SMALL EFFICIENT SCALE AS A

FOUNDATION FOR WALRASIAN EQUILIBRIUM

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INTRODUCTION

We are concerned with a formal description of perfect competition in which small efficient scale and the entry of firms occupy a central role. The set of aggregate technological possibilities for the economy is obtained by summing the production sets of a very large number of productive units, interpreted here as firms. These units are most efficient when their output is small (infinitesimal) relative to demand, and the classical case of U-shaped average cost is admitted in the analysis. Also, because efficient scale is small, firms have only an infinitesimal effect on price when confined to the region in which they make positive profit. This enables one to capture the notion that the demand curve appears flat to a firm, while at the same time demand price may change substantially with substantial changes in aggregate quantity. The mass of firms active in an equilibrium is determined by the conditions of supply and demand. Small changes in aggregate demand will typically change the list of firms which are present in an equilibrium. Since changes in taste cause some firms to leave the market and others to enter, and since there are usually firms "on the margin" of entry, entry plays an important role in the explanation of value.

Despite the fact that our description of a perfectly competitive economy fits quite nicely with the ordinary neo-classical conception, the reader will see that both in detail and interpretation it differs in some important ways from modern formal competitive theory (see; e.g. Debreu [3]). For example, we will argue that the convexity of the set of aggregate production possibilities is irrelevant for the existence of competitive equilibrium. With small efficient scale and the absence of externalities, the aggregate production set is necessarily convex (Richter's Theorem), and with standard assumptions equilibrium exists. With small efficient scale and externalities, the aggregate production set is in general not convex, but again under general conditions equilibrium will exist. Small efficient scale, which only in the absence of externalities guarantees convexity, is the proper requirement for the existence of perfectly competitive equilibrium.

The argument of the paper proceeds as follows. We begin by defining a private ownership perfectly competitive economy $\mathcal{E} = (X_i, \omega_i, \succsim_i, Z(\cdot), s_i(\cdot))$ and defining its equilibria. Throughout the analysis preferences $\succsim_i$ are convex and initial endowments ω_i are interior to the convex comsumption sets X_i. The framework differs from that of standard competitive theory in that there are a continuum of productive units whose output must be integrated to generate a quantity that is significant when compared with demand. The set $Z(\beta)$ is the technology available to unit β, and $s_i(\beta)$ is the share of firm β owned by i. The notion of a variable profit assignments economy $\mathcal{E}' = (X_i, \omega_i, \succsim_i, Z, w(\cdot))$ is then introduced. In addition to the m consumers this requires a specification of the aggregate technology Z, and for each p, m numbers $w_i(p)$, which indicate how max $p \cdot Z$ is shared among consumers: $\Sigma w_i(p) = \max p \cdot Z$. Variable profit assignments economies are of interest in their own right as they capture the notion of a regime in which the aggregate rent from the process of production is distributed to consumers according to a rule which depends on prices. (For example, an individual's share of max $p \cdot Z$ may depend on his wage.) Standard techniques from competitive analysis are sufficient to show that (loosely) every variable profit assignments economy $\mathcal{E}' = (X_i, \omega_i, \succsim_i, Z, w(\cdot))$ has an equilibrium provided Z is convex and $\{w_i(\cdot)\}$ are continuous for prices at which max $p \cdot Z$ is well defined. This is Theorem 1.

In this paper we are concerned with particular variable profit assignments economies. These economies; e.g., $\mathcal{E}' = (X_i, \omega_i, \succsim_i, Z, w(\cdot))$ are associated with perfectly competitive private ownership economies; e.g., $\mathcal{E} = (X_i, \omega_i, \succsim_i, Z(\cdot), s_i(\cdot))$, by defining Z as the integral of the correspondence $\beta \to Z(\beta)$, and for every price p, and every individual i, defining $w_i(p)$ as the integral of i's share of the profits made by each productive unit. When Z is well defined it will be convex (by Richter's Theorem) and mild conditions on the functions $s_i(\cdot)$ will guarantee that the functions $w_i(\cdot)$ have the continuity property necessary for the application of Theorem 1. The central point here is that the convexity of Z does not in any way depend on the convexity of the sets $Z(\beta)$. From a descriptive point of view, the convexity of Z is a consequence of the fact that efficient increases in output require a larger number of active productive units, these units "come in" at minimum average cost, and in the order "lower minimum average cost comes in first." Thus, we have the rather classical description of increases in output being achieved by the successive addition of small scale productive units, with the necessity of using on the margin a unit which is less efficient than those which preceeded it. If the production possibilities available to each firm are shared by an unbounded measure of similar firms, then constant returns to scale obtains in the aggregate. But, if firms differ greatly in their maximum efficiency, then the boundry of the aggregate production set will exhibit substantial curvature with changes in scale. In the latter case it is typical that economic rents will accrue to the productive sector. The ith agents share of these rents is a function of price, since price determines which firms are profitable. Some price vector may make every one of the firms which an agent owns unprofitable (then his share of economic rents will be zero), while another price vector may result in high profit for several of the firms which he owns. Even in economies with only two firms and two consumers, if the firms are not identical, and if each firm is not held equally by each individual, then the share of aggregate profit which is distributed to each agent will vary with price.

