THE COST-OF-LIVING INDEX:

Algebraical Theory

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PREFACE

According to the concept of the cost of living, it is the cost of maintaining at given prices a given standard of living.

Commodities are consumed in the process of living. They are consumed in different amounts, which together define a possible composition of consumption. The standard of living is considered as a relation determined by preferences between such possible compositions.

Thus the cost of living is made intelligible by a preference relation between the possible compositions of consumption.

But the preferences can only be known through their effects on choice, as expressed in expenditures. Hence expenditure data, for a variety of different occasions, are to be the basis for measurement of the cost of living, in respect to the prices found in one of these occasions and the standard of living found in another. This then is the cost of living with these as the base and object occasions.

The history of the cost of living problem has almost entirely been a search for an algebraical formula, involving the expenditure data for the base and object occasions, which could with good reason be considered as an index of the cost-of-living. An index of such a form and with such a justification was never found. All the same, the theory set out in this memorandum can, with a suitable enlargement of ideas, be considered a continuation of that traditional pursuit. However, one may well ask: why should the index be given by an algebraical formula; why should it involve just the expenditure data in the base and object occasions; why, in view of the necessary indeterminacy arising from incomplete knowledge of preferences, should it give a point-determination; and, finally, why should it always be properly definable, seeing that it depends for its meaning on preferences, the existence of which requires a consistency of behaviour that may not be borne out by the data?

The method which is to be adopted for pursuing the problem is to consider all preference systems, of the normal type, that are consistent with the expenditure data for a variety of occasions. If the data are inconsistent, there are no such preference systems and there is no proper basis for proceeding with the measurement. Otherwise there will be an infinite class of such preference systems. Relative to any one, there is a point-determination of the cost of living, with any pair of occasions taken as base and object. It can be shown that as the preference system ranges in the considered infinite class, this point-determination ranges in an open interval, the extremities of which can be calculated from the data. The calculation is not by an algebraical formula, but by a combinatorial process, involving the solution of linear inequalities, and the determination of the extremes of linear functions restricted by linear inequalities.

Thus, in the framework of such a formulation, there is a negative answer to all those questions.

This does not mean that there is no algebraical approach. But, if an algebraical formula is found, then what kind of meaning can it have, which can be made the source for its interpretation as an index? It is assumed that it must have interpretation on some algebraic model of a preference system which is consistent with the data.

It is found that, for any four occasions, and more than four commodities, there always exists an infinite class of quadratic functions whose gradients at the four given consumption points are in the corresponding price-directions. Subject to a certain criterion, given by algebraical inequalities, there will exist among these an infinite subclass of quadratics, which are increasing and convex, and thus measure a normal preference scale in a convex neighborhood containing the consumption points in the four occasions. Relative to any member of this class, there is a determination of the cost of living, for any one of the occasions in respect to any other. The determination describes an open interval, as the preference system describes its class, the extremities of which can be calculated by an algebraical formula. However, it happens that though there is this variance in the determination of cost of living in respect to these preference systems, there is a strict invariance in the determination of the order of standard of living.

This algebraically determined interval, subject to the mentioned criterion, will lie within the combinatorially determined absolute interval, which requires only the consistency condition. A case for the use of the algebraical method is that it is simpler in its calculations, has interpretation in terms of a constructive model, and that it has a further interpretation in terms of the combinatorial method, from which it can be derived, by taking a special form of solution, which is called the median form of solution, of a system of inequalities involved in the combinatorial method.

In fact, this median form of solution will be made the genesis of the algebraical method in the context of the inequalities which are the basis for the combinatorial method. Thus, the quadratic property of median solutions is demonstrated, and then made the basis for the algebraical method.

The method is in no way dependent on any notion of approximation.

In fact, there are no already well-defined preferences to which approximations can be made. The problem is altogether one of arriving at a definition in terms of the data which is in harmony with the preconceived ideas of the sense of the cost-of-living question. The calculations made are exact, in relation to certain underlying models, the existence of which can be demonstrated, though they need never be practically constructed.

The peculiarity of four as the number of occasions which enter into the algebraical algorithm can only be appreciated directly from the algebra. A formula relating to an exact algebraical model must be based on data for some definite number of occasions, which here turns out to be four, irrespective of the number of commodities. It is different with the combinatorial method, since it has no dependence on any constructive model, and is perfectly general.

The independence, in the algebraical algorithm, of this number of occasions from the number of commodities is important, in that the number of commodities may have to be considered just as an indication of the refinement with which consumption is characterized by separation into different components which, in principle, could be greater or less without much disturbing the resulting measurements.

Any obvious attempt to obtain a more general, or even a different and still adequate, algebraical approach, seems to present formidable difficulties. But the simplest approach which is also adequate to the essential structure of the question is also all that is wanted.

The traditional index-number, of the Paasche-Laspeyres type, will be seen to have an important role. But instead of just the one, from the base to the object occasion, which provides the concept for the conventional cost-of-living index, the twelve such numbers defined between four occasions are used. Since preferences are set in a somewhat elaborate structure, a corresponding elaborateness is to be expected in any analysis which exploits them.

A final algebraic formula is not shown explicitly. It is more natural to leave it implicit in the simultaneous solution of certain algebraical equations.

The computations involved can easily be carried out on a desk computer. But it is convenient to have a programme for an automatic computer, and certainly necessary if the calculations are to be made repeatedly. A programme has already been prepared by Mr. Harold Samuels for the IBM 650. It will be given in another memorandum, together with results and interpretations for a number of examples with observed data, and some further examples in two dimensions which will provide graphical illustration.

In contrast to the algebraical algorithm, evaluation of the formulae for the absolute interval obtained by the combinatorial method, in the memorandum which is to follow this, presents a difficult problem. Though these formulae exhaust the cost-of-living measurement problem on the supposition of consistent data, and present a perfectly definite principle of calculation, no method is as yet available for the practical evaluation. However, here, by algebraical means, and with data satisfying a condition not much stronger than consistency, it is possible to calculate a sub-interval of the combinatorially determined absolute interval.

The question, importance of which has been emphasized to me by Professor T. C. Koopmans, about what to do when the data are inconsistent, belongs to a further, statistical development of the subject, in which the inconsistencies are reconciled as far as possible by viewing them as arising from disturbance of a consistent system. The framework is here for approaching that problem; but so far the concern has been only for when consistency is provided, or even the stronger condition which is the basis for the algebraical method now to be described.

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1. Compensating factors

Let $\mathcal{E}$ be a normal expenditure system, therefore with preference relation $P_{\mathcal{E}}$ that is a scale; and so also with indifference relation $\tilde{P}_{\mathcal{E}}$ that is an equivalence. Two compositions x, y that are equivalent in the scale are also said to represent the same standard, as decided by the system. A consumer, having consumption with composition represented by x, is exactly compensated in the scale, when subject to the combination of losses and gains represented by the vector $y - x$, which exchange x for another composition y. This need not imply an ultimate indifference between x and y. Indeed, any two objects which can be distinguished from each other could, in principle, be distinguished in preference; though the preferences which produce the distinction may not be implicit in the expenditure system. However, equivalence between x and y in the scale implies that, whatever the preference between them, if either x or y alone is subject to a loss or gain in any single component, however small, then this equivalence is destroyed, and replaced by the preference corresponding to this loss or gain. It is impossible to observe indifference: all that can be observed in behaviour is preference, and indifference here is just the absence of observed preference, in no way implying an ultimate absence of preference.

The relative cost of a composition y on a balance u is defined by $u'y$, with x and u corresponding in the system. Since the relative cost of the composition x on the balance u is unity, it is the cost of y expressed as a fraction of the cost of x, at the same prices.

Now, given any two compositions x and y, there exists a unique composition x^*, equivalent to y in $\mathcal{E}$, and belonging in $\mathcal{E}$

to a balance u^* which is parallel to the balance u belonging to x. The relative cost, on the balance u corresponding to x, of all compositions z not inferior to x, attains an absolute minimum at $z = x^*$. Thus, with the definition

$$\rho_{x,y} = \min \{u'z \; ; \; x\bar{P}_{\xi} z\} ,$$

there follows

$$\rho_{x,y} = u'x^*$$

together with

$$\rho_{x,y} \leq u'z$$

and provided $x\bar{P}_{\xi} z$.

$$\rho_{x,y} = u'z \Longleftrightarrow z = x^*$$

The number $\rho_{x,y}$, defined for any composition x taken in relation to another y, will be called the compensating ratio, for x in respect to y, or with x and y as object and base.

For a consumer whose expenditures on different occasions are in accordance with the system, $\rho_{x,y}$ gives the minimum expenditure required, at prices of the object x-occasion, to purchase a composition which represents a standard not inferior to that of the base y-occasion, the expenditure in the object occasion being taken as unit. At the prices in the x-occasion, with any greater expenditure a better composition could be attained, provided the prices in the occasions are not parallel; and with any smaller expenditure, it would be impossible to obtain a composition that was not inferior to y. A consumer in the x-occasion, spending in accordance with the system, has to compensate expenditure in the ratio ρ_{xy}, to achieve equivalence with the y-occasion at the same prices. This equivalence is, more strictly, the threshold of non-inferiority. With that compensation made, the composition attained is x^*, equivalent to

y, but not identical with it, unless the prices in the two occasions be parallel.

If the preference relation $xP_\xi y$ between x and y is to be called <u>standard-of-living</u>, then $\rho_{x,y}$ may be called the <u>cost-of-living</u>, with x as object and y as base.

