i) - D\alpha_i \alpha_i' \nabla g(\alpha_i)) = 0$ or $(W^\circ:\Phi_i)z + (-\alpha_i' \nabla g(\alpha_i)) D \alpha_i + \nabla g(\alpha_i) = 0$. But this means that $g(\cdot)$ is identifiable if and only if $\nabla g(\alpha_i)$ is a linear combination of the columns of $(W^\circ:\Phi_i:D \alpha_i)$ or $\rho(W^\circ:\Phi_i:D \alpha_i:\nabla g(\alpha_i)) = \rho(W^\circ:\Phi_i: D \alpha_i)$.

for all α_i giving $\alpha_i'(W^\circ:\Phi_i) = 0$. If $g(\cdot)$ is a vector-valued function then we have

COROLLARY 6.1: Under the normalization rule $\alpha_i' D \alpha_i = 1$, then $g(\cdot)$ is identifiable at $(B^\circ, \Gamma^\circ, \Sigma^\circ)$ if and only if

$$\rho(W^\circ:\Phi_i:D \alpha_i:G'(\alpha_i)) = \rho(W^\circ:\Phi_i:D\alpha_i)$$

locally for all α_i satisfying $\alpha_i'(W^\circ:\Phi_i) = 0$ and $\alpha_i' D \alpha_i = 1$.

To examine the identifiability of all the coefficients in the equation we may set $g(\alpha_i) = \alpha_i$ whereupon $G(\alpha_i) = I_{M+K}$. By Corollary 6.1, then α_i will be identifiable if and only if $\rho(W^\circ:\Phi_i:D \alpha_i:I_{M+K})$ $= \rho(W^\circ:\Phi_i:D \alpha_i)$ or $\rho(W^\circ:\Phi_i:D \alpha_i) = M+K$ for $\alpha_i'(W^\circ:\Phi_i) = 0$ and $\alpha_i' D \alpha_i = 1$. But $D \alpha_1$ is independent of the columns of $(W^\circ:\Phi_i)$ when $\alpha_i' D \alpha_i = 1$ thus, we have the following familiar result.

COROLLARY 6.2: Under the normalization rule $\alpha_i' D \alpha_i = 1$ the coefficients of the i^{th} equation, α_i, are identifiable at $(B^\circ, \Gamma^\circ, \Sigma^\circ)$ if and only if $(W^\circ:\Phi_i) = M+K - 1$.

Consider the linear function $\alpha_i't$ where $\alpha_i^\circ t \neq 0$, then by Corollary 6.1 $\alpha_i't$ is identifiable if and only if $\rho(W^\circ:\Phi_i:D \alpha_i:t) = \rho(W^\circ:\Phi_i:D \alpha_i)$ for all $\alpha_i'(W^\circ:\Phi_i) = 0$ and $\alpha_i' D \alpha_i = 1$. Since $\alpha_i^\circ t \neq 0$ while $\alpha_i^\circ(W^\circ:\Phi_i) = 0$, then t is independent of the columns of $(W^\circ:\Phi_i)$ and the identifiability condition becomes $\rho(W^\circ:\Phi_i:t:D \alpha_i) = \rho(W^\circ:\Phi_i:t)$ for all $\alpha_i'(W^\circ:\Phi_i) = 0$ and $\alpha_i' D \alpha_i = 1$. Now the quadratic restriction $\alpha_i' D \alpha_i = 1$ will only rescale any nontrivial α_i and not change $\rho(W^\circ:\Phi_i:t:D \alpha_i)$ hence $\rho(W^\circ:\Phi_i:t:D \alpha_i)$ $= \rho(W^\circ:\Phi_i:t)$ for $\alpha_i \neq 0$ satisfying $\alpha_i'(W^\circ:\Phi_i) = 0$ in an equivalent condition. Let P_i be a matrix such that $P_i'(W^\circ:\Phi_i) = 0$ and

$\rho(P_i) + \rho(W^\circ:\Phi_i) = M+K$, then P_i spans the null space of $(W^\circ:\Phi_i)$ and any α_i such that $\alpha_i'(W^\circ:\Phi_i) = 0$ is spanned by the columns of P_i. Thus $\rho(W^\circ:\Phi_i:t:D\ \alpha_i) = \rho(W^\circ:\Phi_i:t)$ for $\alpha_i'(W^\circ:\Phi_i) = 0$ if and only if $\rho(W^\circ:\Phi_i:t:DP_i) = \rho(W^\circ:\Phi_i:t)$. But $\rho(W^\circ:\Phi_i:DP_i) = M+K$, so $\alpha_i't$ is identifiable if and only if $\rho(W^\circ:\Phi_i:t) = M+K$, which is equivalent to

COROLLARY 6.3: Suppose $\alpha_i^{\circ\prime}t \neq 0$, then $\alpha_i't$ is identifiable at $(B^\circ, \Gamma^\circ, \Sigma^\circ)$ under the quadratic normalization $\alpha_i'\ D\ \alpha_i = 1$ if and only if $\rho(W^\circ:\Phi_i) = M+K$.