Under rather mild assumption on the correspondence $\beta \to Z(\beta)$, there is a natural correspondence between the equilibria of a private ownership economy $\mathcal{E} = (X_i, \omega_i, \succsim_i, Z(\cdot), s_i(\cdot))$ and the equilibria of the associated variable profit assignments economy $\mathcal{E}' = (X_i, \omega_i, \succsim_i, Z, w_i(\cdot))$. Specifically, $(p, x, z(\cdot))$ is an equilibrium of $\mathcal{E}$ implies $(p, x, \int z(\beta)d\beta)$ is an equilibrium of $\mathcal{E}'$, and with some technical assumptions (p, x, z) for $\mathcal{E}'$ implies there exists an equilibrium of $\mathcal{E}$, $(p, x, z(\cdot))$, such that $\int z(\beta)d\beta = z$. Similarly, there is a natural correspondence between the Pareto efficient allocations of $\mathcal{E}$ and the Pareto efficient allocations of $\mathcal{E}'$. If the allocation $(x, z(\cdot))$ is Pareto efficient for $\mathcal{E}$, then $(x, \int z(\beta)d\beta)$ is Pareto efficient for $\mathcal{E}'$, and if the allocation (x, z) is Pareto efficient for $\mathcal{E}'$ then there exists a Pareto efficient allocation of $\mathcal{E}$, $(x, z(\cdot))$, such that $z = \int z(\beta)d\beta$. With these observations in hand the existence of equilibrium for the perfectly competitive private ownership economy $\mathcal{E}$ is established; this is Theorem 2. The striking feature of the statement of the result is the absence of the assumption that the production sets $Z(\beta)$ are convex. The general argument of the proof is that (a) the production set Z of the associated variable profit assignments economy $\mathcal{E}'$ is convex (Richter's Theorem), (b) equilibrium exists for $\mathcal{E}'$ (Theorem 1), and (c) an equilibrium for $\mathcal{E}'$ is naturally associated with an equilibrium for $\mathcal{E}$. Theorem 3 gives conditions under which equilibria of a perfectly competitive private ownership economy $\mathcal{E} = (X_i, \omega_i, \succsim_i, Z(\cdot), s_i(\cdot))$ are Pareto efficient. The idea of the proof is (a) every equilibrium of $\mathcal{E}$ naturally corresponds to an equilibrium of the associated variable profit assignments economy

$\mathcal{E}'$, (b) equilibria of $\mathcal{E}'$ are Pareto efficient (standard), and (c) Pareto efficient allocations of $\mathcal{E}'$ naturally correspond to Pareto efficient allocations of $\mathcal{E}$. Finally, Theorem 4 and its corollary establish that every Pareto efficient allocation of a perfectly competitive private ownership economy $\mathcal{E}$ is an equilibrium subject to a suitable assignment of ownership. Again, the striking feature of the statement is the absence of the assumption that production sets are convex. The general argument of the proof is (a) each Pareto efficient allocation of $\mathcal{E}$ naturally corresponds to a Pareto efficient allocation of $\mathcal{E}' = (X_i, \omega_i, \gtrsim_i, Z, w_i(\cdot))$, (b) the production set Z is convex (Richter's Theorem), (c) every Pareto efficient allocation of $\mathcal{E}'$ is an equilibrium of $\mathcal{E}'$ for a suitable assignment of ownership (standard), and (d) every equilibrium of $\mathcal{E}'$ naturally corresponds to an equilibrium of $\mathcal{E}$. The formal analysis is followed by six remarks which explain some consequences of our formulation, included here is a reinterpretation of the classical theorems of welfare economics. Externalities and taxation are introduced into the analysis, and a potential cause of market failure is identified which does not appear in the Arrow-Debreu theory.

Define $\Delta = \{ p \in R^\ell_+ : \sum_i p_i = 1 \}$; integration is in the sense of Lebesgue. For integrals of correspondences see Hildenbrand [4], section D II.