From the compensating ratio function $\rho_{x,y}$, the original normal scale from which it is constructed can be recovered. For

$$\rho_{x,y} > \rho_{x,z} \iff yP_\xi z \ ;$$

from which it follows that $\varphi(z) = \rho_{x,z}$ is a gauge for the system, associated with an arbitrary point x.

2. Completions of a configuration

Let $\mathfrak{F}$ denote a given expenditure configuration, such as might be obtained from expenditure data of some consumer.

Questions are to be asked about the consumer, on the basis of the information represented by $\mathfrak{F}$. The method of treating these questions will be to view $\mathfrak{F}$ embedded in a normal expenditure system, in terms of which the questions are made intelligible. A latitude in the answering of the questions lies in the variety of the normal systems containing $\mathfrak{F}$.

However, in the dispersion of admissible answers, forming an interval, there is, under proper conditions, singled out a more limited, algebraically determined interval, with a peculiar structural position within this wider, absolute interval. It also has peculiar properties, in terms of underlying systems, which have a constructive algebraical form within the unconstructable totality of normal systems, which give the framework for a greater power of interpretation.

Accordingly, let Ω denote the class of all normal expenditure systems, of the same dimension as $\mathfrak{F}$. Then let $\Omega_{\mathfrak{F}}$ denote the sub-class of all that are completions of $\mathfrak{F}$. Thus:

$$\mathcal{E} \in \Omega_{\mathfrak{F}} \equiv \mathcal{E} \in \Omega \wedge \mathfrak{F} \subset \mathcal{E} \ ;$$

that is, $\mathcal{E}$ is an expenditure system of the class $\Omega_{\mathfrak{F}}$ if it is normal, and contains $\mathfrak{F}$ as one of its configurations.

In a corresponding fashion, let Π denote the class of normal preference systems, and $\Pi_{\mathfrak{F}}$ the sub-class that contains the preference relation $P_{\mathfrak{F}}$ of $\mathfrak{F}$, that is,

$$P \in \Pi_{\mathfrak{F}} \equiv P \in \Pi \wedge P_{\mathfrak{F}} \subset P \ ;$$

then

$$\mathcal{E} \in \Omega_{\mathfrak{F}} \Longleftrightarrow P_{\mathcal{E}} \in \Pi_{\mathfrak{F}} \ .$$

The normal expenditure systems are in a one-to-one correspondence with the normal preference systems:

$$\mathcal{E} \in \Omega \Longleftrightarrow P_{\mathcal{E}} \in \Pi \ .$$

The condition $\Omega_{\mathfrak{F}} \neq 0$, that there exist normal completions of $\mathfrak{F}$, by which condition $\mathfrak{F}$ is called <u>normal</u>, immediately implies that $P_{\mathfrak{F}}$ is an order, which is the condition that $\mathfrak{F}$ be <u>consistent</u>. For if $\mathcal{E} \in \Omega_{\mathfrak{F}}$ then $P_{\mathfrak{F}} \subset P_{\mathcal{E}}$, where $P_{\mathcal{E}}$ is an order and $P_{\mathfrak{F}}$ transitive; and this implies that $P_{\mathfrak{F}}$ is an order. The converse is also true, and will be proved elsewhere; that is, if $\mathfrak{F}$ is consistent, then it is normal. Thus the conditions of consistency and normality for an expenditure configuration are to be taken as equivalent. The criterion for the consistency condition is given by the Houthakker acyclicity condition, which depends just on the cross-structure of the given configuration. What is important here is that this ensures the existence of normal completions of that configuration.

3. Induced structure

Let $\mathfrak{F} = [U;X]$ be a consistent configuration, so the preference relation $P_{\mathfrak{F}}$ is an order, and $\Omega_{\mathfrak{F}} \neq 0$. Let $\xi \in \Omega_{\mathfrak{F}}$ be any of the therefore existing normal completions of $\mathfrak{F}$, so that $P_{\mathfrak{F}} \subset P_{\xi}$.

It may be impossible to make any direct assertion about order, let alone compensation structure, for $\mathfrak{F}$. For example, if all the cross-deviations are positive, that is $D_{rs} > 0$, which is a real possibility with any data, then the preference order may be null: $P_{\mathfrak{F}} = \bigcirc$. In this case, nothing at all can be asserted about a necessary ordering of the elements. It is true, at the other extreme, $P_{\mathfrak{F}}$ may turn out to be a complete order; in which case order is fully determined. But generally, $P_{\mathfrak{F}}$ will be a partial order, intermediate between a null and a complete order, which will admit refinement to a complete order, or, more generally, to a scale. But the scale is not unique: it is indeterminate in the variety of all scales consistent with that partial order, as can be shown.

Now, relative to the arbitrary normal completion ξ of $\mathfrak{F}$, there is determined a scale $S_{\mathfrak{F}}(\xi)$, by applying the scale P_{ξ} just to the elements of $\mathfrak{F}$, which is necessarily a refinement of $P_{\mathfrak{F}}$. It will be called the scale in $\mathfrak{F}$ induced by the normal completion ξ. It can be shown that as ξ varies, then $S_{\mathfrak{F}}(\xi)$ varies through all the scale refinements of $P_{\mathfrak{F}}$. Thus it turns out that no further specification of order is obtained by taking the normally induced scales rather than the scale refinements of $P_{\mathfrak{F}}$. The operation is thus of negative significance, just so far as order is concerned.

Compensation structure, however, is a much more strict and elaborate concept, which not only gets a determination by induction through any normal completions, but has such an induction as its only vehicle of definition.

Thus, with the normal completion ξ of $\mathfrak{F}$, there is formed the compensating ratio function $\rho_{x,y} = \rho_{x,y}(\xi)$, and this is applied to the elements of $\mathfrak{F}$, to obtain the array $\rho_{\mathfrak{F}}(\xi) = \{\rho_{rs}\}$, where $\rho_{rs} = \rho_{x_r,x_s}$ $(r,s = 1,\ldots,k)$ which define the <u>compensation structure of $\mathfrak{F}$ induced by ξ</u>. As ξ varies in $\Omega_{\mathfrak{F}}$, the array $\rho_{\mathfrak{F}}(\xi)$ also undergoes variation. It will appear elsewhere that there exist two limiting arrays $\rho'_{\mathfrak{F}}$, $\rho''_{\mathfrak{F}}$ with elements ρ'_{rs}, ρ''_{rs} with the property that, for any r,s and any ρ^*_{rs}, there exists an $\xi^*\epsilon\Omega_{\mathfrak{F}}$ such that $\rho_{rs}(\xi^*) = \rho^*_{rs}$ if and only if

$$\rho'_{rs} < \rho^*_{rs} < \rho''_{rs} .$$

Thus the elements of neither of these arrays belong to a normally induced compensation structure; but together they give the extremities of open intervals $I_{rs} = [\rho'_{rs}, \rho''_{rs}]$ which bound the elements ρ_{rs} in any normally induced compensation structure. These may be called the <u>absolute intervals</u>. The final problem is the computation of these limiting arrays, or any particular elements in them. But here a more special problem is being investigated, which does, however, have to be viewed in relation to the general problem.

Instead of considering the totality Ω of normal expenditure systems of the same dimension as $\mathfrak{F}$, and then the class $\Omega_{\mathfrak{F}}$ of those which are completions of $\mathfrak{F}$, one may start with some special class $\Omega^* \subset \Omega$, and consider the completions $\Omega^*_{\mathfrak{F}}$ of $\mathfrak{F}$ which belong to Ω^*. According to the narrowness of the class Ω^*, there will be a narrowing down of the varieties of order and compensation structure induced on $\mathfrak{F}$ relative to completions in Ω^*. Thus $S_{\mathfrak{F}}(\xi^*)$ may not exhaust all the scale refinements of $P_{\mathfrak{F}}$ as ξ^* ranges in Ω^* ; and the elements ρ^*_{rs} of $\rho_{\mathfrak{F}}(\xi^*)$ may range in intervals I^*_{rs} narrower than the absolute intervals I_{rs} .

If Ω^* consists of expenditure systems on some algebraic model, then it becomes an algebraical problem to find the criterion for $\Omega^*_{\mathfrak{F}} \neq 0$, and to determine the arrays $\rho^{*\prime}_{\mathfrak{F}}$, $\rho^{*\prime\prime}_{\mathfrak{F}}$ of limits of the intervals I^*_{rs} , defined under this criterion.

For example, Ω^* could be the class of normal expenditure systems determined by normal preference functions on some convex region containing the base elements of $\mathfrak{F}$ which are given by quadratic function, increasing and convex in that region. The algebraical criterion for $\Omega^*_{\mathfrak{F}} \neq 0$ will be found in this case. And it will turn out, somewhat remarkably, that $S_{\mathfrak{F}}(\xi^*)$ is invariant as ξ^* ranges in $\Omega^*_{\mathfrak{F}}$, thus defining a unique scale induced on $\mathfrak{F}$ relative to the class $\Omega^*_{\mathfrak{F}}$. Moreover, $\rho_{\mathfrak{F}}(\xi^*)$ turns out to be a one-parametric family of arrays. There is a natural parameter M defined with an algebraically determined critical value $\hat{M}$, giving the Ω^*-induced compensation structures in the parametric forms $\rho_{\mathfrak{F}}(M)(\hat{M} < M < \infty)$ in which they appear with the monotonicity property

$$\rho_{rs}(M) < \rho_{rs}(N) \qquad (M > N) .$$

Moreover, the interval limits, excluded by the admissible range of M , are derived as limits obtained by letting M tend to the limits of its range. Thus

$$I^*_{rs} = (\tilde{\rho}_{rs}, \hat{\rho}_{rs}) ,$$

where

$$\hat{\rho}_{rs} = \lim_{M \downarrow M} \rho_{rs}(M) = \rho_{rs}(\hat{M})$$

$$\tilde{\rho}_{rs} = \lim_{M \uparrow \infty} \rho_{rs}(M) .$$

The most advantageous approach is not direct, but is made through the concept of a median solution, which links the algebraical method with the combinatorial method; and then through consideration of the existence and the variety of quadratics which can be associated with such a solution.

