ECONOMIC EQUILIBRIUM AND STEADY-STATE GROWTH

WITH INTERGENERATIONALLY-DEPENDENT PREFERENCES

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1. INTRODUCTION

One of the more interesting extensions of Arrow-Debreu general equilibrium theory concerns the role of externalities. Considerable effort has been devoted to refining the conditions for existence of equilibria with externalities, and to establishing the relationship between equilibria and optima. This paper is concerned with the characterization of equilibrium growth paths in a decentralized economy in which externalities arise from a particular form of interrelated preferences; namely, where each generation cares about the welfare of other generations. Behaviour under these circumstances raises important questions for macroeconomic analysis, but, while the general issue has been raised frequently in the literature, there has been little formal analysis of the problem. The main objective of this paper is to show that with certain restrictions on the nature of the equilibrium, the economy with intergenerationally-dependent preferences can be formulated in a way that leads to an economically interpretable steady state.

The literature on intergenerationally-dependent preferences has developed in two strands. One approach, following Becker (1974), formalizes the concern about other generations by "nested" utility functions: for example, if an individual in generation t cares about the welfare of his immediate descendants, their maximum attainable utility will appear as arguments in his utility function. This approach has been followed by Barro (1974), and more recently by Drazen (1978) and Buiter (1979) in examining the question of whether or not government bonds are perceived as net wealth. An alternative approach that builds on the analysis of consistent plans by Strotz (1956) and Pollack (1968) includes the consumption levels of the relevant individuals in the decision-maker's utility function. This approach is used by Goldman (1979) to examine the question of consistent plans with interdependent preferences. Under certain conditions[1]

the two approaches can be shown to be equivalent. The choice of one or the other should therefore be dictated by the needs of the problem. For the purpose of characterization the nested utility function approach is more convenient and for this reason I will consider only models of this type.

The paper is divided into two main sections. Section II presents the analysis of behaviour when the individual cares about the welfare of only one other generation -- either his immediate forebearer[2] or his immediate desdendant. I will refer to this as "one-sided." Section III considers the more difficult but more interesting "two-sided" case in which the individual cares about both his immediate forebearer and descendant. For the analysis to be tractable, the two-sided case requires some restrictions on how the individual views the behaviour of his forebearer and descendant. These restrictions are formalized in the notion of an "intertemporally asymmetric Nash economy." The interpretation of the steady-state equilibrium under these restrictions is shown to be a logical extension of the one-sided equilibrium. Section IV is a summary of the paper.

II. GIFT AND BEQUEST BEHAVIOUR: THE ONE-SIDED CASE

The first problem that must be faced in setting up an analysis with interdependent preferences is the choice of an appropriate concept of equilibrium. The possibilities range from a cooperative equilibrium where the agents effectively gather and discuss their joint behaviour prior to the start of the economy, to a competitive solution where the individual ignores the effects of his behaviour (including transfers) on other generations and simply imputes to himself some undefined "utility from giving" as a motive for making transfers. The latter is inconsistent with the nested utility function specification. The former, while it is of some interest as a type of "super-rational" solution,

defeats the purpose of decentralizing the exchange process.[3]

Between these two extremes, there are a number of possibilities. The concept that I will use here is essentially a Nash equilibrium in that the individual in period t has a perception of how other generations will respond to his behaviour, and he incorporates the response into his decision-making process. In the one-sided case, there are no behavioural feedbacks between generations. Because of this, it is not necessary to impose any restrictions on either the perceptions that the individual forms or the way in which he uses his perceptions. In the two-sided case examined in Section III some restrictions will be required.

The economy consists of overlapping generations of identical individuals[4], each of whom lives for two periods (denoted by superscripts 1 and 2). Saving takes the form of accumulating real capital. This enables individuals to allocate consumption intertemporally given the constraint that they work in the first period and retire in the second. All individuals have perfect foresight with respect to aggregate parameters. Firms produce a single, non-depreciating output according to a constant-returns-to-scale production function, with labor and capital as inputs. The output can be either consumed or used in production (as capital). Population grows at an exogenously given rate, the economy is fully decentralized, and all markets are competitive.

II(a)   The Individual's Choice Problem: Bequests

Consider the case in which the individual cares only about his descendant. This concern is modeled by including the maximum utility attainable by his descendant as an argument in the individual's utility function. The individual is able to express his concern for the descendant and to influence his behaviour through bequests. For analytical convenience, I will assume the individual's utility function to be additively separable with respect to his descendant's

utility.[5] The utility function for a typical member of generation t will thus be of the form

(1) $U_t = V(c_t^1, c_t^2) + U_{t+1}^*/(1 + \delta)$

where c^1 and c^2 are first- and second-period consumption, U_{t+1}^* is the descendant's indirect utility function as perceived by the individual, and δ is his interpersonal discount rate. The utility function is assumed to be strictly quasi-concave in c^1 and c^2, with $V_1(0,c^2) = V_2(c^1,0) = \infty$, and $V_1(\infty,c^2) = V_2(c^1,\infty) = 0$. Individuals are indexed by the period in which they are born,[6] and the time subscript t will be dropped whenever there is no ambiguity.

The individual's budget constraint is

(2) $c^1 + a \leq w$

$c^2 + (1 + n)b_t \leq (1 + r_{t+1})(a + b_{t-1})$

where w is wage income, a is holdings of real capital, r_{t+1} is the (perfectly foreseen) one-period real rate of return on capital purchased in period t, b_{t-1} is the bequest left by his forebearer (as the bequest is allocated in one period and received in the next, I assume that it is invested in real capital for the interim period), and b_t is the per capita bequest that he leaves for his descendant.

The first-order conditions for a utility maximum of (1) subject to (2) are[7]

(3) $V_1/V_2 = 1 + r_{t+1}$

(4) $(1 + n)V_2 \geq \dfrac{\partial U_t}{\partial b_t}$, if $>$ then $b_t = 0$

where $V_{i,t+j}$ will be the general notation for the derivative of $V(\cdot)$ with

respect to its ith argument for a member of generation $t+j$. The Nash equilibrium concept and the individual's perceptions of how his descendant will behave become operative in evaluating the marginal utility from a bequest $\partial U_t/\partial b_t$. In the one-sided case, we can assume that the individual has full knowledge of his descendant's behaviour. One unit of bequests provides his descendant with $(1 + r_{t+2})b_t$ of second-period endowment. The descendant can consume this extra endowment as c_{t+1}^1 or c_{t+1}^2, leave it as a bequest to his heir, or some combination of the three.