THE MODEL

a. A private ownership perfectly competitive economy $\mathcal{E} = (X_i, \omega_i, \succsim_i, Z(\cdot), s_i(\cdot))$ is:

a1) for each consumer $i = 1,2,\cdots, m$, a consumption set $X_i \subset R^\ell$, an initial endowment vector $\omega_i \in R^\ell$, and a complete preference preordering $\succsim_i \subset X_i \times X_i$,

a2) for each firm $\beta \in [0, \infty)$, a nonempty production set $Z(\beta) \subset R^\ell$,

a3) for each (i, β) a non-negative number $s_i(\beta)$ which indicates the fraction of firm β owned by individual i. For each β, $\sum_i s_i(\beta) = 1$.

b. An equilibrium for the private ownership perfectly competitive economy $\mathcal{E} = (X_i, \omega_i, \succsim_i, Z(\cdot), s_i(\cdot))$ is a triple $(p^*, x^*, z^*(\cdot)) \in \Delta \times \Pi X_i \times$ (integrable functions from $[0, \infty)$ to R^ℓ, where $z^*(\beta) \in Z(\beta)$ for all β) satisfying

b1) $p^* \cdot x_i^* = p^* \cdot \omega_i + \int s_i(\beta) p^* \cdot z^*(\beta) d\beta$, $i = 1,2,\cdots, m$,

b2) $\sum_i x_i^* \leq \int z^*(\beta) d\beta + \sum_i \omega_i$ and $p^* \cdot (\sum_i x_i^* - \int z^*(\beta) d\beta - \sum_i \omega_i) = 0$,

b3) $x_i \succ_i x_i^*$ implies $p^* \cdot x_i > p^* \cdot \omega_i + \int s_i(\beta)\, p^* \cdot z^*(\beta) d\beta$, $i = 1,2,\cdots, m$,

b4) $p^* \cdot z > p^* \cdot z^*(\beta)$ implies $z \notin z(\beta)$, for a.e. β.

c. Given the private ownership perfectly competitive economy $\mathcal{E} = (X_i, \omega_i, \succsim_i, Z(\cdot), s_i(\cdot))$, the associated variable profit assignments A-D economy $\mathcal{E}' = (X_i, \omega_i, \succsim_i, Z, w_i(\cdot))$ is defined by

c1) $Z = \int Z(\beta) d\beta$

c2) for all $p \in \Delta$, $w_i(p) = \begin{cases} \int s_i(\beta) \sup \{p \cdot z : z \in Z(\beta)\} d\beta & \text{if it exists and is finite} \\ 0 & \text{otherwise.} \end{cases}$

d. An equilibrium for the variable profit assignments A-D economy $\mathcal{E}' = (X_i, \omega_i, \succsim_i, Z, w_i(\cdot))$ is a triple $(p^*, x^*, z^*) \in \Delta \times \Pi X_i \times Z$ satisfying

d1) $p^* \cdot x_i^* = p^* \cdot \omega_i + w_i(p^*)$, $i = 1,2,\cdots, m$,

d2) $\sum_i x_i^* \leqq z^* + \sum_i \omega_i$ and $p^* \cdot (\sum_i x_i^* - z^* - \sum_i \omega_i) = 0$,

d3) $x_i \succ_i x_i^*$ implies $p^* \cdot x_i > p^* \cdot \omega_i + w_i(p^*)$, $i = 1,2,\cdots, m$,

d4) $p^* \cdot z > p^* \cdot z^*$ implies $z \notin Z$,

d5) $p^* \cdot z^* = \sum_i w_i(p^*)$.

e. An allocation for the private ownership perfectly competitive economy $\mathcal{E} = (X_i, \omega_i, \succsim_i, Z(\cdot), s_i(\cdot))$ is a pair $(x, z(\cdot)) \in \Pi X_i \times$ (integrable functions from $[0,\infty)$ to R^ℓ, where $z(\beta) \in Z(\beta)$ for all β) satisfying $\sum_i x_i \leqq \int z(\beta)\, d\beta + \sum_i \omega_i$.
The allocation $(x, z(\cdot))$ is Pareto efficient if there does not exist an allocation $(x', z'(\cdot))$ for $\mathcal{E}$ with the property that $x_i' \succsim_i x_i$ for all i and $x_k' \succ_k x_k$ for some k.

f. An allocation for the variable profit assignments A-D economy $\mathcal{E}' = (X_i, \omega_i, \succsim_i, Z, w_i(\cdot))$ is a pair $(x,z) \in \Pi X_i \times Z$ satisfying $\sum_i x_i \leqq z + \sum_i \omega_i$.
The allocation (x,z) is Pareto efficient if there does not exist an allocation (x',z') for $\mathcal{E}'$ with the property that $x_i' \succsim_i x_i$ for all i and $x_k' \succ_k x_k$ for some k.