First, since the considered order and compensation structures are expressed by numbers, some clarification will be made of the relation between these numbers in their different roles.

Given a normal expenditure system $\mathcal{E}$, its preference relation $P_{\mathcal{E}}$ is a partial order, but of that special type which is called a scale, which reduces to a complete order of equivalence classes. The direct construction of the relation $P_{\mathcal{E}}$, following the rule of its definition, is impossible, since it involves the construction of all possible base-preference chains of unrestricted finite length. However, the existence is known of a differentiable function φ with the property

$$\varphi(x) > \varphi(y) \iff xP_{\mathcal{E}}y ,$$

by which it completely represents $P_{\mathcal{E}}$. Moreover, the differential form $u'dx$ is integrable, and

$$\lambda u'dx = d\varphi$$

where λ is some integrating factor, gives a construction for such a function φ , which is called a <u>gauge</u> for $\mathcal{E}$ and λ the <u>conjugate multiplier</u>. If $\omega(t)$ is any differentiable increasing function, then $\varphi^* = \omega(\varphi)$ is another such gauge and $\lambda^* = \omega'\lambda$ the conjugate multiplier, where $\omega' = \omega'(\varphi)$ is the derivative of ω . It is always possible to choose φ so as to be convex on some compact domain. But, generally, φ will be a function just with convex levels.

The compensating ratio function is determined as the single-valued solution $\theta = \rho_{x,y}$ of the equation

$$\psi(\frac{u}{\theta}) = \psi(v) ,$$

where $\psi(u) = \varphi(\xi(u))$ and where $x = \xi(u)$, $y = \xi(v)$, first obtained as a function of u,v and then converted into a function of x,y .

Now

$$\varphi(y) - \varphi(x) = \int_1^{\rho_{x,y}} \lambda(\frac{u}{\theta}) \frac{d\theta}{\theta}$$

considering $\lambda = \lambda(x) = \lambda(\xi(u))$ as a function of u . Therefore, with λ continuous, if $\delta_{x,y} = \rho_{x,y} - 1$ is "infinitesimally small," there is the relation

$$\varphi(y) - \varphi(x) = \lambda\delta_{x,y} ,$$

but, generally, this relation by no means holds finitely.

Now let φ,λ be a gauge and conjugate multiplier for some normal completion $\mathcal{E}$ of $\mathcal{F}$. The corresponding level and multiplier sets $\Phi = \{\varphi_r\}$, $\Lambda = \{\lambda_r\}$ are determined by

$$\varphi_r = \varphi(x_r) , \lambda_r = \lambda(x_r) .$$

If $D_{rs} = u_r'x_s - 1$, it can be shown that the conditions

$$\lambda_r > 0 , \lambda_r D_{rs} > \varphi_s - \varphi_r$$

are necessary and sufficient for $\{\Phi, \Lambda\}$ to be constructable relative to some convex gauge of some normal completion of $\mathcal{F}$. For the scale $S = S_{\mathcal{F}}(\mathcal{E})$ on $\mathcal{F}$, induced relative to $\mathcal{E}$, there is the condition

$$\varphi_r > \varphi_s \iff x_r S x_s .$$

Now the completion $\mathcal{E}$ of $\mathcal{F}$ has a compensation ratio function $\rho_{x,y}$ which induces compensation ratios for $\mathcal{F}$ given by

$$\rho_{rs} = \rho_{x_r, x_s} .$$

From remarks just made, provided the figures belonging to r and s are considered close to each other, the relation

$$\delta_{rs} = \frac{\varphi_s - \varphi_r}{\lambda_r}$$

where $\delta_{rs} = \rho_{rs} - 1$, could be considered to hold correspondingly closely. But, apart from this inference, there is no way of constructing the compensating ratios from the levels and multipliers. Indeed, there is an infinity of systems $\mathcal{E}$ which complete $\mathcal{F}$, produce identical sets of levels and multipliers on $\mathcal{F}$, but different compensating ratios. The possible level and multiplier sets are constructed from the cross-structure of $\mathcal{F}$ alone; and generally there is no way of inferring compensation structure from them alone. The approximation principle only states what is, conceptually, the local relation between levels, multipliers, and compensating ratios, and it yields nothing for a discrete set of figures, such as compose any configurations obtained from observed expenditure data. In order to achieve an analysis of compensation structure for the configuration, it is necessary to go beyond information contained just in cross-structure, and take into account, unreduced, the entire information presented by the configuration itself.

4. Median levels and multipliers

The normal system of levels and multipliers (φ_r, λ_r) are determined as the solutions of the normal inequalities

$$\lambda_r > 0 \; , \; \lambda_r D_{rs} > \varphi_s - \varphi_r \; .$$

Thus it is asked that the differences $\varphi_r - \varphi_s$ belong to the intervals $[-\lambda_r D_{rs}, \lambda_s D_{sr}]$, these intervals being positive subject to the positive interval condition

$$\lambda_r D_{rs} + \lambda_s D_{sr} > 0 \; .$$

Now there is an infinite variety of solutions to the normal inequalities, which will be called normal solutions. There is the liberty to be more restrictive. It may be asked, more stringently, that these differences all lie, if possible, at the mid-points of these intervals; that is,

$$\varphi_r - \varphi_s = \frac{1}{2}(\lambda_s D_{sr} - \lambda_r D_{rs}) .$$

These will be called the median equations. Any set of levels and multipliers which satisfy them will be called a median solution. The positive multiplier and positive interval condition is necessary and sufficient for a median solution also to be a normal solution.

Thus in searching for normal solutions it is possible first to search for a median solution, and then see if the positive interval condition holds. If it does, then a normal median solution has been obtained. Such a normal solution, with the median property that the level differences lie not just anywhere in the prescribed intervals, but at their mid-points, is to be taken as a model form of solution. Any solution can be characterized through its deviation from this median model, even if no exact median solution exists. However, the more fundamental question is to know of the existence, and the variety, of median solutions.

The median consistency of a configuration may be defined through the existence of a normal median solution. It is a condition generally stronger than consistency.

Enquiry will have to be made into the possibility of finding median solutions. It will appear equivalent to the possibility of finding multipliers which obtain a condition of cycle-reversibility. It will also appear that for a configuration of four figures, a median solution always exists, and is essentially unique. For fewer than four figures, median solutions exist, with an essential indeterminacy. But with more than four, exact median solutions do not generally exist. The question which then naturally arises, of the determination of solutions which conform as exactly as possible to the median model, is tied to the more general investigation of the analysis of configurations which can even be inconsistent. Therefore, it is most fittingly excluded from this investigation, which is committed to the consistent preference hypothesis in its exact form. This hypothesis must nevertheless always be recognized as of limited application, since expenditure configurations, such as may be obtained from observed data, and in any case in principle, need not conform to this model of consistency. The consumer, as pictured, is at liberty to purchase in the most disorderly manner.

However, analysis is the investigation of order, and economic analysis is, classically, committed to the existence of that order in the form that is expressed by consistent preferences. And that hypothesis, without being given an algebraical form, involving more restrictive, but constructive models, is not tractable for algebraical treatment, such as is wanted now. The peculiar concept of a median solution, joined with the calculations which will proceed on the basis of it, is singled out by a peculiar workability, for the purpose of putting the hypothesis in a restricted algebraical form which is an effective basis for developing an algebraical method for the questions now considered.

5. Determination of medians

If (φ_r, λ_r) is a median solution, then for any r, s, t there are the relations

$$\varphi_r - \varphi_s = \frac{1}{2}(\lambda_s D_{sr} - \lambda_r D_{rs})$$
$$\varphi_s - \varphi_t = \frac{1}{2}(\lambda_t D_{ts} - \lambda_s D_{st})$$
$$\varphi_t - \varphi_r = \frac{1}{2}(\lambda_r D_{rt} - \lambda_t D_{tr}) .$$

By addition of these, the levels are eliminated, and there is obtained the relation

$$\lambda_r D_{rs} + \lambda_s D_{st} + \lambda_t D_{tr} = \lambda_r D_{rt} + \lambda_t D_{ts} + \lambda_s D_{sr}$$

for the multipliers. Let

$$S_{rst} = \lambda_r D_{rs} + \lambda_s D_{st} + \lambda_t D_{tr} ,$$

then this relation is

$$S_{rst} = S_{rts} .$$

The number S_{rst}, formed from any set of multipliers λ_r, for a configuration with given cross-deviations D_{rs}, depends just on the cyclic order of r,s,t. There are two cyclic orders for three elements. Thus when r,s,t are permuted, S_{rst} changes its value just when the cyclic order is reversed, yielding two generally distinct values corresponding to every r,s,t. The considered condition on the multiplier is that these two generally distinct values obtained should be identical.