Lemma 1: The marginal utility that t expects $t+1$ to derive from a unit of bequests from t to $t+1$ can be expressed entirely in terms of $t+1$'s marginal utility from second-period consumption as

$$\frac{\partial U_{t+1}^*}{\partial b_t} = (1 + r_{t+2})V_{2,t+1}$$

Proof: By definition, $t+1$'s utility gain from db_t is the sum of the utilities that he derives from his disposal of the proceeds:

$$\frac{\partial U_{t+1}^*}{\partial b_t} = V_{1,t+1} \cdot \frac{dc_{t+1}^1}{db_t} + V_{2,t+1} \cdot \frac{dc_{t+1}^2}{db_t} + \frac{\partial U_{t+1}}{\partial b_{t+1}} \cdot \frac{db_{t+1}}{db_t}$$

If the descendant is at a utility maximum, he will be indifferent between these options at the margin provided they are already at positive levels;[8] that is, from his first-order conditions $(1 + n)V_{2,t+1} = (1+n)V_{1,t+1}/(1 + r_{t+2}) \geq \partial U_{t+1}/\partial b_{t+1}$. From the descendant's budget, $(1 + r_{t+2})dc_{t+1}^1 + dc_{t+1}^2 + (1 + n)db_{t+1} = (1 + r_{t+2})db_t$. Therefore,

$$\frac{\partial U_{t+1}^*}{\partial b_t} = (1 + r_{t+2})V_{2,t+1} \qquad \Box$$

Note that the individual can calculate $\partial U_{t+1}^*/\partial b_t$ without needing to know how his descendant disposes of the proceeds. Since the individual is affected only by U_{t+1}^*, he is indifferent to how his descendant spends the bequest as long as

he does it optimally. This is a consequence of the one-sided nature of the model. The ability to express the marginal utility from a bequest in terms of the marginal utility from consumption is fundamental to our ability to interpret the steady state of a system with operative transfers.[9] Lemma 1 and equation (1) allow us to write (4) as

(4') $\quad V_2'/V_{2,t+1} \geq (1 + r_{t+2})/(1 + n)(1 + \delta)$

In this form, the decision to leave a bequest is seen to involve a trade-off between the individual's utility from consumption and his descendant's utility from consumption. If the system converges so that a solution to (3) and (4') exists then $v_{2,t}$ can be written in solution form as a function of w_t , r_{t+1} and b_{t-1} . Further, the functional form of this solution will be the same for all periods t . In his analysis of bequest behaviour, Drazen (1978) uses this convergence along with an assumption of static expectations to write (4') as

(4") $\quad V_{2,t+1}(w,r,b_t)(1 + r)/(1 + \delta)(1 + n) \leq V_{2,t}(w,r,b_{t-1})$

He then interprets this condition as generating positive, zero or negative desired bequests according to whether $(1 + \delta)$ is less than, equal to, or greater than $(1 + r)/(1 + n)$.

This interpretation is too simplistic. Firstly, the precision of the relationship is lost if we drop the assumption of static expectations. Secondly, and more importantly, while r is given to the individual, it is determined in the general equilibrium by individual actions and is influenced by the bequest motive. The investigation of the conditions that lead to operative bequests is a general- rather than a partial-equilibrium problem. Thirdly, the equality of $(1 + \delta)$ and $(1 + r)/(1 + n)$ does not imply zero bequests. The level of bequests in any period t depends on two sets of factors: the initial conditions (the capital intensity and the level of bequests from generation $t-1$) , and the

nature of the steady state (in particular, whether or not steady-state bequests are positive). These two sets of factors determine the solution form of V_2 as well as the time paths of w, r and b. If the steady state involves positive bequests then it is likely that bequests will be positive along the time path converging to the steady state. The problem requires a more detailed analysis of the aggregate temporary and steady-state equilibria.

II(b)   The Aggregate Economy and Bequests in the Steady State

The aggregate economy consists of overlapping generations of identical individuals, and cost-minimizing firms. The firms face a standard neoclassical production technology, population grows at a constant rate and all markets are competetive. These assumptions are formalized in the following relationships [10]

(5) $N_t = N_0(1 + n)^t$

(6) $y_t = f(k_t)$, $f' > 0$, $f'' < 0$, $f'(0) = \infty$, $f'(\infty) = 0$

(7) $r_t = f'(k_t)$

(8) $w_t = f(k_t) - k_t f'(k_t)$

where N_t is population, n is the population growth rate, y_t is per capita output, k_t is the capital-labor ratio, and (7) and (8) reflect the assumption of competition. The temporary equilibrium of the economy is described by equilibrium in the capital market

(9) $a_t + b_{t-1} = (1 + n)k_{t+1}$

The steady-state definitions of first- and second-period consumption are found by substituting the steady-state equivalents of (7) - (9) into the budget constraint (2)

(10) $c^1(k,b) = w(k) - (1 + n)k + b$

$c^2(k,b) = (1 + n)(1 + r(k))k - (1 + n)b$

The steady-state properties of the system can be analysed by substituting (10) into the marginal conditions (3) and (4'). This provides the steady-state marginal conditions.

(11) $\hat{V}_1(k,b)/\hat{V}_2(k,b) = 1 + r(k)$

(12) $(1 + n)(1 + \delta) \geq 1 + r(k)$

where $\hat{V}_i(k,b) \equiv V_i(c^1(k,b), c^2(k,b))$ for $i = 1,2$.

Before examining the properties of this system, it is necessary to impose a restriction on the feasible values of the interpersonal discount rate in the steady state

$$U = V(c^1, c^2) + U/(1 + \delta)$$

$$= \frac{(1 + \delta)}{\delta} \cdot V(c^1, c^2)$$

For the steady state to exist in the quadrant of positive consumptions, steady-state utility must be positive and bounded.[12] This leads to the following assumption

Assumption 1: The interpersonal discount rate, δ, is strictly positive.

As a reference point when discussing optimality I will follow the literature on growth theory and define an optimal steady state as one that maximizes steady-state utility for a representative generation.[13] As shown by Diamond (1965) we can think of this optimum as resulting from a social planner who maximizes $V(c^1, c^2)$ subject to the social budget constraint

$$c^1 + c^2/(1 + n) \leq f(k) - nk$$

The social planner's decision leads to the following optimality conditions

$$V_1/V_2 = 1 + n$$

$$f' = n$$

where the second of these conditions requires that the capital intensity be at the golden-rule level k_G. I will refer to a decentralized steady-state equilibrium in which $k > k_G$ as overcapitalized and to one in which $k < k_G$ as undercapitalized.

A decentralized steady-state equilibrium in the bequest model is a point (k^*, b^*), $0 < k^*$, $0 \le b^*$, that satisfies (11) and (12). The main qualitative property of this equilibrium is contained in the following proposition

Proposition 1: If the individual cares about his descendant only, the steady-state equilibrium with operative bequests is undercapitalized with $k < k_G$.

Proof: From (12) and Assumption 1, operative bequests imply that $n < r = f'(k)$. □

Equations (11) and (12) suggest that provided the solution for b from (11) is non-negative, the bequest motive determines the steady-state interest rate according to the standard modified golden rule of optimal growth theory: That is, modified to account for time preference.[14] The steady-state level of bequests is then determined by the "within-life" endogenous marginal rate of time preference.[15] The modified golden rule capital intensity is sub-optimal ~~is sub-optimal~~ in terms of the optimality criterion stated above.

The nature of the steady-state equilibrium with bequests is illustrated in Figure 1 in which PQ is the steady-state consumption possibilities frontier under the assumption that individuals receive and leave no bequests. It is found by solving (10) for c^2 as a function of c^1 with b set equal to zero. The effect of introducing positive steady-state bequests is to slide the steady-state

consumption possibilities frontier from PQ downward to the right along a line with slope $-(1 + n)$; that is, a unit of b increases c^1 by one unit and decreases c^2 by $(1 + n)$ for all c^1 and c^2. The new locus for a given quantity of bequests will lie below PQ (for example, $P'Q'$ in Figure 1(a)). Negative desired bequests would shift PQ in the opposite direction at the rate $(1 + n)$. With bequests constrained to be non-negative, the attainable consumption frontier is BAQ as shown in Figure 1(a) where the point A corresponds to the consumption levels on PQ generated by the modified golden rule level of capital intensity. A steady-state equilibrium with positive bequests will occur if there exists a point on AB at which an indifference curve is tangent to the line with slope $-(1 + r(k_A))$, where $r(k_A)$ is the solution to (12). Such an equilibrium is shown as C in Figure 1(a) where the bequest level is DC.