THEOREMS

For reference we state an interchangability lemma.

LEMMA 0: If the graph of the correspondence $\beta \to Z(\beta)$ is measurable and $\int Z(\beta)d\beta \neq \emptyset$ then for every $p \in R^{\ell}$ $\sup\{p\cdot z : z \in \int Z(\beta)d\beta\} = \int \sup\{p\cdot z : z \in Z(\beta)\}d\beta$.

Proof: The proof is similar to that of Proposition D II 6 of Hildenbrand [4] with minor modification since the measure is not a probability measure.

LEMMA 1. Given the private ownership perfectly competitive economy $\mathcal{E} = (X_i, \omega_i, \succsim_i, Z(\cdot), s_i(\cdot))$ and the associated variable profit assignments A-D economy $\mathcal{E}' = (X_i, \omega_i, \succsim_i, Z, w_i(\cdot))$, then, (i) $(p^*, x^*, z^*(\cdot))$ is an equilibrium of $\mathcal{E}$ implies $(p^*, x^*, \int z^*(\beta)d\beta)$ is an equilibrium of $\mathcal{E}'$.
If the correspondence $\beta \to Z(\beta)$ is closed valued, has a measurable graph, and for each i $s_i(\cdot)$ is measurable, then
(ii) (p^*, x^*, z^*) is an equilibrium for $\mathcal{E}'$ implies there exists an equilibrium of $\mathcal{E}$, $(p^*, x^*, z^*(\cdot))$, such that $\int z^*(\beta)d\beta = z^*$.

Proof: Trivial using the interchangability lemma for part (ii).

LEMMA 2: Given the private ownership perfectly competitive economy $\mathcal{E} = (X_i, \omega_i, \succsim_i, Z(\cdot), s_i(\cdot))$ and the associated variable profit assignments A-D economy $\mathcal{E}' = (X_i, \omega_i, \succsim_i, Z, w_i(\cdot))$, then (i) the allocation $(x, z(\cdot))$ is Pareto efficient for $\mathcal{E}$ implies $(x, \int z(\beta)d\beta)$ is Pareto efficient for $\mathcal{E}'$, and (ii) the allocation (x,z) is Pareto efficient for $\mathcal{E}'$ implies there exists a Pareto efficient allocation of $\mathcal{E}$, $(x, z(\cdot))$, such that $z = \int z(\beta)d\beta$.

Proof: Trivial.

THEOREM 1: Given the private ownership perfectly competitive economy $\mathcal{E} = (X_i, \omega_i, \succsim_i, Z(\cdot), s_i(\cdot))$, the associated variable profit assignments A-D economy $\mathcal{E}' = (X_i, \omega_i, \succsim_i, Z, w_i(\cdot))$ has an equilibrium if:

(i) for every i, X_i is closed, convex, bounded below,

$\succsim_i$ is continuous[1], convex[2], and there is no satiation consumption,

$\omega_i \in \text{Int } X_i$,

(ii) Z is closed, $Z \cap (-Z) = \{0\}$, and $Z \supset (-R^{\ell}_+)$, and

(iii) there is a compact cube $K \subset R^{\ell}$ such that K^{m+1} contains all allocations (x,z) in its interior[3] and for every i, w_i is continuous and non-negative on $\hat{\Delta} = \{p \in \Delta: \max p\cdot(Z\cap K) = \sup p\cdot Z\}$ and for every $p \in \hat{\Delta}$, $\sum_i w_i(p) = \max p\cdot(Z \cap K)$

Proof: By Richter's theorem $Z = \int Z(\beta)d\beta$ is convex, so by (ii) there is a $p \in \Delta$ such that $p\cdot Z \leqq 0 = p\cdot 0$, and $\hat{\Delta}$ is nonempty. The set $\hat{\Delta}$ is also closed so for each i there is a continuous function $\bar{w}_i: \Delta \to R_+$ such that $\bar{w}_i(p) = w_i(p)$ for all $p \in \hat{\Delta}$ and $\sum_i \bar{w}_i(p) = \max p\cdot(Z \cap K)$ for all $p \in \Delta$. By application of a standard technique (the abstract economies approach) due to Arrow and Debreu [ ] the variable profit assignments A-D economy $\mathcal{E}'_k = (X_i \cap K, \omega_i, \succsim_i, Z \cap K, \bar{w}_i(\cdot))$ has an equilibrium (p^*, x^*, z^*) which is also an equilibrium of $\mathcal{E}'$.