The number S_{rst} may be called a cycle coefficient of the configuration, for a cycle of three elements r,s,t corresponding to a set of multipliers λ_r. The considered invariance of value when the cyclic order is reversed defines the condition of 3-cycle reversibility on the multipliers.

THEOREM 1. 3-cycle reversibility is a necessary and sufficient condition for multipliers to belong to a median solution; and the levels which belong with them are determined uniquely, but for an arbitrary additive constant.

The necessity has already been shown. To prove the sufficiency, suppose now that $\{\lambda_r\}$ is a set of multipliers with 3-cycle reversibility. One of the numbers φ_r to be found can be chosen arbitrarily, say φ_m. Then φ_r $(r \neq m)$ can be determined from

$$\varphi_r - \varphi_m = \frac{1}{2}(\lambda_m D_{mr} - \lambda_r D_{rm}) .$$

By subtracting this from the same expression for φ_s , and using the 3-cycle reversibility relation, it is verified that

$$\varphi_r - \varphi_s = \frac{1}{2}(\lambda_s D_{sr} - \lambda_r D_{rs}) ,$$

so a median solution $\{\varphi_r, \lambda_r\}$ has been found, to which the multipliers $\{\lambda_r\}$ belong.

It follows that in order to find median levels and multipliers there only have to be found multipliers with 3-cycle reversibility; and any such multiplier may be joined with essentially unique levels to obtain a median solution.

In order to decide if a median solution is also a normal solution, it only has to be inspected for the normal condition $\lambda_r > 0$, together with the positive interval condition

$$\lambda_r D_{rs} + \lambda_s D_{sr} > 0 ,$$

which, like the reversibility condition, applies to the multipliers alone.

Now, for a normal solution, there are the relations

$$\lambda_r D_{rs} > \varphi_s - \varphi_r$$
$$\lambda_s D_{st} > \varphi_t - \varphi_r$$
$$\lambda_t D_{tr} > \varphi_r - \varphi_t ,$$

which, by addition, give

$$S_{rst} > 0 ,$$

which may be called the <u>3-cycle positivity condition</u>. Thus the 3-cycles are not only reversible, for a normal solution, but also positive. Hence there may be added:

THEOREM 2. <u>The 3-cycles are all positive for a normal solution.</u>

The numbers

$$C_{rst} = S_{rst} - S_{tsr}$$

are antisymmetric for permutation of r,s,t preserving value when cyclic order is preserved, and changing sign when it is reversed. They may be called antisymmetric cycle coefficients. They give the statement

$$C_{rst} = 0$$

for cycle reversibility.

6. Cycle reversibility and positivity

More generally, given any elements $r,s,t,\ldots,q$ in the cyclic order determined from this order, with the first element following the last, there may be formed the cycle coefficient

$$S_{rst\ldots p} = \lambda_r D_{rs} + \lambda_s D_{st} + \ldots + \lambda_q D_{qr} ,$$

corresponding to any set of multipliers $\{\lambda_r\}$. The general condition of cycle reversibility is now defined by the condition that this coefficient on any cycle, of any number of elements, should remain unchanged when the cyclic order is reversed, thus:

$$S_{rst\ldots p} = S_{p\ldots tsr} .$$

Now any cycle can be expressed as a sum of 3-cycles and 2-cycles, and the reverse cycle is the sum of these 3-cycles and 2-cycles reversed. But there is no change in reversing a 2-cycle since there are no distinct cyclic orders for two elements. Thus, reversibility in respect to three cycles implies reversibility in respect to all cycles. Hence, to elaborate on the significance of 3-cycle reversibility:

THEOREM 1. *For general cycle reversibility, 3-cycle reversibility is sufficient.*

Now a general condition of cycle positivity is defined by

$$S_{rst\ldots p} > 0 .$$

The following is easily established:

THEOREM 2. <u>All cycles are positive provided they are reversible and all intervals are positive.</u>

For if

$$S_{rst\ldots q} = S_{q\ldots tsr} ,$$

then

$$2S_{rst\ldots p} = S_{rst\ldots q} + S_{q\ldots tsr}$$

$$= S_{rs} + S_{st} + \ldots + S_{qr} > 0 ,$$

provided intervals $S_{rs} > 0$.

7. Four-base determinacy

Let $\mathcal{F} = \{E_r\}$ be a configuration of four figures E_r , indexed by $r = \alpha,\beta,\gamma,\delta$. The cross-structure is then specified by the 4×4 array $D_{\mathcal{F}} = \{D_{rs}\}$. The cycle-reversibility condition for a set of multipliers $\{\lambda_r\}$ is given by

$$C_{rst} = 0$$

where

$$C_{rst} = \lambda_r(D_{rs} - D_{rt}) + \lambda_s(D_{st} - D_{sr}) + \lambda_t(D_{tr} - D_{ts}) = 0$$

and where r,s,t is any of the four sets of three distinct elements taken from the four elements $\alpha,\beta,\gamma,\delta$ without regard for order, since the same equation $C_{rst} = 0$ is obtained when r,s,t are permuted. Thus there are the conditions

$$C_{\beta\gamma\delta} = 0,\ C_{\alpha\delta\gamma} = 0,\ C_{\delta\alpha\beta} = 0,\ C_{\gamma\beta\alpha} = 0 ,$$

providing a simultaneous system of four homogeneous linear equations for the ratios of the four multipliers $\lambda_\alpha, \lambda_\beta, \lambda_\gamma, \lambda_\delta$. But there is the identity

$$C_{\beta\gamma\delta} + C_{\alpha\delta\gamma} + C_{\delta\alpha\beta} + C_{\gamma\beta\alpha} = 0 .$$

Therefore the equations are automatically consistent and can be solved for ratios of the λ's . Thus, in matrix form, the system is

$$\begin{pmatrix} 0 & D_{\beta\gamma} - D_{\beta\delta} & D_{\gamma\delta} - D_{\gamma\beta} & D_{\delta\beta} - D_{\delta\gamma} \\ D_{\alpha\delta} - D_{\alpha\gamma} & 0 & D_{\gamma\alpha} - D_{\gamma\delta} & D_{\delta\gamma} - D_{\delta\beta} \\ D_{\alpha\beta} - D_{\alpha\delta} & D_{\beta\delta} - D_{\beta\alpha} & 0 & D_{\delta\alpha} - D_{\delta\beta} \\ D_{\alpha\gamma} - D_{\alpha\beta} & D_{\beta\alpha} - D_{\beta\gamma} & D_{\gamma\beta} - D_{\gamma\alpha} & 0 \end{pmatrix} \begin{pmatrix} \lambda_\alpha \\ \lambda_\beta \\ \lambda_\gamma \\ \lambda_\delta \end{pmatrix} = 0 \; ;$$

and the matrix of the system is singular, since its rows sum to zero.

Moreover, in general, any three equations of the system are independent, since any three rows of the matrix of the system are, in general, independent. Therefore the matrix of the system is generally of rank 3; and the system determines uniquely the three independent ratios for the four λ's .

THEOREM 1. For a four figure configuration, multipliers which obtain cycle-reversibility always exist, and their ratios with each other are in general uniquely determined.

Now with such a set of multipliers $\{\lambda_r\}$, uniquely determined, but for multiplication by an arbitrary constant, there is determined a set of levels $\{\varphi_r\}$, uniquely determined but for multiplication and addition with arbitrary constants such that $\{\varphi_r, \lambda_r\}$ is a median solution, satisfying

$$\varphi_r - \varphi_s = \tfrac{1}{2}(\lambda_s D_{sr} - \lambda_r D_{rs}) \; , \quad (r \neq s, \; r,s = \alpha,\beta,\gamma,\delta) \; .$$

If $\{\varphi_r, \lambda_r\}$ is any such solution, then any other one is of the form $(\omega + \sigma\varphi_r, \; \sigma\lambda_r)$, obtained from it by a change of origin, defined by ω , and of scale, defined by σ . In these terms:

THEOREM 2. For a four-figure configuration, median levels and multipliers exist, and are uniquely determined but for an indeterminacy in origin and magnitude of scale.

8. Median consistency

Let $\mathcal{F}$ now be any configuration, of some k figures. It has appeared that if $k \geq 4$ there is in general no exact median solution unless $k = 4$; but if there is one, it is essentially unique. Also, if $k < 4$, then there is always a variety of such solutions.

Suppose a median solution $\{\lambda_r, \varphi_r\}$ exists. For normality, the λ's must be determined with positive ratios, so they can be chosen positive:

$$\lambda_r > 0 .$$

Then it is required further that

$$\lambda_r D_{rs} + \lambda_s D_{sr} > 0 .$$

Now define the median scale S , corresponding to that median solution, by

$$x_r S x_s \equiv \varphi_r > \varphi_s .$$

If $k \geq 4$, S is generally unique if it exists; and it always exists for $k = 4$. The normality conditions imply

$$D_{rs} \leq 0 \Rightarrow D_{sr} > 0 ,$$

which is a statement of the well-known Samuelson condition, that is a part of the Houthakker condition. From this, together with $\lambda_r > 0$, it follows that

$$D_{rs} \leq 0 \Longrightarrow \varphi_s - \varphi_r = \tfrac{1}{2}(\lambda_r D_{rs} - \lambda_s D_{sr}) < 0 ,$$

and therefore that

$$P_{\mathcal{F}} \subset S .$$

Now since S is a scale, and $P_{\mathcal{F}}$ transitive, $P_{\mathcal{F}}$ must be an order, so $\mathcal{F}$ is consistent. Thus, for any configuration $\mathcal{F}$, median consistency or the existence of a normal median solution, implies consistency. Moreover, the median scale S is then a consistent scale of the configuration, being consistent with its preference order $P_{\mathcal{F}}$.