If such a tangency does not exist on AB, bequests will be zero and a solution on AQ will occur. This case is illustrated in Figure 1(b) where the solution to (11) and (12) results in a tangency at C' which involves a negative desired bequest. The zero-bequest equilibrium is at a point such as E on AQ. The actual outcome depends on a combination of the production technology and preferences. One obvious implication that can be drawn from Figure 1 is that for a given technology, the likelihood of operative bequests increases as the discount rate decreases and as the within-life preference for c^1 over c^2 increases.
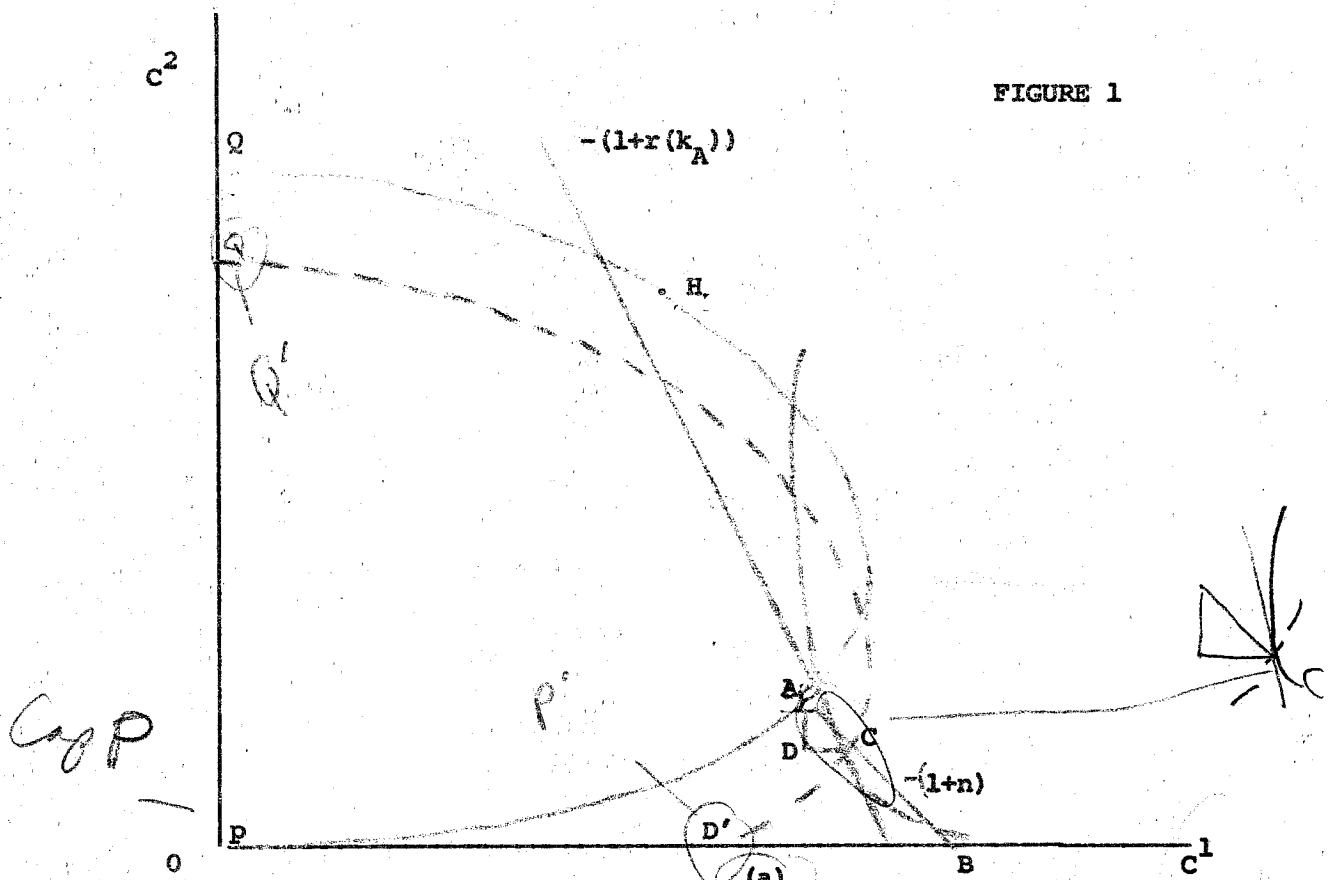
II(c) Gifts

The analysis of a situation in which the individual cares about his forebearer but not his descendant is very similar to that in Section II(b) above. The individual's utility function in this model is assumed to be of the form

$$(13) \quad U_t = V(c^1, c^2) + U^*_{t-1}/(1 + \delta)$$

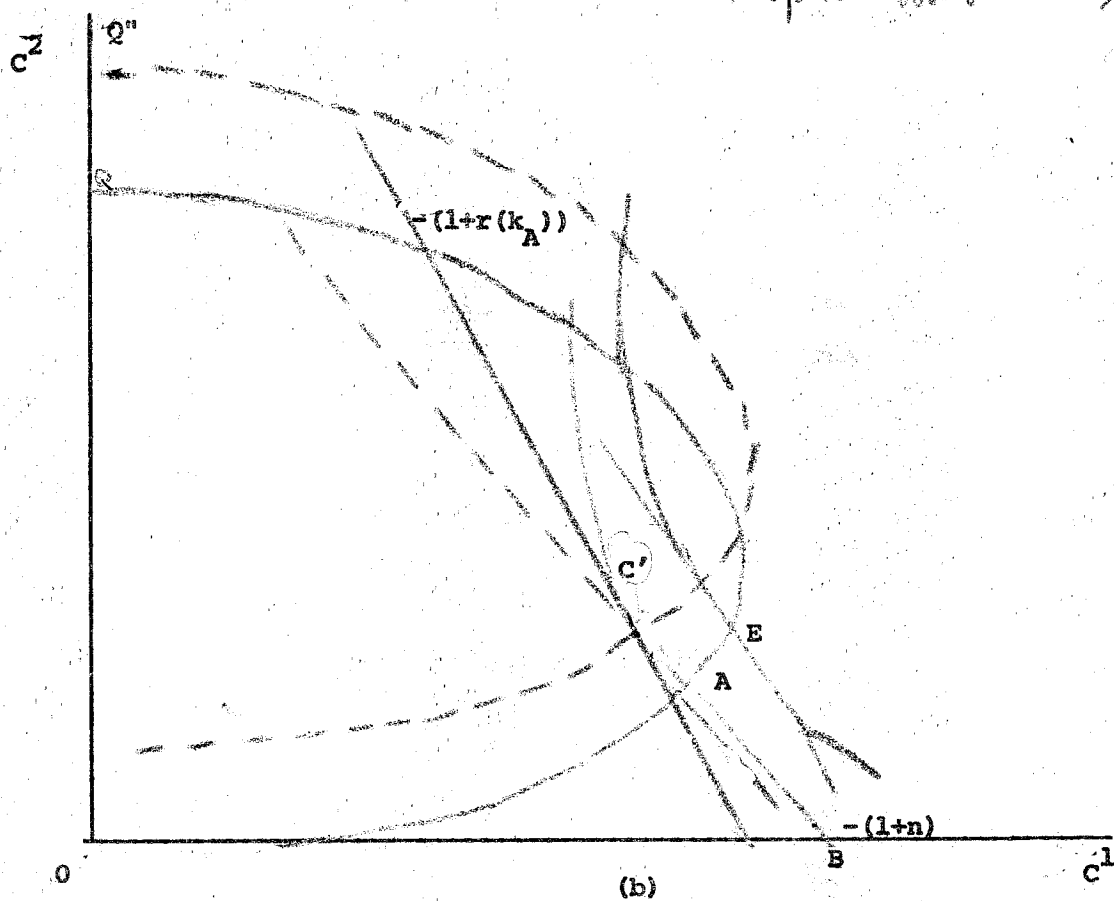where U^*_{t-1} is the perceived maximum attainable per capita utility of his immediate