THEOREM 2: The perfectly competitive private ownership economy $\mathcal{E} = (X_i, \omega_i, \succsim_i, Z(\cdot), s_i(\cdot))$ has an equilibrium if:

(i) for every i, X_i is closed, convex, bounded below,

$\succsim_i$ is continuous, convex, and there is no satiation consumption,

$\omega_i \in \text{Int } X_i$,

1. $\succsim_i$ is continuous if for every $x_i' \in X_i$, the sets $\{x_i \in X_i: x_i \succsim_i x_i'\}$ and $\{x_i \in X_i: x_i' \succsim_i x_i\}$ are closed in X_i.

2. $\succsim_i$ is convex if $x_i' \succ_i x_i$ implies $\lambda x_i' + (1-\lambda) x_i \succ_i x_i$ for all $\lambda \in (0, 1)$

3 Note that the definition of allocation requires feasibility of (x, z). By a standard argument (i) and (ii) imply that such a K exists.

(ii) the correspondence $\beta \to Z(\beta)$ is closed valued with a measurable graph and $0 \in Z(\beta)$ for all β,

(iii) $\int Z(\beta)d\beta = Z$ is closed,[4] $Z \cap (-Z) = \{0\}$ and $Z \supset (-R^{\ell}_{+})$, and

(iv) for every i, the function s_i is measurable.

Proof: Consider the associated variable profit assignments A-D economy $\mathcal{E}' = (X_i, \omega_i, \succsim_i, Z, w_i(\cdot))$. By (i), (iii) and a standard argument there exists a compact cube $K \subset R^{\ell}$ such that K^{m+1} contains all allocations (x, z) in its interior. Using this K, assumption (iii) of Theorem 1 holds by (ii) and (iv). By Theorem 1 there is an equilibrium of $\mathcal{E}'$ and by Lemma 1(ii) there is an equilibrium of $\mathcal{E}$.

THEOREM 3: Suppose that the perfectly competitive private ownership economy $\mathcal{E} = (X_i, \omega_i, \succsim_i, Z(\cdot), s_i(\cdot))$ satisfies the condition that X_i contains no point of local satiation for each i. If $(p^*, x^*, z^*(\cdot))$ is an equilibrium for $\mathcal{E}$, then $(x^*, z^*(\cdot))$ is Pareto efficient.

Proof: Consider the associated variable profit assignments A-D economy $\mathcal{E}' = (X_i, \omega_i, \succsim_i, Z, w_i(\cdot))$. By Lemma 1 (i) $(p^*, x^*, \int z^*(\beta)d\beta)$ is an equilibrium of $\mathcal{E}'$, so by a standard argument $(x^*, \int z^*(\beta)d\beta)$ is Pareto efficient in $\mathcal{E}'$. Thus by Lemma 2 (ii), $(x^*, z^*(\cdot))$ (and any allocation $(x^*, z(\cdot))$ such that $\int z(\beta)d\beta = z^*$) is Pareto efficient in $\mathcal{E}$.

THEOREM 4: Let the private ownership perfectly competitive economy $\mathcal{E} = (X_i, \omega_i, \succsim_i, Z(\cdot), s_i(\cdot))$ be such that:

(i) for every i, X_i is convex and $\succsim_i$ is continuous and convex, and

(ii) the correspondence $\beta \to Z(\beta)$ is closed valued and has a measurable graph.

Given a Pareto efficient allocation $(x^*, z^*(\cdot))$, where x^*_k is not a satiation consumption for some k, there is a non-zero price vector $p^* \in R^{\ell}$ such that

4 The closedness of Z is not implied by (ii). Additional assumptions could be made about the correspondence $\beta \to Z(\beta)$ to insure that Z is closed.

(iii) for each i, x_i^* minimizes $p^* \cdot x_i$ on $\{ x_i \in X_i : x_i \succsim_i x_i^* \}$

(iv) for a.e. β, $p^* \cdot z > p^* \cdot z^*(\beta)$ implies $z \notin Z(\beta)$.