THEOREM. Median consistency implies consistency, and that the corresponding median scale, generally complete, is a refinement of the preference order, which is generally incomplete.

Thus, for a four-figure configuration, there is always defined a scale which is in general a complete order, since the median levels are in general distinct. Moreover, with median normality, this is a refinement of the preference order. The preference order is generally a partial order; it may even be null, giving no order distinction at all, since it is possible that every $D_{rs} > 0$. But even in this case, the median scale is defined, and, in general, is a complete order, but never conflicting with the preference order even when that order is not null.

When standard-of-living is the only issue, it may be decided in a unique fashion according to the median scale, under the normality condition. But there is still not the means for decision on the cost-of-living question. To this end, there has to be investigation of the expenditure systems in which configurations that admit medians can be embedded.

9. Quadratic property of medians

Let $\mathcal{F} = \{E_r\}$ be a configuration with figures $E_r = [u_r;x_r]$ and cross-structure $D_{\mathcal{F}} = \{D_{rs}\}$, where $D_{rs} = u_r{}'x_s - 1$; and suppose it has a median with level and multiplier sets $\Phi = \{\varphi_r\}$, $\Lambda = \{\lambda_r\}$, satisfying

$$\varphi_r - \varphi_s = \frac{1}{2}(\lambda_s D_{sr} - \lambda_r D_{rs}) .$$

Associate with the point x_r the vector

$$g_r = u_r \lambda_r \qquad (r = 1,\dots,k) ,$$

so as to form a vector configuration $\{X,G\}$ with base and object sets $X = \{x_r\}$, $G = \{g_r\}$. Then, with

$$\lambda_r = x_r' g_r , \text{ since } u_r' x_r = 1 ,$$

the median equations give

$$\begin{aligned} \varphi_r - \varphi_s &= \tfrac{1}{2}(\lambda_s(u_s' x_r - 1) - \lambda_r(u_r' x_s - 1)) \\ &= \tfrac{1}{2}(g_s'(x_r - x_s) - g_r(x_s - x_r)) \\ &= \tfrac{1}{2}(x_r - x_s)'(g_r + g_s) . \end{aligned}$$

Now form the <u>skeleton</u> $\Sigma = \{X,G,\Phi\}$, which associates the scalar φ_r and the vector g_r with the point x_r , and is defined to <u>admit</u> any differentiable function φ , with gradient g , whose value and gradient at x_r is φ_r and g_r ; that is, such that

$$\varphi(x_r) = \varphi_r , \; g(x_r) = g_r .$$

Now the condition

$$\varphi_r - \varphi_s = \tfrac{1}{2}(x_r - x_s)'(g_r + g_s)$$

is necessary and sufficient for the skeleton to admit a quadratic function φ (Res. Mem. No. 20); and the condition that such a skeleton can be constructed on the expenditure configuration $\mathcal{F}$ is equivalent to the existence of multipliers which obtain cycle-reversibility.

THEOREM. <u>Let</u> u_r <u>be a vector associated with a point</u> x_r , <u>such that</u> $u_r' x_r = 1$ $(r = 1,\dots,k)$, <u>and let</u> $D_{rs} = u_r' x_s - 1$. <u>Let</u> $\varphi = \varphi(x)$ <u>be a differentiable function, with gradient</u> $g = g(x)$; <u>let</u>

$$\varphi(x_r) = \varphi_r , \; g(x_r) = g_r , \; \lambda_r = x_r' g_r ,$$

<u>and let</u>

$$\Lambda = \{\lambda_r\} \text{ and } \Phi = \{\varphi_r\} .$$

Let Λ^* denote any non-trivial solution of the equation

$$\lambda_r^* D_{rs} + \lambda_s^* D_{st} + \lambda_t^* D_{tr} = \lambda_t^* D_{ts} + \lambda_s^* D_{sr} + \lambda_r^* D_{rt}$$

and $(\hat{\Lambda}, \hat{\Phi})$ any non-trivial solution of the equations

$$\hat{\varphi}_r - \hat{\varphi}_s = \tfrac{1}{2}(\hat{\lambda}_s D_{sr} - \hat{\lambda}_r D_{rs}) .$$

(A) Solutions (Λ^*) exist if and only if solutions $(\hat{\Lambda}, \hat{\Phi})$ exist.

(A') For every solution Λ^* there exists a solution $(\hat{\Lambda}, \hat{\Phi})$ with $\hat{\Lambda} = \Lambda^*$.

(A") Every solution $(\hat{\Lambda}, \hat{\Phi})$ provides a solution Λ^* with $\Lambda^* = \hat{\Lambda}$.

(B) If $k > 4$ there are generally no solutions of these equations.

(B') If $k = 4$ there exists a solution $(\hat{\Lambda}, \hat{\Phi})$ which, generally, is linearly unique, in that any other solution is of the form

$$(\{\lambda_r \sigma\} , \{\varphi_r \sigma + \omega\}) \quad (\sigma \neq 0) .$$

(B") If $k < 4$, there exists a variety of solutions, without linear uniqueness.

(C) Solutions $(\hat{\Lambda}, \hat{\Phi})$ are identical with the (Λ, Φ) for quadratics such that

$$(1 - u_r x_r')g_r = 0 .$$

If such quadratics exist, they are generally linearly unique only for $k \geq n$.

COROLLARY. The existence of a median solution is necessary and sufficient for the existence of quadratics whose gradients at the consumption points are parallel to the price-directions. This condition is automatic only for $k \leq 4$.

COROLLARY. If a variety of quadratics are constrained to have their gradients determined in a common direction at not less than four points, then also their values and the magnitudes of their gradients are essentially determined at those points, the indeterminacy corresponding just to addition and multiplication with a constant.

If $k < n$, the property of admitting a quadratic is a porism, in that if one is admitted, then so will be an infinity, these forming a parametric family of dimension depending generally just on k and n.

10. Median closure

Let $\mathcal{F} = [U;X]$, where $U = \{u_r\}$, $X = \{x_r\}$, $(r = 0,1,\ldots,k-1)$, be an expenditure configuration, with cross-deviations $D_{rs} = u_r'x_s - 1$, admitting a median-solution with multipliers $\Lambda = \{\lambda_r\}$; and define

$$g_r = u_r\lambda_r \ .$$

To any point

$$x_\alpha = x_1\alpha_1 + \ldots + x_k\alpha_k$$

in the convex closure $\hat{X}$ of X , where

$$\alpha_1, \ \ldots \ , \alpha_k \geq 0 \ , \ \alpha_1 + \ldots + \alpha_k = 1 \ ,$$

let there correspond the vector

$$g_\alpha = g_1\alpha_1 + \ldots + g_k\alpha_k \ ;$$

and define

$$\lambda_\alpha = x_\alpha' g_\alpha \ .$$

Now, provided $\lambda_\alpha \neq 0$, take

$$u_\alpha = \frac{g_\alpha}{\lambda_\alpha}$$

as a vector corresponding to x_α , such that

$$u_\alpha' x_\alpha = 1 \ .$$

Provided $X = \{x_r\}$, $G = \{g_r\}$ are simplicial, and every $\lambda_r \neq 0$, the correspondence

$$x_\alpha \longrightarrow u_\alpha$$

defining u as a function

$$u(x) = u_\alpha \qquad (x = x_\alpha)$$

of x on the convex closure $\hat{X}$ of X , is invertible. There is obtained the continuous configuration whose figures are $[u_\alpha;x_\alpha]$ include those of $\mathcal{F}$, and whose base set $\{x_\alpha\}$ is the convex closure of the base set $\{x_r\}$ of $\mathcal{F}$. This continuous configuration may be denoted by $\bar{\mathcal{F}}$, and called the median closure of $\mathfrak{F}$.

It is easy to see the following:

THEOREM. If $\mathfrak{F}$ is a median configuration, then so is every configuration $\mathfrak{F}^* \subset \bar{\mathfrak{F}}$ in its median closure, and $\bar{\mathfrak{F}}^* \subset \bar{\mathfrak{F}}$.

In other words, the median closure of a median configuration is always a continuation of the median closure of any configuration in its own median closure.