FIGURE 1

c^2

Q

$-(1+r(k_A))$

H

P'

A

C

D

$-(1+n)$

D'

P

0

B

c^1

(a)

c^2

Q''

Q

$-(1+r(k_A))$

C'

E

A

$-(1+n)$

B

0

c^1

(b)

forebearer. His budget constraint is

(14) $c^1 + a + g_t/(1 + n) \leq w$

$c^2 \leq (1 + r_{t+1})a + g_{t+1}$

where g_{t+1} is the level of gifts received in the second period of life from his descendant, and g_t is the per capita gift that he gives to his forebearer.[16] The first-order conditions for a maximum of (13) subject to (14) are

$V_1/V_2 = 1 + r_{t+1}$

$V_2(1 + r_{t+1}) \geq (1 + n)\dfrac{\partial U_t}{\partial g_t}$, if > then $g_t = 0$

where the second condition makes use of the equality in the first. [At this point, we face a decision on how to specify the individual's perceptions of his forebearer's behaviour. One possible formulation of the problem is to take all past actions as given. When the individual announces his plans at the beginning of his first period, his forebearer is already in the second period of his life so that when the individual evaluates the effects of his marginal decisions, he will consider only second-period responses from his forebearer on the grounds that the past cannot be changed. An alternative formulation of the problem takes the viewpoint that given perfect foresight, the forebearer will have anticipated all possible decisions that the individual may make and will have acted accordingly. Under this formulation, the individual will include all possible reactions of his forebearer, regardless of whether or not they have already occurred. In this and the subsequent section, I will use the former approach which I will call "intertemporally asymmetric." Both the symmetric and asymmetric formulations are consistent with perfect foresight in that agents know the true behavioural rules that other agents follow. The difference between them is in the way in which they specify the behavioural rules. The qualitative results of the analysis are

independent of the formulation chosen, although the quantitative properties of the two-sided models are different.[17]

We can now extend Lemma 1 to the one-side case with gifts.

Lemma 2: The marginal utility that t expects $t-1$ to derive from a unit of bequests from t to $t-1$ can be expressed entirely in terms of $t-1$'s marginal utility from second-period consumption as

$$\frac{\partial U^{*}_{t-1}}{\partial g_t} = V_{2,t-1}$$

Proof: A unit of gifts from t to $t-1$ provides the forebearer with g_t of extra second-period endowment. When t announces his plans, $t-1$ is already in the second period of his life. He therefore has only one option available to him -- he must consume the gift as c^2_{t-1}.[18]

The steady-state expressions for first- and second-period consumption in this economy are

$$c^1(k,g) = w(k) - (1 + n)k - g/(1 + n)$$

$$c^2(k,g) = (1 + n)(1 + r(k))k + g$$

In combination with Lemma 2, these definitions lead to the following steady-state marginal conditions

(15) $\hat{V}_1(k,g)/\hat{V}_2(k,g) = 1 + r(k)$

(16) $(1 + n)/(1 + \delta) \leq 1 + r(k)$

where $\hat{V}_i(k,g) \equiv V_i(c^1(k,g), c^2(k,g))$, $i = 1,g$. If Assumption 1 is satisfied,[19] the steady-state equilibrium (k^*, g^*), $0 < k^*$, $0 \leq g^*$ of the system (15) and (16) has the following property:

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<u>Proposition 2</u>:   In an economy in which the individual cares about his forebearer only, the steady-state equilibrium with operative gifts is overcapitalized with $k > k_G$.

<u>Proof</u>:   From (16), operative gifts imply that $n > r$.   □

add?

Again, it is the intergenerational transfer motive (if operative) that determines the steady-state capital intensity. In this case, however, the capital intensity is determined by an inverted modified golden rule condition.[20] As well as being sub-optimal in terms of the reference criterion stated above, this equilibrium is inefficient in the Phelps-Koopmans sense. In terms of Figure 1, the attainable frontier now consists of the zero-transfer locus from P up to the point corresponding to the consumption allocation determined by the inverted modified golden rule capital intensity (H in Figure 1(a)). From this point, the frontier will be a linear segment exchanging c^1 for c^2 at the rate $(1 + n)$. As was the case for bequests, the conditions that determine whether or not steady-state gifts will be positive depend upon the production technology and preferences. In this case, a strong within-life preference for c^1 over c^2 and a high discount rate will reduce <u>the</u> likelihood of operative gifts.

III.   GIFT AND BEQUEST BEHAVIOUR: THE TWO-SIDED CASE

The above analyses of steady-state behaviour are instructive but limited. Why should individuals care about one but not the other of their related generations? Each individual's life overlaps with that of his immediate forebearer and descendant. It is more realistic to assume that he cares about both. ~~The questions of interest in this case become:~~ *In this case there are two questions of interest. Firstly, under* what conditions will either, neither, *Secondly* or both of the intergenerational transfer mechanisms be operative? ~~And~~, what are

I take responsibility!!

the implications of two-sided caring for the steady-state capital intensity?

The analysis in the two-sided case is made considerably more complex by feed-back chains; the individual cares about his descendant and the descendant cares about him. In this situation the individual will not necessarily be indifferent to the way in which *his descendant spends his bequest because one possibility* is that part of the bequest is returned to him as a gift. My understanding of the model is that for each individual to know every other individual's actual behaviour-al reactions, and for him to keep track of all feedbacks, leads to intractable an-alytical problems. For this reason, I will deal only with a model in which certain restrictions are imposed on the way in which the individual perceives the behaviour of others. The main restriction is that the individual knows the budget and first-order conditions of all other generations but does not (or is not able to) compute their actual behaviour. In defense of this restriction, it should be noted that the step from knowing the budget and first-order conditions of every other genera-tion to being able to solve for their actual behaviour is non-trivial. This re-striction can be viewed as a (non-stochastic) rational expectations equilibrium in which the information set available to each individual is limited. Given that they all have the same information set, and know that others behave on the basis of this information set, the equilibrium will not involve any misperceptions or unfulfilled expectations. The full nature of the equilibrium concept is contained in the fol-lowing definition.

Definition 2: An economy with intergenerationally-dependent preferences will be called an "intertemporally asymmetric Nash economy" if each individual bases his actions on his perceptions of the behaviour of other generations where these per-ceptions obey the following assumptions:

(a) The individual announces his plans at the beginning of his first period.

(b) When he makes his plans, he knows the utility levels of all past generations,

up to and including his immediate forebearer.

(c) When evaluating the effect of marginal changes in his actions, the individual ignores the effects on generations that are already dead.

(d) While the individual can affect the utility of other generations through all his actions, he can influence their behaviour only through direct transfers.

(e) The individual treats his received transfers as parametric.

Assumptions (a) and (b) are not restrictive and have been implicit in the preceding analysis of the one-sided case. Assumption (c) is the source of asymmetry as mentioned in Section II above. Assumption (d) is equivalent to imposing additive separability on the individual's utility function with respect to the utility of other generations. Assumption (e) is the restriction that the individual does not know the actual behavioural response of other generations to his marginal actions. This assumption, together with (d), is essential to our ability to express the marginal utility of a transfer in terms of a marginal utility from consumption.