Proof: Consider the associated variable profit assignments A-D economy $\mathcal{E}' = (X_i, \omega_i, \succsim_i, Z, w_i(\cdot))$. The allocation $(x^*, \int z^*(\beta)d\beta)$ is Pareto efficient in $\mathcal{E}'$ by Lemma 2(i), and Z is convex by Richter's theorem so by a standard argument there exists a non-zero $p^* \in R^\ell$ such that (iii) holds and $[p^* \cdot z > p^* \cdot \int z^*(\beta)d\beta$ implies $z \notin Z]$. Then (iv) easily follows using (ii) and the interchangability lemma.

Corollary: If the conditions of Theorem 4 hold and $x_i^* \in \text{int } X_i$ for each i and $(-R_+^\ell) \subset Z = \int Z(\beta)d\beta$, then $(p^*, x^*, z^*(\cdot))$ is an equilibrium for some assignment of endowments and ownership shares.

Proof: Trivial.

Remark 1

If for each $j \in \{1, \cdots, n\}$ and $\beta \in [0, \infty)$ we define $Z(j, \beta)$ to be the production set of firm j, β (with ownership shares $s_i(j, \beta)$), then $Z_j = \int Z(j, \beta) d\beta$ is convex. If we let $w_i(p, j)$ be consumer i's income from industry j at prices p, and $w_i(p) = \sum_{j=1}^{n} w_i(p, j)$ then the variable profit assignments A-D economy $\mathcal{E}' = (X_i, \omega_i, \succsim_i, Z_1, \cdots, Z_n, w_i(\cdot))$ is quite similar to a standard Arrow-Debreu economy. However, the interpretation is different. Here Z_j is an industry production set, where the industry is made up of a continuum of infinitesimal firms, each of which has no effect on price. Thus the industry acts competitively, and as prices vary the industry output varies both because active firms change their productions and because firms enter and leave the market. This corresponds to the classical intuition about perfect competition.

Novshek and Sonnenschein [7] treated a special case where for each j, $Z(j, \beta)$ is the same non-convex production set for all β . While each firm had efficient scale bounded away from zero (in the scale of the firm), the industry production set was a convex cone. These are the analogs of U-shaped average cost and horizontal supply in partial equilibrium. They consider a sequence of finite economies $\mathcal{E}_k$ with non-infinitesimal firms which converges to the perfectly competitive economy $\mathcal{E}$. In the finite economies firms set quantity in order to maximize profit, recognizing their effect on price.[5] They show that if $\mathcal{E}$ has an equilibrium satisfying a condition called DSD, then whenever the efficient scale of the firms is sufficiently small relative to the market (k is sufficiently large) a Cournot-Nash equilibrium with free entry exists for $\mathcal{E}_k$, and the set of Cournot-Nash equilibria with free entry converges to the set of (perfectly competitive) equilibria of $\mathcal{E}$ which satisfy DSD.[6] These results suggest that in the intuitive model of perfect competition, with entry and exit of many small (infinitesimal) firms producing the changes in output, only those equilibria which satisfy DSD are true equilibria.

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5. Prices are such that the excess demand of the price taking consumer sector equals the asserted quantity actions of the firms.

6. Some technical qualifications are needed; in particular the inverse demand function F must be C^2 in a neighborhood. This of course includes the requirement that F is defined in a neighborhood of the Cournot equilibrium.

From this perspective, other equilibria are artifacts of a perfectly competitive specification in which firms are regarded as points rather than infinitesimals.

The DSD condition is a "generalization" of Marshallian stability: at quantities "greater" than equilibrium, demand price is "less" than supply price, and thus prices provide the correct entry and exit signals. If $F(z_1,\cdots,z_n)$ is the function giving the prices that arise when industry productions are $z_1,\cdots,z_n$, and $(p^*,x^*,z_1^*, z_2^*,\cdots,z_n^*)$ is an equilibrium of $\mathcal{E}$, then $p^* = F(z_1^*,\cdots,z_n^*)$ and $p^* \cdot z_j^* = 0$ for all j since Z_j is a cone. The DSD condition requires that for each j, a "small increase in output" to $(1 + \lambda) z_j^*$ for λ small, positive, must lead to prices $F(z_1^*,\cdots,(1 + \lambda) z_j^*,\cdots,z_n^*) = p(\lambda)$ such that $p(\lambda)\cdot z_j^* \leqq 0$ so that entry is unprofitable. The DSD condition is not a pathological condition. It is satisfied at an equilibrium of $\mathcal{E}$ if a weak local version of the weak axiom of revealed preference holds at the equilibrium or if the consumer sector acts as a single consumer near the equilibrium. However, it is also not pathological for a specific equilibrium of $\mathcal{E}$ to fail to satisfy the DSD condition. Failure of DSD can occur in a two commodity economy in which the aggregate production set is a cone and utility functions are homogeneous.