Proceeding further, the median condition is necessary and sufficient for the existence of a vector c and a symmetric matrix B such that

$$x_r - c = Bg_r \ .$$

In this case, also

$$\begin{aligned} x_\alpha - c &= Bg_\alpha \\ &= Bu_\alpha \lambda_\alpha \ . \end{aligned}$$

Consider the correspondence $x = x(u)$ defined by

$$x - c = Bu\lambda \ ,$$

where

$$\lambda = \frac{1-u'c}{u'Bu} \ , \text{ so that } \ u'x = 1 \ .$$

It gives

$$x(u) = x_\alpha \quad (u = u_\alpha) \ ,$$

so that it extends the median closure $\bar{\mathfrak{F}}$ of $\mathfrak{F}$. It is continuous, so that if $u_\alpha \supset 0$, there exists a neighborhood N of X such that $u(x) \supset 0$ $(x \in N)$. But, given $x_r \supset 0$, $g_r \supset 0$, and necessary and sufficient condition for this is $\lambda_r > 0$. In this case, there is obtained an expenditure system $x(u)$, determined by the vector c and matrix B . Provided the matrix $\{(x_r-x_o)'(g_r-g_o)\}$ is regular, the matrix B can be chosen regular. Then the system can be inverted to give $u(x) \supset 0$ for every $x \in N$. And since B is symmetric, this system is integrable. For, with $A = B^{-1}$

$$\lambda u' dx = d\varphi \quad , \quad \text{where} \quad \varphi = \frac{1}{2}(x-c)' A (x-c) \; .$$

Moreover, every such system must contain $\bar{\mathfrak{F}}$. Accordingly:

THEOREM. *If* $\mathfrak{F} = [U;X]$ *is a median expenditure configuration, with base and object set* $X = \{x_r\}$ *and* $U = \{u_r\}$, *and multipliers* $\Lambda = \{\lambda_r\}$ $(r = 0,1,\ldots,k-1)$, *and if* $g_r = u_r \lambda_r$, *then, if and only if the matrix* $\{(x_r - x_o)'(g_r - g_o)\}$ *is regular, and* $\lambda_r > 0$, *there exist invertible and integrable expenditure systems, whose integrals are quadratic functions, in which* $\mathfrak{F}$ *can be embedded; and all these extend the median closure* $\bar{\mathfrak{F}}$ *of* $\mathfrak{F}$.

11. *Quadratic consistency*

The question to ask now is whether among such quadratics there are any that are convex in, and are therefore normal preference functions over the convex closure of X , and therefore determining normal expenditure systems in which $\mathfrak{F}$ can be embedded.

The normality conditions for the multipliers in a median solution are

$$\lambda_r > 0 \; , \; \lambda_s D_{sr} + \lambda_r D_{rs} > 0 \; ;$$

and since $u_r \supset 0$, prices and expenditure being positive, these become the conditions

$$g_r \supset 0 \; , \; (x_r - x_s)'(g_r - g_s) < 0 \; .$$

The condition $g_r \supset 0$ is necessary and sufficient for any quadratic φ , with gradient g_r at x_r , to be monotone increasing in the convex closure X of $X = \{x_r\}$. Assuming multipliers which obtain reversibility, the condition $(x_r - x_s)'(g_r - g_s) < 0$ is necessary and sufficient for the existence of a convex function φ with gradient g_r at x_r ;

but it is not sufficient for the existence of such a function which is also quadratic. It has appeared that there exist such functions which are quadratic but not convex. Now a necessary and sufficient condition for the existence of a convex quadratic among these is given by the stronger condition

$$(x_\alpha - x_\beta)'(g_\alpha - g_\beta) < 0$$

for all distributions α, β (Res. Mem. No. 20); and this is equivalent to the negative definiteness of the matrix $G_o'X_o$, square and of order $k-1$, where

$$G_o = \{g_r - g_o\}, \quad X_o = \{x_r - x_o\} \qquad (r = 1, \ldots, k-1)$$

are matrices of order $n \times (k-1)$ (Res. Mem. No. 20).

THEOREM 1. <u>If</u> $\mathcal{F}$ <u>is any expenditure configuration with</u> k <u>figures</u> $[u_r; x_r]$ $(r = 0, \ldots, k-1)$, <u>if</u> $g_r = u_r \lambda_r$ <u>for any multipliers</u> λ_r, <u>and if</u>

$$G_o = \{g_r - g_o\}, \quad X_o = \{x_r - x_o\}$$

<u>then the multipliers obtain cycle-reversibility if and only if the matrix</u> $G_o'X_o$ <u>is symmetric. Under this condition, there exist quadratic functions</u> φ <u>with gradient</u> g_r <u>at</u> x_r. <u>A necessary and sufficient condition for these quadratics to be monotone increasing in the convex closure of the points</u> x_r <u>is that</u> $\lambda_r > 0$.

Let the <u>quadratic consistency</u> of the configuration $\mathcal{F}$ be defined by the existence of a quadratic function which is a normal preference function in the neighbourhood of the convex closure of its base points x_r and which is compatible with it.

COROLLARY. <u>The quadratic consistency of</u> $\mathcal{F}$ <u>is equivalent to the existence of multipliers</u> $\lambda_r > 0$ <u>such that</u> $G_o'X_o$ <u>is symmetric and negative definite.</u>

The three forms of consistency which have been defined for an expenditure configuration have the following relations:

THEOREM 2. The conditions of consistency, median consistency, and quadratic consistency for an expenditure configuration are increasingly restrictive, each being implied by the following one, and no pair being equivalent.

12. Quadratic maxima

A convex quadratic φ is specified by its centre c , which is the unique point at which its gradient vanishes, by its initial value M , which is the value it takes at its centre, and by a negative definite characteristic matrix A , which give it in the form

$$\varphi = M + \frac{1}{2}(x-c)'A(x-c) \ .$$

Thus, the gradient is

$$g = A(x-c) \ ,$$

and it is seen that

$$g(c) = 0 \ , \ \varphi(c) = M \ .$$

Also, with A negative definite

$$\varphi(x) \leq M \ ,$$

with equality if and only if $x = c$. So M is the absolute maximum of φ , attained at its centre c .

If $\mathfrak{F}$ is a quadratically consistent configuration of no less than four figures, then, but for a positive factor of proportionality, there exists an essentially unique set of multipliers $\lambda_r > 0$ such that $g_r = u_r\lambda_r$ determines the gradient at x_r for all the admissible quadratic preference functions. It should be noted that quadratic consistency is possible, but with a zero measure of likelihood, for more than four figures, and with a positive measure of likelihood for no more than four.

This multiplier set $\Lambda = \{\lambda_r\}$ may be joined with a level set $\Phi = \{\varphi_r\}$, unique but for an additive constant, so as to obtain a median solution $\{\Lambda, \Phi\}$. Then φ_r determines the value at x_r of all these functions, but for multiplication and addition with constants.

Thus every admissible quadratic preference function φ, suitably normalized by multiplication and addition with constants, is such that

$$\varphi(x_r) = \varphi_r , \quad g(x_r) = g_r ,$$

where g is the gradient of φ. One may ask of these quadratics, thus normalized, what is the locus of their centres, and what is the range of their initial values.

The answer (Res. Mem. No. 20) is that their centres c must satisfy the condition

$$G_o'x_o - X_o'g_o = G_o'c ,$$

together with the condition

$$g_o'(c-\hat{c}) > 0 ,$$

where

$$\hat{c} = x_o - X_o(G_o'X_o)^{-1}X_og_o .$$

Now, further, it appears that the initial, or maximum value M of any such quadratic is determined by a function $M = M(c)$ of its centre, given by

$$M(c) = \varphi_o - \frac{1}{2}(x_o-c)'g_o .$$

Let

$$\hat{M} = M(\hat{c}) ,$$

so that

$$\hat{M} = \varphi_o + \frac{1}{2}g_o'X_o(G_o'X_o)^{-1}X_o'g_o .$$

Then, from the inequality which applies to c, there follows the inequality

$$M > \hat{M}$$

specifying the range of M, which thus can be any finite number greater than $\hat{M}$. From the negative definiteness of $G_o'X_o$ it follows that

$$\hat{M} > \varphi_r$$

for every $r = 0,1,\ldots,k-1$.

THEOREM. _If they exist, the maxima of the convex quadratics_ φ, _with gradient_ g, _constrained by the conditions_

$$\varphi(x_r) = \varphi_r \,, \; g(x_r) = g_r \quad (r = 0,1,\ldots,k-1) \,,$$

describe all values greater than

$$\varphi_o + \tfrac{1}{2}g_o'X_o(G_o'X_o)^{-1}X_o'g_o$$

where

$$X_o = \{x_r - x_o\} \,, \; G_o = \{g_r - g_o\} \,;$$

and a necessary and sufficient condition for the existence of such quadratics, given the vectors g_r, _and for some values_ φ_r, _is that_ $G_o'X_o$ _be symmetric and negative definite, and, given this condition, then that_

$$\varphi_r - \varphi_o = \tfrac{1}{2}(x_r - x_o)'(g_r + g_o) \,.$$

It is noted that one element x_o is here distinguished from the k elements x_r $(r = 0,1,\ldots,k-1)$. But the distinction is arbitrary; and nothing is changed if the elements are permuted, so that x_o is replaced by any other element x_r. Thus, given the symmetry of $G_o'X_o$, it follows from

$$\varphi_r - \varphi_o = \tfrac{1}{2}(x_r - x_o)'(g_r + g_o)$$

that

$$\varphi_r - \varphi_s = \tfrac{1}{2}(x_r - x_s)'(g_r + g_s) \,;$$

and from this alone there follows again the symmetry of $G_o'X_o$, or of the same matrix with any interchange of elements, which may leave any x_r in place of x_o . Similarly, for the expression giving the lower bound of the maxima, the distinction of the 0-element from the rest is inessential.

13. Linear price systems

The expenditure needed to purchase any amount ξ of a simple commodity at a given market price π is determined by the market equation

$$\epsilon = \pi\xi \ .$$

Thus, with π fixed, there follow the relations

$$\frac{\xi}{\epsilon} = \frac{1}{\pi} , \ \frac{\partial\xi}{\partial\epsilon} = \frac{1}{\pi} ,$$

which shows that, in the nature of the market mechanism, with a fixed price for every unit, irrespective of the number of units, there is no distinction between average price, marginal price and market price for a simple commodity. However, when generalization is made for a composite commodity by means of a normal preference function which is to measure a level for a composite amount, these price concepts become distinct from each other.