III(a)   The Individual's Choice Problem

In this section I will retain the assumption that the individual's utility function is additively separable with constant interpersonal discount rates. As mentioned above, the separability is necessary for the Nash equilibrium concept to be consistent with actual outcomes. The constant discount rate assumption is unnecessary but convenient. The qualitative results are unchanged by relaxing this constancy. Consider then a utility function for individual t of the form

$$(17) \quad U_t = V(c^1, c^2) + U^*_{t+1}/(1 + \delta_d) + U^*_{t-1}/(1 + \delta_f)$$

where δ_d and δ_f are the interpersonal discount rates applied to the descendant and forebearer respectively. The indivual's budget constraint is

(18) $c^1 + a + g_t/(1 + n) \leq w$

$c^2 + (1 + n)b_t \leq (1 + r_{t+1})(a + b_{t-1}) + g_{t+1}$

The first-order conditions for a maximum of (17) subject to (18) are

(19) $\dfrac{\partial U_t}{\partial c_t^1} / \dfrac{\partial U_t}{\partial c_t^2} = 1 + r_{t+1}$

(20) $(1 + n)\dfrac{\partial U_t}{\partial c_t^2} \geq \dfrac{\partial U_t}{\partial b_t}$, if $>$ then $b_t = 0$

(21) $\dfrac{\partial U_t}{\partial c_t^1}/(1 + n) \geq \dfrac{\partial U_t}{\partial g_t}$, if $>$ then $g_t = 0$

The notation $\partial U_t/\partial c_t^i$ is used to indicate the sum of marginal utilities derived by t from consuming a unit of c_t^i. In the one-sided case, $\partial U_t/\partial c_t^i = V_{i,t}$. In the two-sided case, $\partial U_t/\partial c_t^i > V_{i,t}$ due to the presence of utility feedbacks. The feedbacks will be defined more precisely below.

__Assumption 2__: The product of the interpersonal discount factors is greater than units; $\delta_d \delta_f > 1$.

This restriction is necessitated once again by the requirement that steady-state utility be positive and bounded. In the steady state,

$U = V(c^1, c^2) + U/(1 + \delta_d) + U/(1 + \delta_f)$

$= V(c^1, c^2)/[1 - 1/(1 + \delta_d) - 1/(1 + \delta_f)]$

The condition for positive and bounded utility is that $1 > 1/(1 + \delta_d) + 1/(1 + \delta_f)$, or equivalently, $\delta_d \delta_f > 1$. This condition implies that if the individual cares about both his forebearer and descendant then the sum of the weights that he gives them in his utility function must be less than the weight that he gives to his own utility.[21]

III(b)   Expression of Transfer Utilities in Terms of Consumption Utilities

In the one-sided cases above, the interpretation of the steady state was made possible by reducing the marginal conditions associated with transfers to a comparison of consumption utilities available in two related generations. This principle will be followed in this section. The problem in this case is made more complex by the fact that all marginal utility terms involve an infinite chain of feedback effects on utility; for example, if individual t consumes a unit of c_t^2 he obtains $V_{2,t}$ directly. This increase in his utility increases utility for $t-1$ and $t+1$ by $V_{2,t}/(1 + \delta_f)$ and $V_{2,t}/(1 + \delta_d)$ respectively.[22] The utility gain for $t-1$ feeds back on t. The utility gain for $t+1$ feeds back on both t and $t+2$. In other words, t's consumption sets up an infinite chain of utility effects.[23] *Assumption 2 is sufficient to ensure convergence of this chain.* We can now define the marginal utility from a unit of c_t^2 as

$$(22) \quad \frac{\partial U_t}{\partial c_t^2} = V_{2,t} + \frac{\partial U_{t+1}^*}{\partial c_t^2}/(1 + \delta_d) + \frac{\partial U_{t-1}^*}{\partial c_t^2}/(1 + \delta_f)$$

where

$$(23) \quad \frac{\partial U_{t-1}^*}{\partial c_t^2} = \frac{\partial U_t^*}{\partial c_t^2}/(1 + \delta_d)$$

and

$$(24) \quad \frac{\partial U_{t+i}^*}{\partial c_t^2} = \frac{\partial U_{t+i+1}^*}{\partial c_t^2}/(1 + \delta_d) + \frac{\partial U_{t+i-1}^*}{\partial c_t^2}/(1 + \delta_f)$$

for all $i > 0$. The role of the Nash equilibrium concept in defining these marginal utilities should be clear. If a change in t's utility altered the behaviour (in particular, the transfer behaviour) of other generations, (22) would be considerably more complicated.[24] The relationships in (22) - (24) are illustrated in Figure 2 where the convergence assumption is reflected in the decline in marginal utility as generations become more distant from t. A similar set of relationships

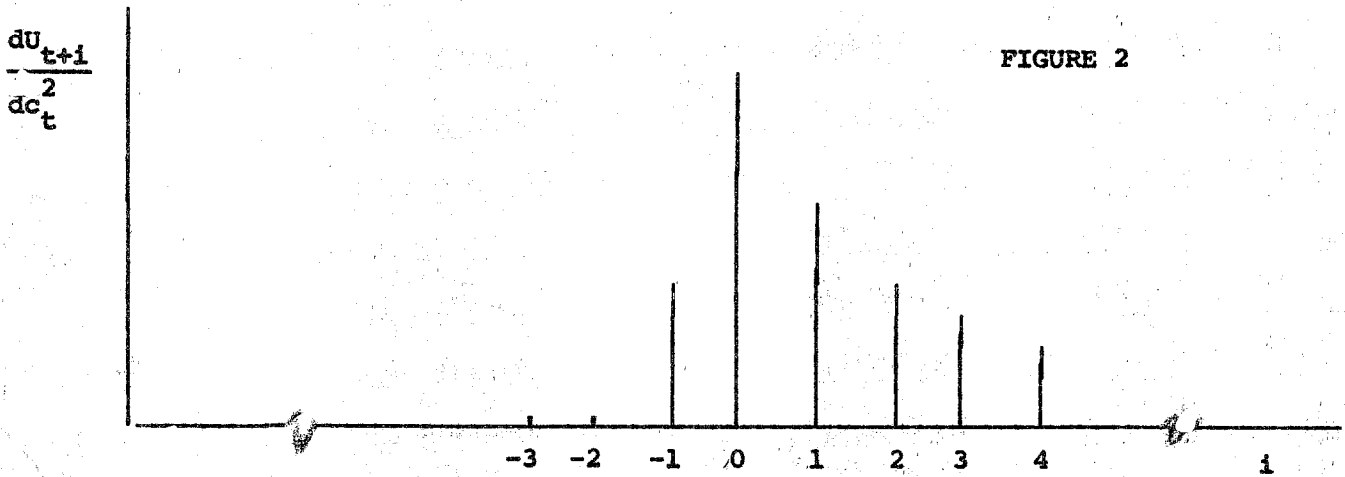defines the chain of marginal utilities resulting from a unit of c_t^1 .

$$\frac{dU_{t+i}}{dc_t^2}$$

FIGURE 2

-3 -2 -1 0 1 2 3 4   i

If bequests are operative, they involve a transfer of resources from the individual to his descendant. As was the case with t's consumption, t-1 is affected by the utility link only; if t gets utility from leaving a bequest then t-1 derives utility from t's utility. The evaluation of the effect on t+1 , however, must account for the fact that t+1's behaviour will change because he receives an increase in his endowment. From Definition 2, the only thing that matters to t in his calculation of $\partial U_t/\partial b_t$ is the utility that t+1 derives from the bequest.[25] The expression of this utility requires an extension of Lemma 1.

