THE n-PERSON BARGAINING GAME

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I. Introduction

When we consider a bargaining situation among persons, many complicated factors will be found which affect the final outcome to each participant of that bargaining. Even in the case of bargaining between two players, many theories have been proposed [1,6,7,8] which have given rise to much controversy [2,9].

In this paper we should like to generalize the idea of the Nash solution for a two-person bargaining game to the n-person case. Along this line of generalization, Harsanyi [3] has developed an interesting theory. We received recently his revised paper [4] on this problem, in which he replies to criticisms raised by Isbell [5]. The purpose of this paper is to develop a more generalized treatment of n-person bargaining games which includes Harsanyi's and Isbell's results in part as special cases.

II. Summary

In our n-person bargaining model, we take into consideration the preliminary meetings among the members of each subset—that is, of each coalition S —of the all-player set N as well as the plenary meeting among all the players. Even though the final payoff to each player may be determined at the plenary meeting, we shall assume that the final payoff u_i is constructed by adding up the incremental payoffs Δ_i^S which could be attributed to the effect of a player i advancing from one coalition to a larger one. On the basis of this final payoff structure, we define the equilibrium strategy set as the one which satisfies the internal balance among the incremental payoffs (Definition 1).

In this case we take into consideration explicitly the dependence of the outcome space which a coalition S could attain on the strategy chosen by the complementary coalition $\overline{S} = N - S$.

Our first task then was to prove the existence of such an equilibrium strategy set under certain assumptions (Theorem 1). The second task was to choose an optimal strategy set among equilibrium strategy sets. For that we defined the deflated final payoff v_i which could be rewritten as the weighted sum of the quantities which one coalition wishes to maximize and its complementary coalition wishes to minimize. Looking at this expression of the deflated final payoff, we defined the optimal strategy set by requiring the realization of the maximin property between each pair of complementary coalitions. However we made clear that the optimal strategy set will depend only on the maximin property between the one-player coalition and its complementary $(n-1)$-person coalition and not on the maximin property among the other pairs of complementary coalitions (Theorem 2).

III. The structure of the n-person bargaining game

We consider the structure of the n-person bargaining game Γ as follows.

Let N be the set of n players $1,2,\ldots,n$ of the game Γ. We call every subset S of N —including one member subsets and all member subset N, but not an empty set—a coalition. In playing the game Γ, each coalition $S \subseteq N$ announces its joint strategy Z^S. This strategy Z^S may be cooperative or non-cooperative, but at any rate, it must be one element of the given strategy space Θ^S of the coalition S. Let Θ be the product space of all Θ^S, $S \subseteq N$.

Each element Z of Θ consists of (2^n-1) strategies Z^S, that is

$$Z = \{Z^S;\ S \subseteq N\}\ , \text{ where } Z^S \in \Theta^S\ .$$

If a strategy set $Z(\in \Theta)$ is announced by all coalitions $S \subseteq N$, then a final payoff u_i to each player i, $i = 1,2,\ldots,n$, is determined by the all player strategy $Z^N(\in Z)$ according to the rule of the game Γ, that is

$$u_i = f_i(Z^N)\ ,\ i = 1,2,\ldots,n\ .$$

This situation may be considered to represent the fact that a final payoff to each member of the game will be determined at the plenary meeting among all members $1,2,\ldots,n$. It would appear, therefore, that the strategies announced by the coalitions—in other words the strategies decided at the various preliminary meetings among the members of the coalitions $S \subset N$ —have no effect on the final payoffs u_i, $i = 1,2,\ldots,n$. But, we assume that the final payoffs u_i, $i = 1,2,\ldots,n$, are constructed by adding up many components which depend on the strategies chosen by the coalitions S in the following way.

(i) Corresponding to a pair of strategies $Z^{(i)}$ and $Z^{N-(i)}$, chosen by a one-person coalition (i) and its complementary coalition $N-(i)$ respectively, the rule of the game guarantees a payoff $\sigma_i^{(i)} \equiv \Delta_i^{(i)}$ to player i. This will be written:

$$(1) \qquad \sigma_i^{(i)} \equiv \Delta_i^{(i)} = f_i(Z^{(i)},\ Z^{N-(i)}),\quad i = 1,2,\ldots,n\ .$$

This payoff $\Delta_i^{(i)}$ may be interpreted as the amount which player i can guarantee by himself.

(ii) Corresponding to a pair of strategies $Z^{(ij)}$ and $Z^{N-(ij)}$, the rule of the game guarantees a payoff $\sigma_\mu^{(ij)}$ to each player μ in the coalition (i,j). This will be called a dividend to player-μ from a coalition (i,j) and can be written:

$$(2) \qquad \sigma_{\mu}^{(ij)} = f_{\mu}(Z^{(ij)}, Z^{N-(ij)}), \quad \mu = i,j .$$

In this case, we define the quantities $\Delta_i^{(ij)}$ and $\Delta_j^{(ij)}$ by the following equations:

$$(3) \qquad \Delta_i^{(ij)} = \sigma_i^{(ij)} - \Delta_i^{(i)}, \quad \Delta_j^{(ij)} = \sigma_j^{(ij)} - \Delta_j^{(j)} ,$$

that is

$$(4) \qquad \sigma_i^{(ij)} = \Delta_i^{(i)} + \Delta_i^{(ij)} , \quad \sigma_j^{(ij)} = \Delta_j^{(j)} + \Delta_j^{(ij)} .$$

Then, $\Delta_i^{(ij)}$ [or $\Delta_j^{(ij)}$] may be interpreted as a net incremental payoff to player i [or j] which is attributable to the formation of a coalition (i,j).