Relative to a normal preference function φ for a composite commodity, there is obtained a level

$$X = \varphi(x)$$

for any composition x . Given the market price vector p and the expenditure e , the composition x is not directly determined just by the market constraint

$$p'x = e \ ,$$

as in the simple case, but as the unique equilibrium, in respect to the normal preference system under that constraint. It is determined, accordingly, by the condition

$$g = u\lambda \ ,$$

where g is the gradient of φ , $u = \frac{p}{e}$, and $\lambda = x'g$. This condition determines x as a function of $u = \frac{p}{e}$, and, correspondingly, X as a function of $\frac{p}{e}$. Then the partial derivative of X with respect to e with p fixed is defined as given by

$$\frac{\partial X}{\partial e} = \frac{1}{P} ,$$

where

$$P = \frac{\lambda}{e}$$

defines the marginal price, for attaining a level X , as measured by the gauge φ , when the market prices are p . A small increment dX in the level X attained at price p is achieved by an increment de in the expenditure e , given by

$$de = PdX .$$

But it cannot generally be asserted that

$$f - e = P(Y-X)$$

where, with prices fixed at p , X and Y are the levels of the composition attained by expenditures e and f ; in other words, that the marginal price P is independent of expenditure e , and is purely a function of market price p . However, should this be the case, and with prices fixed, the level attained will be a linear function of the expenditure. In this case, therefore, a linear price system may be said to operate. When the prices p are fixed, and P is known, the change in expenditure to achieve a given change in level is then decided, and is independent of the level at which the change is made. This is in direct analogy with the market equation for a simple commodity, when the simple physical amount is used as the measure of level.

While for a simple commodity there can, quite trivially, be only one expenditure system; and, for this expenditure system, while a variety of gauges are associated with it, the one defined by physical amount is singled out in a most obvious way, there is quite a different situation for composite commodities. Firstly, a particular normal expenditure system has to be involved; and, without empirical or other external guidance, there is a variety of undifferentiated possibilities. Then, even when there is a commitment to a particular normal expenditure system, the possible gauges, though equivalent to each other, as belonging to the same system, again form a variety. Should one gauge happen to determine a linear price system, the others which are not linear functions of it will not. So, given a normal expenditure system, it may be asked whether or not it possesses a gauge which provides a linear price system, this being of particular importance in the method for approaching the problem being considered. It will appear that should a system have a quadratic gauge, then it is possible to find another gauge which is not quadratic, but which has the linear price property. An importance for this is that any four-figure configuration can be embedded in a system with a quadratic gauge. In fact, this can be done in an infinity of ways. By analysis of this construction, it will be possible to elaborate further on the problem of the construction of a cost-of-living index.

14. Generalized market equation

Consider a system in which any composition x, in a compact convex region C_o, is assigned a level $\varphi(x)$ by a quadratic function φ, which, in C_o, is strictly increasing, and has convex level surfaces. But a quadratic with convex levels in a neighbourhood must be a convex function. Thus φ must be convex, and hence of the form

$$\varphi = M + \frac{1}{2}(x-c)'A(x-c) ,$$

where A is negative definite, and therefore with inverse $B = A^{-1}$. The gradient of φ is

$$g = A(x-c) .$$

The associated expenditure system is determined by the equilibrium condition

$$g = u\lambda ,$$

where $\lambda = x'g$ since $u'x = 1$. Hence the system is

$$A(x-c) = u\lambda , \text{ where } \lambda = x'A(x-c) .$$

Inversely,

$$x - c = Bu\lambda , \text{ where } \lambda = \frac{1-u'c}{u'Bu} .$$

Thus u is determined as a function of x ; which inverts to give x as a function of u . Moreover, it follows that

$$(x-c)'A(x-c) = (1-u'c)\lambda ,$$

whence

$$\frac{2(\varphi-M)}{\lambda} = 1 - u'c ,$$

and

$$\frac{2(\varphi-M)}{\lambda^2} = u'Bu .$$

It appears that if $\psi(u) = \varphi(x)$ is the function derived from φ by substituting x as a function of u , then

$$\psi(u) = M + \frac{1}{2}\frac{(1-u'c)^2}{u'Bu} .$$

It is noted that, substituting $u = \frac{p}{e}$,

$$\frac{\lambda}{e} = \frac{e-p'c}{p'Bp} ,$$

and this is not purely a function of p , but depends also on e ; so the linear price condition is not satisfied.

However, if, instead of the quadratic, there is taken the negative square root of the negative of its principal part as a measure of level, representing an equivalent scale, then it appears that the linear price condition is satisfied.

For, define

$$X = -\{-(x-c)'A(x-c)\}^{\frac{1}{2}} ,$$

so that

$$X^2 = -2(\varphi-M) ;$$

and let $\frac{1}{U} = \dot{X}$ denote the corresponding multiplier, where $\dot{X} = e\frac{\partial X}{\partial e}$. Then the operation $e\frac{\partial}{\partial e}$, where $u = \frac{p}{e}$, gives

$$2X\dot{X} = -2\dot{\varphi} ;$$

and, with $\lambda = \dot{\varphi}$, this gives

$$\frac{X}{U} = -\lambda .$$

Therefore

$$U^2 = \frac{X^2}{\lambda^2} = -\frac{2(\varphi-M)}{\lambda^2}$$

$$= -u'Bu .$$

Accordingly, with

$$u = \frac{p}{e} , \quad U = \frac{P}{e} ,$$

it appears that

$$P^2 = -p'Bp .$$

Thus P is independent of e , and is purely a function of p . Therefore the linear price condition is satisfied.

With p given, the level attained by any expenditure e is given by

$$X = -\{-2(\varphi-M)\}^{\frac{1}{2}}$$

$$= -\{-\frac{(1-u'c)}{u'Bu}\}^{\frac{1}{2}}$$

$$= -(1-u'c)(-u'Bu)^{-\frac{1}{2}} .$$

Therefore, with $u = \frac{p}{e}$, and $P = -(-p'Bp)^{\frac{1}{2}}$

$$X = -(e-p'c)(-p'Bp)^{-\frac{1}{2}}$$

$$= \frac{e-p'c}{P} .$$

Thus there is established the relation

$$PX = e - p'c ,$$

in generalization of the form of the simple market equation

$$\pi\xi = \epsilon .$$

With p fixed, and therefore P fixed, the change in level ΔX corresponding to a change of expenditure Δe satisfies

$$P\Delta X = \Delta e ,$$

in perfect generalization, preserving exactly the form, of the corresponding relation

$$\pi\Delta\xi = \Delta\epsilon ,$$

obtained from the simple market equation.

It is noted that ΔX depends only on Δe , and not on X , or, equivalently, e . Thus it can be said that, whatever the initial level, the change in expenditure required to bring about a given change in level depends only on the prices. The changes in expenditure and level are proportional, and the prices enter through their ratio.

Thus, with such a system applying to a composite commodity, there is obtained a perfect generalization of the market system for a simple commodity, The price index P , a function of the price vector p , has exactly the role of the simple market price π for a simple commodity, where, instead of purchase being for a direct simple physical amount, it is for a level associated with a composition of several physical amounts.

Now suppose on two occasions o,1 the prices are p_o, p_1 and the expenditures are e_o, e_1 , and it is known that the indices of price and level attained according to such a system are P_o, P_1 and X_o,X_1 . Then it is possible to calculate the expenditure ϵ_{o1} needed to attain, at the prices of occasion o , the level of occasion 1 . Thus

$$\epsilon_{o1} - e_o = P_o(X_1 - X_o) ,$$

Hence

$$\rho_{o1} = 1 + U_o(X_1 - X_o) ,$$

where

$$\rho_{o1} = \frac{\epsilon_{o1}}{e_o} , \quad U_o = \frac{P_o}{e_o} .$$

The number ρ_{o1} is the ratio in which expenditure in occasion o has to be adjusted in order that, with prices remaining the same, the level of occasion 1 is attained. It thus measures the cost-of-living with 1 and o as base and object occasions.

15. Ranging the cost-of-living

The position is now clear for developing an algebraical formula for ranging the cost of living. It involves the expenditure data for four occasions. It assigns limits to the cost of maintaining, at the prices of any one of the four occasions, the standard of living attained in any other one.

Together with the algebraical formula, there is a criterion of quadratic consistency given by certain algebraic inequalities, by which it is possible to extrapolate the expenditures on a normal quadratic model. Since the criterion is given by algebraic inequalities, there is always a positive measure of likelihood that it will be satisfied.

When it is satisfied, there always exists an infinity of such quadratic models extrapolating the data. Each of these will give a cost-of-living determination; and the totality of such determinations is identical with the open interval obtained from the formula.

Now, quadratic consistency implies consistency; and, given consistency, there is a larger class of normal preference systems, including the quadratic class, which range the cost of living in a larger, absolute interval, containing the interval corresponding to the quadratic class. Thus the algebraic formula, given quadratic consistency, can be considered as ranging the cost of living within the absolute interval. The absolute interval cannot be calculated by algebraical means, but requires combinatorial methods, which present a much more elaborate computation problem.

Thus, though the absolute interval, which is defined under the condition of consistency, cannot be determined algebraically, under the stronger conditions of quadratic consistency it is possible to calculate a sub-interval of it by algebraical means.