Lemma 3: In the two-sided case, the marginal utility that t expects t+1 to derive from a unit of bequests received from t can be expressed in terms of t+1's marginal utility from consumption as

$$\frac{\partial U_{t+1}^*}{\partial b_t} = (1 + r_{t+2})\frac{\partial U_{t+1}}{\partial c_{t+1}^2}$$

Proof: Generation t+1's marginal utility from a bequest can be defined as

$$\frac{\partial U_t^*}{\partial b_t} = \frac{\partial U_{t+1}}{\partial c_{t+1}^1} \cdot \frac{dc_{t+1}^1}{db_t} + \frac{\partial U_{t+1}}{\partial c_{t+1}^2} \cdot \frac{dc_{t+1}^2}{db_t} + \frac{\partial U_{t+1}}{\partial b_{t+1}} \cdot \frac{db_{t+1}}{db_t} + \frac{\partial U_{t+1}}{\partial g_{t+1}} \cdot \frac{dg_{t+1}}{db_t}$$

From t+1's first-order conditions, this can be written as[26]

$$\frac{\partial U_{t+1}^*}{\partial b_t} = \frac{\partial U_{t+1}}{\partial c_{t+1}^2} \frac{dc_{t+1}^2}{db_t} \left[\frac{dc_{t+1}^2}{db_t} + (1 + r_{t+2}) \frac{dc_{t+1}^1}{db_t} + (1 + n) \frac{db_{t+1}}{db_t} + \frac{(1 + r_{t+2})}{(1 + n)} \cdot \frac{dg_{t+1}}{db_t} \right]$$

From t+1's budget, the terms in the square brackets sum to $1 + r_{t+2}$.

We can define the marginal utility that t expects to derive from a unit of

b_t as

(25) $\quad \dfrac{\partial U_t}{\partial b_t} = \dfrac{\partial U_{t+1}^*}{\partial b_t}/(1 + \delta_d) + \dfrac{\partial U_{t-1}^*}{\partial b_t}/(1 + \delta_f)$

where

(26) $\quad \dfrac{\partial U_{t-1}^*}{\partial b_t} = \dfrac{\partial U_t^*}{\partial b_t}/(1 + \delta_d)$

and

(27) $\quad \dfrac{\partial U_{t+1}^*}{\partial b_t} = (1 + r_{t+2})\dfrac{\partial U_{t+1}}{\partial c_{t+1}^2}$

Substituting (26) and (27) into (25) yields

(28) $\quad \dfrac{\partial U_t}{\partial b_t} = \gamma_{bt}(1 + r_{t+2})\dfrac{\partial U_{t+1}}{\partial c_{t+1}^2}$

where $\gamma_{bt} \equiv (1 + \delta_f)/[(1 + \delta_d)(1 + \delta_f) - 1]$.

If gifts are operative, resources are transferred from t to t-1 . Lemma 2 can also be extended to the two-sided case.

Lemma 4: In the two-sided case, the marginal utility that t expects t-1 to derive from a unit of gifts received from t can be expressed in terms of t-1's marginal utility from consumption as

$$\frac{\partial U_{t-1}^{*}}{\partial g_t} = \frac{\partial U_{t-1}}{\partial c_{t-1}^2}$$

Proof: Generation t-1 can spend the gift on c_{t-1}^2 or bequests back to t ; that is,

$$\frac{\partial U_{t-1}^{*}}{\partial g_t} = \frac{\partial U_{t-1}}{\partial c_{t-1}^2} \cdot \frac{dc_{t-1}^2}{dg_t} + \frac{\partial U_{t-1}}{\partial b_{t-1}} \cdot \frac{db_{t-1}}{dg_t}$$

Generation t-1's first-order conditions and second-period budget constraint then yield Lemma 4.

The marginal utility derived by t from a unit of gifts is defined by

$$(29) \quad \frac{\partial U_t}{\partial g_t} = \frac{\partial U_{t+1}^{*}}{\partial g_t}/(1 + \delta_d) + \frac{\partial U_{t-1}^{*}}{\partial g_t}/(1 + \delta_f)$$

where

$$(30) \quad \frac{\partial U_{t-1}^{*}}{\partial g_t} = \frac{\partial U_{t-1}}{\partial c_{t-1}^2}$$

and

$$(31) \quad \frac{\partial U_{t+i}^{*}}{\partial g_t} = \frac{\partial U_{t+i+1}^{*}}{\partial g_t}/(1 + \delta_d) + \frac{\partial U_{t+i-1}^{*}}{\partial g_t}/(1 + \delta_f)$$

for all i > 0 . Equation (29) does not collapse immediately into an expression

involving only $\partial U_{t-1}/\partial c_{t-1}^2$ as (25) did. In order to obtain this reduction, we make use of the following observation. Equations (24) and (31) define the utility chains for $t+i$ $(i > 0)$ arising from c_t^2 and g_t respectively. In both these instances, the utility that $t+i$ derives from t's action depends crucially upon the utility that t derives. Because these actions do not transfer resources to generations later than t, the only source of utility for $t+i$ is t's utility; if dU_t from the action is zero then so is dU_{t+i}. These pure utility chains are independent of the actions that generate them. If consuming a unit of c_t^1 yields α_t units of utility for t and α_{t+i} units for $t+i$, and a unit of gifts from t to $t-1$ yields α_t units of utility for t, it follows that $t+i$ _will derive_ α_{t+i} units from the action involving the gift. Therefore, any action in t (or $t-1$) that does not involve a change in $t+1$'s resources will generate the following relationship:

$$(32) \quad \frac{\partial U_{t+1}^*}{\partial U_t} = \nu_t$$

where ν_t is a constant that depends upon the time period t, the discount rates, and the utility functions of t and $t+i$ $(i > 0)$, but not on the specific action that generates it. The definition of pure utility chains, implicit in both (24) and (31), and the condition that these chains converge imply that $0 < 1/(1 + \delta_f) < \nu_t < 1$. Substituting (30) and (32) into (29) yields the desired expression for $\partial U_t/\partial g_t$ in terms of $t-1$'s consumption utility

$$(33) \quad \frac{\partial U_t}{\partial g_t} = \gamma_{gt} \cdot \frac{\partial U_{t-1}}{\partial c_{t-1}^2}$$

where $\gamma_{gt} \equiv (1 + \delta_d)/[(1 + \delta_f)(1 + \delta_d - \nu_t)]$. We can also use (32) to express $\partial U_t/\partial c_t^1$ and $\partial U_t/\partial c_t^2$ in terms of the underlying own-utilities from consumption $V_{1,t}$ and $V_{2,t}$

$$(34) \quad \frac{\partial U_t}{\partial c_t^1} = \gamma_c \cdot V_{1,t}$$

(35) $\dfrac{\partial u_t}{\partial c_t^2} = \gamma_c \cdot v_{2,t}$

where $\gamma_c \equiv (1 + \delta_d)(1 + \delta_f)/[(1 + \delta_d)(1 + \delta_f) - (1 + \delta_f)\gamma_t - 1]$.

III(c) The Steady-State Economy in the Two-Sided Case

We now describe the steady-state equilibrium of the economy. From (18), the steady-state consumption levels are

(36) $c^1(k,z) = w - (1 + n)k + z$

$c^2(k,z) = (1 + n)(1 + r)k - (1 + n)z$

where $z \equiv b - g/(1 + n)$ is the net bequest level. Substituting (33) - (35) and (28) into the first-order conditions (19) - (21) and imposing the steady-state restrictions yields the steady-state marginal conditions

(37) $\hat{v}_1(k,z)/\hat{v}_2(k,z) = 1 + r(k)$

(38) $(1 + n)/\gamma_b \geq 1 + r(k)$

(39) $(1 + n)\gamma_g \leq 1 + r(k)$

where $\hat{v}_i(k,z) \equiv v_i(c^1(k,z), c^2(k,z))$.