This condition means that a nonzero linear function of the coefficients of a single equation cannot be identifiable under a quadratic normalization unless all the coefficients of the equation are identifiable. In order to identify a linear combination of coefficients of a single equation we must instead use a linear normalization rule. But, as was pointed out above, a linear normalization is not necessarily neutral since it can restrict the direction as well as scale of α_i. Thus, if we are interested in identifying linear functions of coefficients, the analysis of when a linear normalization is neutral, which was conducted in the previous section, is important.

7. IDENTIFIABILITY OF RATIOS OF COEFFICIENTS

Suppose we are interested in the relative effects of certain variables in the i^{th} equation. Then the identifiability of the ratio of the corresponding coefficients, say α_{ij}/α_{ik}, becomes important when the first equation is unidentifiable. Assume, as in the previous sections, that the linear coefficient restrictions are homogeneous and apply only to single equations. Then α_{ij}/α_{ik} is identifiable at $(B^\circ, \Gamma^\circ, \Sigma^\circ)$ if and only if α_{ij}/α_{ik} is defined and constant for every α_i satisfying $\alpha_i'(W^\circ:\Phi_i) = 0$, $|B(\alpha)| \neq 0$, and the normalization rule (if any). Using a somewhat different approach to that given below, Kelly [5] purports to have developed a sufficient (but not necessary) condition for the identifiability of α_{ij}/α_{ik}.

A complication that has been overlooked by Kelly, however, is the possibility that $\alpha_{ik} = 0$ for some α_i satisfying observational equivalence and the prior restrictions. In this case α_{ij}/α_{ik} is not defined for all α_i satisfying $\alpha_i'(W^\circ:\Phi_i) = 0$, $|B(\alpha)| \neq 0$ and the normalization rule and hence is not (globally) identifiable. The best we can hope for is that α_{ij}/α_{ik} is constant <u>where</u> defined for α_i satisfying observational equivalence and the prior restrictions, whereupon α_{ij}/α_{ik} is domain identifiable. This concept is not empty, for if α_{ij}/α_{ik} is domain identifiable, then we know that the ratio will be (globally) identifiable if we are willing to impose the additional prior restriction $\alpha_{ik} \neq 0$. Of course, when $\alpha_{ik} \neq 0$ for all α_i in the relevant parameter space then global and domain identifiability are the same.

Suppose that we impose the normalization rule $\alpha_i' D \alpha_i = 1$, then α_{ij}/α_{ik} is domain identifiable at $(B^\circ, \Gamma^\circ, \Sigma^\circ)$ if and only if $\alpha_{ij}/\alpha_{ik} = \alpha_{ij}^\circ/\alpha_{ik}^\circ$ for all α_i satisfying $\alpha_i'(W^\circ:\Phi_i) = 0$, $\alpha_i' c_i(\alpha) = |B(\alpha)| \neq 0$,

$\alpha_i' D \alpha_i = 1$, and $\alpha_{ik} = \alpha_i' e_k \neq 0$. Now when α_i satisfies $\alpha_i'(W^\circ:\Phi_i) = 0$, $\alpha_i' c_i(\alpha) \neq 0$, and $\alpha_i' e_k \neq 0$ then so will any nonzero scalar multiple of α_i, while α_{ij}/α_{ik} is homogeneous of degree zero and hence constant for any scalar multiple of α_i when $\alpha_{ik} \neq 0$. Thus the choice of scale for α_i is superfluous to the domain identifiability of α_{ij}/α_{ik} and the ratio is domain identifiable at $(B^\circ, \Gamma^\circ, \Sigma^\circ)$ if and only if $\alpha_{ij}/\alpha_{ik} = \alpha_{ij}^\circ/\alpha_{ik}^\circ$ for every α_i satisfying $\alpha_i'(W^\circ:\Phi_i) = 0$, $\alpha_i' c_i(\alpha) \neq 0$, and $\alpha_i' e_k = \alpha_{ik}^\circ$. But this means that α_{ij}/α_{ik} is domain identifiable, with or without a neutral normalization, if and only if α_{ij} is (globally) identifiable under the normalization $\alpha_i' e_k = \alpha_{ik}^\circ$. Let $\Psi_i = (\Phi_i:e_k)$, $\lambda_i' = (0, \alpha_{ik}^\circ)$, and $g(\alpha_i) = \alpha_i' e_j$, then we can apply Theorem 5.1 to obtain.