ALGORITHM. Let $\mathcal{F} = \{E_r\}$ be an expenditure configuration of four figures $E_r = [u_r;x_r]$ $(r = 0,1,2,3)$, with cross deviations determined by

$$D_{rs} = u_r{}'x_s - 1 .$$

Let $\Lambda = \{\lambda_r\}$ be four multipliers whose three independent ratios are determined from the cycle-reversibility equation

$$C_{o12} = 0 , \; C_{o23} = 0 , \; C_{o31} = 0$$

where

$$C_{ors} = (\lambda_o D_{or} + \lambda_r D_{rs} + \lambda_s D_{so}) - (\lambda_s D_{sr} + \lambda_r D_{ro} + \lambda_o D_{os}) .$$

Then let $\Phi = \{\varphi_r\}$ be four levels whose intervals are determined from the median equation

$$\varphi_r - \varphi_o = \tfrac{1}{2}(\lambda_o D_{or} - \lambda_r D_{ro}) .$$

These multipliers and levels are uniquely determined by arbitrarily taking

$$\lambda_o = 1 , \; \varphi_o = 0 .$$

Now let

$$g_r = u_r \lambda_r$$

and form the matrices

$$X_o = \{x_r - x_o\} \ , \ G_o = \{g_r - g_o\}$$

of order $n \times 3$. The 3×3 matrix $X_o{}'G_o$ should be symmetric, by consequence of the conditions determining the numbers λ_r.

The criterion for quadratic consistency is $\lambda_1, \lambda_2, \lambda_3 > 0$, and that $X_o{}'G_o$ be negative definite.

Calculate

$$\tilde{\delta}_{rs} = \frac{\varphi_s - \varphi_r}{\lambda_r} .$$

Now calculate

$$\hat{M} = \varphi_o + \tfrac{1}{2} g_o{}' X_o (X_o{}' G_o)^{-1} X_o{}' g_o .$$

Quadratic consistency provided, it is necessary that

$$\varphi_r < \hat{M} ,$$

so it is possible to calculate

$$\hat{X}_r = -\{-2(\varphi_r - \hat{M})\}^{\frac{1}{2}} ,$$

$$\hat{U}_r = -\frac{\hat{X}_r}{\lambda_r} ,$$

and then

$$\hat{\delta}_{rs} = U_r (X_s - X_r) .$$

With quadratic consistency given, it should be found automatically that

$$\tilde{\delta}_{rs} < \hat{\delta}_{rs} < D_{rs} .$$

Assuming $n > 4$, quadratic consistency implies the existence of an infinity of normal completions of the configurations which belong to quadratic preference functions. Any one of these gives a determination of the fractional change δ_{rs} in the expenditure of occasion r which exactly compensates for the price-change from occasion s. The totality of these determinations describes the open interval

$$\tilde{\delta}_{rs} < \delta_{rs} < \hat{\delta}_{rs} .$$

The explanation of this algorithm is as follows. Let Λ , Φ be determined according to the algorithm, and suppose the quadratic consistency criterion is satisfied. Then it is known that there exists an infinite class of convex quadratic functions φ , with gradient g , such that

$$g(x_r) = g_r \; , \; \varphi(x_r) = \varphi_r \; .$$

All these functions determine normal completions of $\mathcal{F}$, and any such quadratic, which determines a normal completion, can be normalized, by addition and multiplication with a constant, so as to satisfy these conditions.

Also it is known that corresponding to any number $M > \hat{M}$, there exists a subclass of these quadratics with M as maximum value, thus:

$$\varphi = M + \frac{1}{2}(x-c)'A(x-c)$$

where c is some vector and A some negative definite matrix. If

$$X = -\{-(x-c)'A(x-c)\}^{\frac{1}{2}}$$

it follows that

$$X_r = -\{-2(\varphi_r - M)\}^{\frac{1}{2}}$$

and that

$$U_r = -\frac{X_r}{\lambda_r} \; .$$

But it is known that

$$\delta_{rs} = U_r(X_s - X_r) \; .$$

The admissible class of quadratics is thus resolved into subclasses corresponding to every $M > \hat{M}$. All the members in these subclasses induce the same determination of δ_{rs} , which is evaluated by these formulae as a function $\delta_{rs} = \delta_{rs}(M)$ of M .

Now, more explicitly,

$$\delta_{rs} = \frac{2M - X_r X_s}{\lambda_r} - \frac{2\varphi_r}{\lambda_r} \; ;$$

and this varies with M , according to the variation of

$$2\,M - X_r X_s \; ,$$

or according to the variation of

$$M - \{(M-\varphi_r)(M-\varphi_s)\}^{\frac{1}{2}} .$$

By differentiation with respect to M, there is obtained the expression

$$\frac{\{(M-\varphi_r)(M-\varphi_s)\}^{\frac{1}{2}} - \frac{1}{2}\{(M-\varphi_r) + (M-\varphi_s)\}}{(M-\varphi_r)(M-\varphi_s)}$$

The numerator is the defect of the arithmetic mean of two positive numbers $M - \varphi_r$, $M - \varphi_s$ from their arithmetic means. But this is always non-positive, and is zero if and only if the numbers are equal, by the general theorem on the relation between the arithmetic and geometric mean. It follows that if $\varphi_r \neq \varphi_s$, the considered expression, and therefore also δ_{rs}, is a strictly decreasing function of M. It is next asked if the function is bounded, and, if bounded, what is its limit. But

$$M - \{(M-\varphi_r)(M-\varphi_s)\} = \frac{(\varphi_r + \varphi_s) - \frac{\varphi_r\varphi_s}{M}}{1 + \{(1-\frac{\varphi_r}{M})(1-\frac{\varphi_s}{M})\}^{\frac{1}{2}}}$$

$$\longrightarrow \frac{\varphi_r + \varphi_s}{2} \qquad (M \longrightarrow \infty) .$$

Therefore

$$\delta_{rs} \longrightarrow \frac{\varphi_r + \varphi_s}{\lambda_r} - \frac{2\varphi_r}{\lambda_r}$$

$$= \frac{\varphi_s - \varphi_r}{\lambda_r} .$$

Thus, with the definition

$$\tilde{\delta}_{rs} = \frac{\varphi_s - \varphi_r}{\lambda_r} ,$$

there is the proposition

$$\delta_{rs} \downarrow , \quad \delta_{rs} \longrightarrow \tilde{\delta}_{rs} \qquad (M \longrightarrow \infty) .$$

Thus as M increases indefinitely, all the calculable numbers δ_{rs} decrease simultaneously, approaching indefinitely closely, but never finally attaining, their calculable limits $\tilde{\delta}_{rs}$.

Let $\hat{\delta}_{rs}$ denote the upper limit of the number δ_{rs} for the admissible $M > \hat{M}$. By continuity, this is evaluated simply by setting $M = \hat{M}$. Then

$$\delta_{rs} < \hat{\delta}_{rs} \quad (M > \hat{M}) , \quad \delta_{rs} \longrightarrow \hat{\delta}_{rs} \quad (M \longrightarrow \hat{M}) .$$

Thus it may be asserted that

$$\hat{\delta}_{rs} > \delta_{rs} > \tilde{\delta}_{rs} \quad (\hat{M} < M < \infty) ,$$

with limits of δ and limits of M taken correspondingly. Moreover, it is known that

$$\delta_{rs} < D_{rs} \quad (M > \hat{M}) .$$

Hence it follows that

$$\hat{\delta}_{rs} \leq D_{rs} .$$

In general,

$$\hat{\delta}_{rs} < D_{rs} ,$$

provided $p_r \nparallel p_s \quad (r \neq s)$.

Now the following can be asserted:

THEOREM. Any expenditure configuration of four figures can always be embedded in a class of integrable expenditure systems belonging to quadratic functions, provided the median multipliers λ_r are positive. That the symmetric matrix $G_o{}'X_o$ be negative definite is necessary and sufficient for the existence among these of a class of normal expenditure systems, belonging to convex quadratic functions, which induce sets of compensating fractional expenditure adjustments δ_{rs} lying in an open interval whose limits $\hat{\delta}_{rs}$, $\tilde{\delta}_{rs}$ are determined from the algorithm. The induced numbers δ_{rs} form a one-parametric family of arrays $\{\delta_{rs}(M)\}$ $(\hat{M} < M < \infty)$, such that

$$\delta_{rs}(M) < \delta_{rs}(N) \quad (\hat{M} < M < N)$$

and

$$\lim_{M \to \hat{M}} \delta_{rs}(M) = \hat{\delta}_{rs} \ , \ \lim_{M \to \infty} \delta_{rs}(M) = \tilde{\delta}_{rs} \ .$$

It also appears from the expression for δ_{rs} that

$$\varphi_r - \varphi_s = \frac{1}{2}(\lambda_s \delta_{sr} - \lambda_r \delta_{rs}) \ ,$$

is an identity in M. Thus, remarkably, the λ's and φ's are determined from the D's so as to satisfy the relation

$$\varphi_r - \varphi_s = \frac{1}{2}(\lambda_s D_{sr} - \lambda_r D_{rs}) \ .$$

Then the λ's and φ's together with a value of M go to determine the δ's, which turn out then, together with the λ's and φ's, to satisfy the relation in which they take on the role of the D's. An importance might well be found for this identity in regard to the process by which, instead of proceeding directly with measurements for a group of commodities, that group is first formed into subgroups, and then the results of measurement for these subgroups taken independently are combined, as if the measures for these subgroups of commodities were to belong to single commodities. Such a composite process is of obvious practical interest when there is a large and complicated variety of commodities.

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