Lemma 5: In the steady state, (i) $1 > \gamma_b > 1/(1 + \delta_d)$, and

(ii) $1 > \gamma_g > 1/(1 + \delta_f)$.

Proof: From the definition of γ_b , (i) implies that $1 > (1 + \delta_f)/[(1 + \delta_d)(1 + \delta_f) - 1] > 1/(1 + \delta_d)$. By rearrangement, the right-hand inequality can be written as $(1 + \delta_f)(1 + \delta_d) > (1 + \delta_d)(1 + \delta_f) - 1$ which is true. By rearrangement, the left-hand inequality can be written as

$(1 + \delta_d)(1 + \delta_f) > (1 + \delta_f) + 1$ which is true given Assumption 2. The definition of γ_g and the fact that $0 < \nu < 1$ gives $\gamma_g > 1/(1 + \delta_f)$. Now, suppose $\gamma_g \geq 1$; by rearrangement, $0 \geq \delta_f(1 - \nu) - \nu + \delta_d\delta_f$ which implies that $\delta_d\delta_f < 1$. This contradicts Assumption 2.

The steady-state properties of the economy in which the individual cares about both his forebearer and descendant are summarized in the following propositions.

Proposition 3: At most, one of the intergenerational transfer mechanisms can be operative in the steady state.

Proof: The steady state is characterized by two endogenous variables, k and z. These two variables have to satisfy three equilibrium conditions. Unless $\gamma_g = 1/\gamma_b$, at least one of the equations (38) and (39) must hold with a strict inequality. From Lemma 5, both γ_g and γ_b are strictly less than unity so that it can never be the case that $\gamma_g = 1/\gamma_b$.

An intuitive explanation of Proposition 3 can be given as follows. Both gifts and bequests involve a weighted comparison of the consumption utilities of two con-secutive generations. In the steady state, consumption utilities are stationary across generations so that the weights applied in the comparison become the sole determinant of operative transfers. The fact that the individual applies a posi-tive discount rate to both his forebearer and descendant leads to the conclusion that if conditions are conducive to one type of transfer being operative, it is not possible for the conditions to support the other type of transfer. For example, operative bequests involve deferring consumption in time (from the individual to his descendant). For a given set of discount rates and population growth rate, operative bequests require a high interest rate. Because gifts involve shifting

consumption forward in time, operative gifts will require a low interest rate for the same growth and discount rates. Proposition 3 is strictly a long-run condition. In the temporary equilibrium the conditions determining operative gifts and bequests involve variables from different time periods. As these variables are not stationary in the temporary equilibrium, it is possible that both transfers could be operative in the same period; for example, consider a temporary equilibrium that involves a substantial rise in the interest rate from the preceding period.

Proposition 4: A steady state with operative bequests is undercapitalized.

Proof: If bequests are operative, (38) holds with equality. From Lemma 5, $\gamma_b < 1$ so that $n < r$ which implies that the capital intensity is below the golden rule level .

Proposition 5: A steady state with operative gifts is overcapitalized.

Proof: If gifts are operative, (39) holds with equality. From Lemma 5, $\gamma_g < 1$ so that $n > r$.

From Propositions 4 and 5, the two-sided case can be seen to be a natural extension of the two one-sided cases. Quantitatively, the equilibria are different but qualitatively, they are identical.

The actual steady state outcome depends upon the nature of the solution for the net bequest, z , from the two simultaneous pairs of equations (37) and (38), and (37) and (39). With γ_b and γ_g both less than unity, the inequalities can be satisfied for any feasible values of z and k . If the solution for z is positive in both cases then (37) and (38) will determine the equilibrium with positive bequests being operative. If the solution is negative in both then (37) and (39)

will determine the equilibrium with positive gifts being operative. If the solution to the first pair yields z negative and to the second pair z is positive then (37) alone determines the equilibrium with both transfer mechanisms inoperative.

The two-sided case can be illustrated diagrammatically. For a given discount rates, the presence of concern for both forebearer and descendant alters the attainable consumption possibilities frontier. The effect is to reduce the degree of inefficiency associated with both gifts and bequests. In terms of Figure 1(a), both A and H move toward the golden rule point. The new consumption possibilities frontier is illustrated in Figure 3 where BAQ is the bequests-only frontier (U_{t-1}^{*} not in U_t) , PHJ is the gifts-only frontier (U_{t+1}^{*} not in U_t) , and $B'A'H'J'$ is the gifts and bequests frontier.

From Figure 3, it can be seen that having both motives present increases the likelihood that the intergenerational transfer mechanism will be operative, not only because it allows both types of transfer but also because it expands the feasible consumption possibilities frontier   associated with each type of transfer. This last result follows from Lemma 5. Once again, whether or not gifts, bequests, or neither will be operative depends upon a combination of the production technology and preferences.

IV. SUMMARY AND CONCLUSION

In this paper, I have attempted to present a rigorous analysis of economic behaviour when finite-lived individuals care about their immediately related generations. By expressing the marginal utility of a transfer in terms of the recipient's marginal utility from consumption, it is possible were able to write the steady-state system in a way that has a natural economic interpretation. To keep the model tractable in the two-sided case, it is was necessary to restrict the scope of analysis to