THEOREM 7.1: Let $\alpha_{ik}^\circ \neq 0$ then α_{ij}/α_{ik} is <u>domain</u> identifiable at $(B^\circ, \Gamma^\circ, \Sigma^\circ)$ if and only if

$$\rho(W^\circ:\Phi_i:e_k:e_j) = \rho(W^\circ:\Phi_i:e_k).$$

Now α_{ij}/α_{ik} will be (globally) identifiable if and only if it is domain identifiable and $\alpha_{ik} \neq 0$ for all $\alpha_i'(W^\circ:\Phi_i) = 0$ and $|B(\alpha)| \neq 0$ or equivalently the linear normalization $\alpha_i' e_k = \alpha_{ik}^\circ$ is a neutral normalization if and only if $\rho(W^\circ:\Phi_i:e_k:c_i(\alpha)) = \rho(W^\circ:\Phi_i:e_k)$ locally for all α satisfying $|B(\alpha)| \neq 0$ and $\alpha_\ell'(W^\circ:\Phi_\ell) = 0$ $(\ell = 2, 3, \ldots, M)$. Combining the condition for domain identifiability of α_{ij}/α_{ik} and the condition for $\alpha_i' e_k = \alpha_{ik}^\circ$ to be neutral yields the following corollary.

COROLLARY 7.1: Suppose $\alpha_{ik}^\circ \neq 0$, then α_{ij}/α_{ik} is (globally) identifiable at $(B^\circ, \Gamma^\circ, \Sigma^\circ)$ if and only if

$$\rho(W^\circ:\Phi_i:e_k:e_j:c_i(\alpha)) = \rho(W^\circ:\Phi_i:e_k) \tag{7.1}$$

locally for all α_ℓ yielding $\alpha_\ell'(W^\circ:\Phi_\ell) = 0$ where $\ell \neq i$.

Suppose that $(W^\circ:\Phi_i:e_k:e_j:c_i(\alpha))$ has constant rank for all α locally satisfying $\alpha_\ell'(W^\circ:\Phi_\ell) = 0$ where $\ell \neq i$. It follows immediately that α_{ij}/α_{ik} is (globally) identifiable if and only if (7.1) holds for $\alpha = \alpha^\circ$. Now, as we argued in the third section, almost all points in the restricted parameter space will satisfy this condition of regularity. Thus, unless α° is a pathological point in the restricted parameter space, a necessary and sufficient condition for α_{ij}/α_{ik} to be identifiable is that (7.1) be met at α°.

Since $\alpha_i^{\circ\prime}c_i(\alpha^\circ) = |B(\alpha^\circ)| \neq 0$ then at least one element of α_i°, say α_{ik}°, is nonzero. Applying Corollary 7.1, we find that the ratios of all coefficients in the vector to α_{ik}, namely $\frac{1}{\alpha_{ik}}\alpha_1$, is identifiable if and only if $\rho(W^\circ:\Phi_i:e_k:I_{M+K}:c_i(\alpha)) = \rho(W^\circ:\Phi_i:e_k)$ for the appropriate α. Thus $\frac{1}{\alpha_{ik}}\alpha_i$ is identifiable at $(B^\circ, \Gamma^\circ, \Sigma^\circ)$ if and only if $\rho(W^\circ:\Phi_i:e_k) = M+K$. Since $\alpha_i^\circ e_k \neq 0$, then e_k is independent of the columns of $(W^\circ:\Phi_i)$ and we obtain a variation of Corollary 6.2 wherein no normalization rule is imposed.

COROLLARY 7.2: The coefficients of the first equation, α_i, are identifiable up to a scalar multiple if and only if $\rho(W^\circ:\Phi_i) = M+K-1$.

FOOTNOTES

[1] We need only assume that x_t is a vector of predetermined variables to derive the results of this paper, however, the presentation is somewhat simpler with this stronger assumption.

[2] The assumption of normality is overly restrictive. If all we know about u_t is that it has mean zero given any x_t and positive definite covariance, then the same results apply.

[3] To see this we need only look at the largest-order square submatrix that achieves full rank. The determinant of this matrix is a polynomial in its elements and hence analytic in α. Since this determinant is nonzero for some values of α, it is nonzero for all values except on a set of measure zero.

[4] The most frequently used normalization procedure is to set one coefficient of each equation, say α_{ij}, to unity. But this can be written $\alpha_{ij} = \alpha_i'e_j = 1$ which is recognizable as a linear normalization.

[5] Another problem that might arise is that the linear normalization is not consistent with the homogeneous restrictions and observational equivalence. That is, $\alpha_i'\mu_i = 0$ whenever α_i satisfies $\alpha_i'(W^\circ:\Phi_i) = 0$. But the normalization $\alpha_i'\mu_i = \delta$ will not be neutral in such an event, so we need not treat this problem separately.

[6] A closely related normalization is to set $\beta_i'P\beta_i = 1$ where P an MxM positive definite matrix. Since $|B| \neq 0$ then $\beta_i \neq 0$ and this procedure will only rescale the vector. Of course we can define

$$D = \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix}$$

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