When firms in an industry do not have identical production sets the DSD condition must be carefully defined. At an equilibrium $(p^*,x^*,z_1^*,\cdots,z_n^*)$ of $\mathcal{E}$, the DSD condition must be applied to those firms which are not yet active and will therefore depend on the industry outputs z_j^*. If there is a positive measure of firms which are inactive in the equilibrium and each of which can "approximately" produce y, then for DSD to be satisfied, a small change in production of λy for λ small, positive must lead to prices $p(\lambda)$ such that $p(\lambda)\cdot y \leqq 0$ so that entry is unprofitable. If the firms in industry j can be ordered in terms of efficiency (e.g. $Z(j,\beta) \subset Z(j,\beta')$ if $\beta > \beta'$) and only firms $[0, \beta^o]$ are active in equilibrium then DSD must be checked by "entry" of the next most efficient firms, those in $(\beta^o, \beta^o + \epsilon)$, or firms of type $\beta^o +$.

Though DSD is a static condition, a dynamic process involving the entry and exit of firms could be developed. As opposed to the price dynamics of price adjusting tatonnement, prices would change because of the entry and exit of firms in response to the prevailing prices. The stability conditions for this entry dynamics would differ from those for tatonnement in much the same way as Marshallian stability differs from Walrasian stability.

Remark 2

Theorem 3 is often referred to as the first theorem of classical welfare economics; it asserts that competitive equilibria are Pareto efficient. Even with DSD added as a requirement for equilibrium, the conclusion will still of course hold. Theorem 4 is the second theorem of classical welfare economics; it asserts that Pareto efficient allocations are perfectly competitive equilibria subject to a suitable assignment of ownership. Here the result is somewhat different than in the Arrow-Debreu theory. First, we observe that convexity of the set of aggregate production possibilities is not assumed. Such convexity is one of the central hypotheses in standard treatments of the second welfare theorem, but here it is a consequence of small efficient scale and the absence of externalities (Richter's Theorem). Next we note, that our definition of perfectly competitive equilibrium does not require DSD. Even if we succeed (by suitability assigning ownership) to support a given efficient allocation as a perfectly competitive equilibrium, DSD may fail; therefore, prices may give the wrong entry signals, and in this case free entry and exit may drive the economy away from the given efficient allocation. However, suppose that preferences are homothetic. If the efficient allocation is supported as a perfectly competitive equilibrium, and if endowments and ownership are distributed so as to be proportional among individuals, then equilibrium will satisfy DSD. This suggests that rather than convexity, small efficient scale and the possibility of assigning ownership so that prices give the correct entry signals may be the important considerations in formulating the second welfare theorem.

Remark 3

The model can be extended to include externalities. We will indicate how the existence theorem, Theorem 2, can be modified to include production externalities such that each firm's production set depends on the aggregate production of each industry. Using the notation of Remark 1, where z_k is the aggregate production of industry k, let $Z(j,\beta, z_1,\cdots,z_n)$ be the production set of firm j,β when the aggregate productions are $z_1,\cdots,z_n$. Then the aggregate production set (by industry)

$\mathcal{Z} = \{ (z_1,\cdots,z_n) : z_j \in \int Z(j, \beta, z_1,\cdots,z_n)d\beta$ for each $j\}$,

$\mathcal{Z}_j = \{ z_j : (z_1,\cdots,z_n) \in \mathcal{Z}$ for some $z_1,\cdots,z_{j-1}, z_{j+1},\cdots,z_n\}$,

and the aggregate production set $\bar{\mathcal{Z}} = \{ \sum_{j=1}^{n} z_j : (z_1,\cdots,z_n) \in \mathcal{Z}\}$ may not be convex. Let $\hat{\mathcal{Z}}$ (respectively $\hat{\mathcal{Z}}_j$) be the smallest closed convex cone containing $\bar{\mathcal{Z}}$ (respectively $\mathcal{Z}_j$).