FIGURE 3
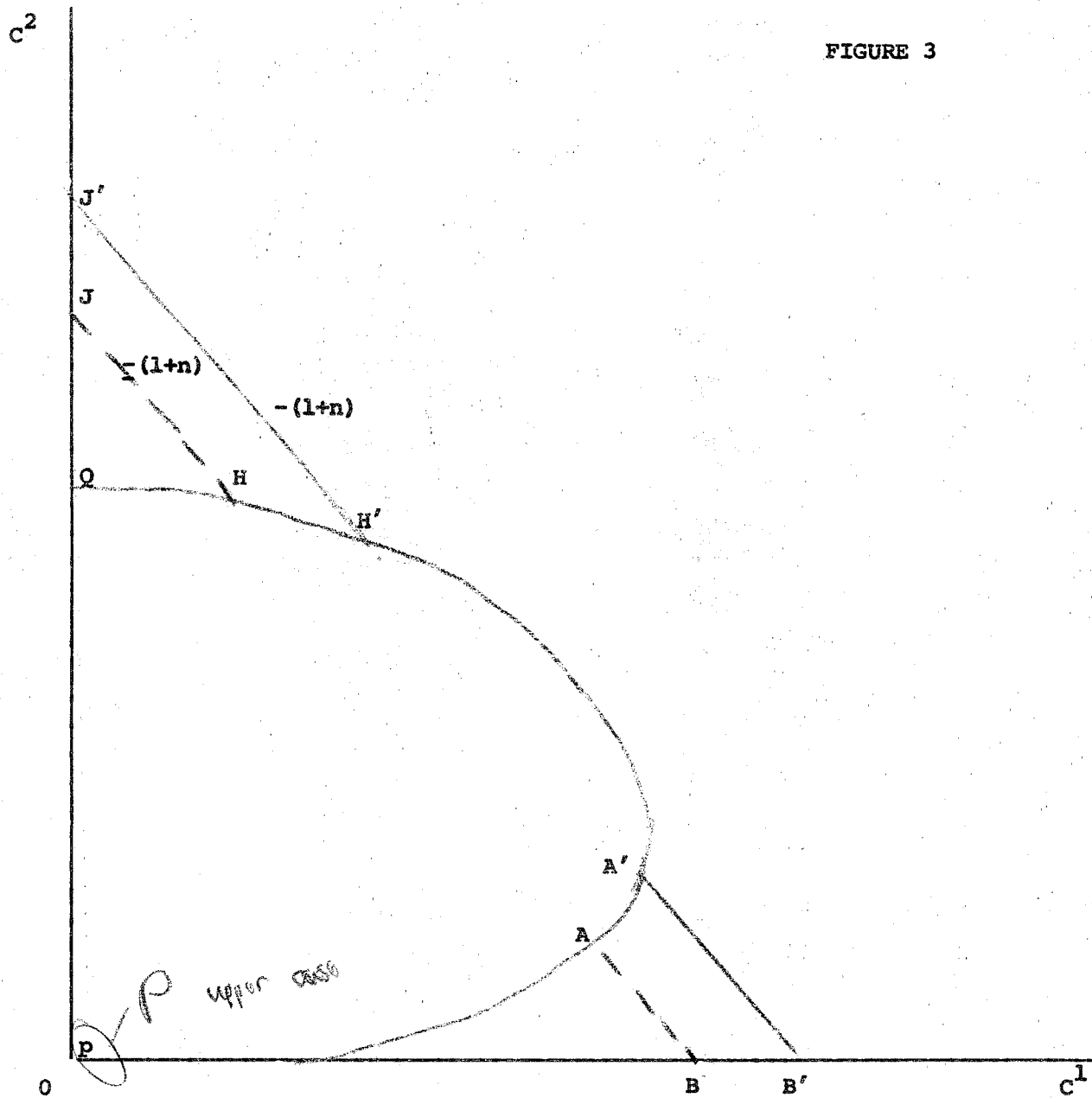
equilibria of the Nash type with imperfect perceptions. The advantage of this restriction is that it allows us to calculate complex marginal utility terms without having to know the exact behavioural reaction of all agents. The conclusions are limited to situations that satisfy this concept of behaviour.

The analysis suggests a number of characteristics that will be displayed by an economy with intergenerationally-dependent preferences of the type considered --

(i) At most, one form of intergenerational transfer will be operative in the steady state.

(ii) The likelihood of one of the transfer motives being operative increases as the interpersonal discount rates decrease.

(iii) Any steady-state configuration of capital intensity, growth rate, and consumption pattern is possible; the actual outcome depends upon both the production technology and preferences.

(iv) If optimality is defined as maximal steady-state utility for each generation then a steady state in which either transfer motive is operative is necessarily sub-optimal. The sub-optimality stems from the fact that transfers are a means of allocating life-time consumption. The rate of transforming consumption intertemporally through transfers, determined by a combination of the population growth rate and interpersonal preferences, exerts a portfolio effect on the rate of return on capital, thus causing it to diverge from its optimal value.

FOOTNOTES

* Reserve Bank of Australia. The work reported in this paper owes a substantial debt to Willem Buiter and Ed Green. Their encouragement and suggestions improved the work immeasurably. Any remaining errors are of course my own responsibility. The views expressed in the paper are my own and do not necessarily reflect the views of my employer.

1. Namely, the condition that steady-state utility be positive and bounded (see Assumptions 1 and 2 below).

2. Throughout the analysis all variables will be expressed in per capita units. As population will be assumed to grow at the constant rate n I will refer to descendants and forebearers as though there were only one of each; the descendant will be $(1 + n)$ times "bigger," and the forebearer $(1 + n)$ times "smaller" than the individual.

3. Buiter (1979) provides one of the few attempts to define the steady state of an economy with gift and bequest behaviour. He appears to have this superrational concept of equilibrium in mind. Consequently, his conclusions are different from those presented in this paper. In particular, the superrational equilibrium with operative intergenerational transfers is necessarily optimal under the definition of optimality introduced below.

4. The basic consumption unit is the household. As the analysis does not distinguish between the household and its individual members, I will refer to it as though it consisted of a single individual.

5. In Carmichael (1979) I show that the properties of the one-sided models are unaltered by the use of a general utility function of the form
$$u_t = v(c_t^1, c_t^2, u_{t+1}^*) .$$
w.c.

6. At times I will use the convenience of labeling an individual by his birth period; for example, "individual t" will be a member of the generation born in period t .

7. The assumptions about $V(\cdot)$ ensure that c^1 and c^2 are positive and that the budget constraint holds with equality.

8. If bequests are at a corner solution, db_{t+1}/db_t will be zero. It is possible that the extra bequest from t may induce an interior solution in which case this derivative will become positive.

9. Note that we could also have chosen to express Lemma 1 in terms of the marginal utility from first-period consumption. This alternative choice would not alter any of the results.

10. Throughout the paper, upper case letters will represent aggregate values and lower case letters will represent per capita values.

11. The term "temporary" is preferred to "short run" as a description of equilibrium as all factors of production are fully employed at all times.

12. The requirement that utility be positive assumes that $V(\cdot)$ is defined to be positive. If we allow for arbitrary definitions of $V(\cdot)$ that include $V(\cdot)$ everywhere less than zero then the restriction of positivity is replaced by the requirement that utility does not change sign for any value of δ .

13. This is not the only possible concept of optimality in a dynamic economy. Nevertheless, it has been used widely and its limitations are well known.

14. This modified golden rule outcome is consistent with the interpretation of bequest behaviour as extending a finite-lived individual into one with an effectively infinite horizon.

15. This is a reversal of roles compared with the short run in which the marginal rate of time preference determines k_{t+1} (through r_{t+1}) and the bequest motive then determines b_t given k_{t+1} .

16. As gifts can be made while both the giver and the recipient are alive, there is no need to assume anything about the form in which the gift is stored. A gift is a current-period transfer from young to old.

17. The symmetric Nash economy for the two-sided case is presented in Carmichael (1979).

18. Note that Lemma 2 would be unchanged by the assumption of intertemporal symmetry. The proof would follow the same lines as that for Lemma 1.

19. Assumption 1 is still necessary for steady-state utility to be positive and bounded.

20. The reason that caring about one's parents does not replicate the steady state of the infinite-lived individual lies in the nature of the discount rate. An infinite-lived individual applies a negative discount rate when discounting consumption forward in time from $t-1$ to t . The reason is that he prefers earlier consumption to later. In the finite-lived case, the discount rate is interpersonal rather than intertemporal. Because descendants consume in the future and forebearers in the present, the implicit time discount rate is symmetric about each time period rather than asymmetric as is the case with infinite-lived individuals.

21. An intuitive interpretation of this restriction is that if the individual cares about his relatives too much he will tend to give his resources away. In the steady-state this will lead to a degenerate solution.

22. Although the forebearer is dead when c_t^2 is consumed, he is affected because t announces his plans in period t .

23. Note that while Definition 2 rules out the need to account for feedback effects in terms of resource transfers, it does not rule out the need to account for utility feedbacks.

24. For example, suppose t's consumption of c_t^2 raised $t+1$'s utility to the point where it caused $t+1$ to stop his promised bequest to $t+2$; this would reduce $t+2$'s endowment and cause another infinite chain of effects. These in turn, would generate more changes.

25. This is the only thing that matters to t in as much as knowledge of $\partial U_{t+1}^{*}/\partial b_t$ enables him to calculate $\partial U_t/\partial b_t$.

26. Again, if either g_{t+1} or b_{t+1} is at a corner solution, its response to db_t will be zero.

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