Condition (i) of Theorem 2 is unchanged while conditions (ii), (iii) and (iv) are modified to: (ii)' for all $(z_1,\cdots,z_n) \in R^{\ell n}$, $j \in \{ 1,\cdots,n \}$, the correspondence $\beta \to Z(j,\beta,z_1,\cdots,z_n)$ is closed valued with a measurable graph and $0 \in Z(j,\beta,z_1,\cdots,z_n)$ for all β,

(iii)' $\mathcal{Z}$ is closed, $\hat{\mathcal{Z}} \cap (-\hat{\mathcal{Z}}) = \{0\}$, $\hat{\mathcal{Z}} \supset (-R^{\ell}_{+})$, and the $\hat{\mathcal{Z}}_j$ are positively semi independent

(iv)' for every i and j, the function $s_i(j,\cdot)$ is measurable. We add a condition which is similar to condition (iii):

(v) $\sum_j \int Z(j,\beta,0,\cdots,0)d\beta \supset (-R^{\ell}_{+})$ and

$$\left(\sum_j \int Z(j,\beta,0,\cdots,0)d\beta \right) \cap \left(-\sum_j \int Z(j,\beta,0,\cdots,0)d\beta \right) = \{0\}.$$

Finally we add a new condition which requires that the effect of externalities on production is "smooth" in the aggregate:

(vi) for all i and j, $Z(j,\beta,z_1,\cdots,z_n)$ and $s_i(j,\beta)Z(j,\beta,z_1,\cdots,z_n)$ are "integrably continuous" in $(z_1,\cdots,z_n)$ i.e. for any nonempty compact cube $K \subset R^{\ell}$ centered at the origin, the correspondences $(z_1,\cdots,z_n) \to K \cap \int Z(j,\beta,z_1,\cdots,z_n)d\beta$ and $(z_1,\cdots,z_n) \to K \cap \int s_i(j,\beta)Z(j,\beta,z_1,\cdots,z_n)d\beta$ are continuous.

In order to prove the existence of an equilibrium with externalities under these conditions we consider an associated economy with

$$w_i(p,z_1,\cdots,z_n) = \begin{cases} \sum_{j=1}^{n} \int s_i(j,\beta) \sup \{p\cdot z : z \in Z(j,\beta,z_1,\cdots,z_n)d\beta & \text{if it exists and is finite} \\ 0 & \text{otherwise.} \end{cases}$$

The existence of an equilibrium for the associated economy follows, as in the proof of Theorem 1, from application of an abstract economy existence theorem, where the industry j production set (constraint set) given $(p,x_1,\cdots,x_m,z_1,\cdots,z_n)$ is $K \cap \int Z(j,\beta,z_1,\cdots,z_n)d\beta$. The existence of an equilibrium for $\mathcal{E}$ then follows as in Lemma 1 (ii).

Remark 4

In the Arrow-Debreu theory the convexity of the set of aggregate productions Z plays a key role in establishing the existence of perfectly competitive equilibrium. As we have observed several times, without externalities, the required convexity is a consequence of Richter's Theorem, and so does not have to be assumed. But with externalities present, Z may not be convex; nevertheless, the previous remark demonstrated that a general existence theorem still obtains. We assert that the convexity of the set of aggregate production possibilities is not relevant to the problem of the existence of competitive equilibrium. The existence of perfectly competitive equilibrium depends fundamentally on small efficient scale, which in any case is the economically natural condition for perfect competition to apply. Downward sloping demand in the appropriate region, so that prices give the correct entry signals, remains an additional possible requirement. These remarks are very much related to the work of John S. Chipman [2] who makes some similar observations in a more specialized model.

Remark 5

McKenzie [5,6] introduces a non-marketed entrepreneurial factor which is private to the firm and owned by the owners of the firm, in order to prove the existence of equilibrium in an economy with convex, non-cone production sets via an existence theorem for an economy with constant returns to scale. Application of a similar technique to the economy $\mathcal{E}$ would require the introduction of a continuum of new commodities, one for each firm.[7] On the other hand, application of the technique to the associated economy $\mathcal{E}'$ would yield "ownership shares" $w_i(p) \,/\, \sum_k w_k(p)$ which vary (perhaps discontinuously) with price.

Remark 6

Clearly perfectly competitive equilibria are not efficient in the presence of externalities of the type introduced in Remark 3. Even though with externalities, the set of aggregate production possibilities is not convex, is it possible to state and prove a theorem that every efficient allocation is an equilibrium subject to the proper assignment of ownership shares and the appropriate Pigovian taxes?[8] This can be achieved ;e.g., by applying the ideas of Shafer and Sonnenschein [8] to the specification of an economy given in Remark 3. One may conclude the convexity of the set of aggregate production possibilities has nothing to do with the second welfare theorem (with externalities and corrective taxes.) Once again, small efficient scale is the appropriate condition. As before, the possibility of not being able to distribute ownership so that DSD is satisfied remains a problem for the result; and with externalities, this may not be possible.

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7. This was pointed out to us by A. Mas-Colell.

8. Again, this is done in a more explicit setting by Chipman [2].

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