(iii) Proceeding in this way, if we take any proper coalition $S \subset N$, a pair of strategies Z^S and $Z^{\overline{S}}$ announced by a coalition S and its complementary coalition $\overline{S}$ ($= N-S$) guarantees a payoff, that is a dividend, σ_i^S to each member i of S according to the rule of the game. This will be written:

$$(5) \qquad \sigma_i^S = f_i(Z^S, Z^{\overline{S}}), \quad i \in S .$$

Of course, in this case, the rule of the game also guarantees a payoff, that is a dividend, $\sigma_j^{\overline{S}}$ to each member j of $\overline{S}$. This will be written:

$$(6) \qquad \sigma_j^{\overline{S}} = f_j(Z^S, Z^{\overline{S}}) , \quad j \in S .$$

For simplicity, we will sometimes express (5) and (6) by

$$\sigma^S = f_S(Z^S, Z^{\overline{S}}) \quad \text{and} \quad \sigma^{\overline{S}} = f_{\overline{S}}(Z^S, Z^{\overline{S}}) ,$$

respectively.

At this stage, assuming that quantities Δ_i^R, that is the net incremental payoffs to player i from the coalitions R, are defined for all proper subsets R of S, we define Δ_i^S by the following equation:

$$(7) \qquad \Delta_i^S = \sigma_i^S - \sum_{\substack{R \ni i \\ R \subset S}} \Delta_i^R ,$$

that is

(8) $$\sigma_i^S = \sum_{\substack{R \ni i \\ R \subseteq S}} \Delta_i^R , \quad i \in S .$$

Then, this quantity Δ_i^S may be interpreted as a net incremental payoff to player i , which results from the fact that player i advanced from $(s-1)$ -person coalitions to an s-person coalition S , where s is the number of the members in the coalition $S(\subset N)$.

(iv) As mentioned above, a strategy Z^N chosen by all-player coalition N determines a final payoff $u_i \equiv \sigma_i^N$ to each player i , that is

(9) $$u_i \equiv \sigma_i^N = f_i(Z^N) , \quad i = 1,2,\ldots,n .$$

But we assume this final payoff σ_i^N , $i = 1,2,\ldots,n$, which results from a strategy set $Z = \{Z^S, S \subseteq N\}$ is made up from the net incremental payoffs Δ_i^S from all coalitions $S \subset N$, of which i is a member, and a net incremental payoff Δ_i^N to player i from all player coalition N , where Δ_i^N is defined through the equation

(10) $$\sigma_i^N = \sum_{\substack{R \ni i \\ R \subseteq N}} \Delta_i^R , \quad i = 1,2,\ldots,n ,$$

as in the case of Δ_i^S , $S \subset N$. We will call a final payoff $u = (u_1, u_2, \ldots, u_n)$ an equilibrium payoff if all net incremental payoffs Δ_i^R , constituting final payoffs u_i , possess some kind of an internal balance which will be defined in the next section.

IV. The equilibrium strategy set and the equilibrium payoff

Let $\xi = \{\xi^S, S \subseteq N\}$, $\xi^S \in \Theta^S$ be a given strategy set in the game Γ . Then for any coalition S and a given fixed strategy $\xi^{\bar{S}}$, where $\bar{S}$ is a complementary coalition of S , we will represent a set of all dividends to the members i in S , resulting from the choice of a strategy $Z^S \in \Theta^S$, that is

(11) $$f_S(Z^S, \xi^{\bar{S}}) \equiv \{f_i(Z^S, \xi^{\bar{S}}), \quad i \in S\}$$

as a point in s-dimensional Euclidean space E^s , where s is the number of the members in S .

Let $P^S(\xi^{\bar{S}})$ be the set of all these points, that is

(12) $$P^S(\xi^{\bar{S}}) = \{f_S(Z^S, \xi^{\bar{S}});\ Z^S \in \Theta^S\} .$$

We should like to remark here that this outcome space $P^S(\xi^{\bar{S}})$ will depend on the strategy $\xi^{\bar{S}}$ chosen by the complementary coalition $\bar{S}$ of S . We assume that $P^S(\xi^{\bar{S}})$ is a compact and convex subset of E^s , and the upper-right boundary of $P^S(\xi^{\bar{S}})$ is represented by the equation

(13) $$H(x_i, x_j, \ldots, x_k\ ;\ \xi^{\bar{S}}) = 0 ,$$

where $S = (i,j,\ldots,k)$. We will denote this upper-right boundary by $H^S(\xi^{\bar{S}})$. For simplicity, we assume the differentiability of the function H with respect to each variable x_μ in it.

Now, in defining the equilibrium strategy set, we will discuss successively the two-person subgame ,..., the s-person subgame ,..., and the final n-person game in the following way:

(i) At first, we consider the two-person subgame $\Gamma^{(ij)}(\xi^{N-(ij)})$, for all two-person coalitions (i,j) in the game Γ , under the condition that the strategy chosen by the complementary coalition $N-(ij)$ is $\xi^{N-(ij)}$.

In this subgame $\Gamma^{(ij)}(\xi^{N-(ij)})$ between two players i and j , they can choose any strategy $Z^{(ij)}$ in $\Theta^{(ij)}$. If they choose a strategy $Z^{(ij)}$, then the resulting payoffs to each player i and j are given by (11) with $S = (i,j)$. Accordingly the outcome space of this subgame is given by (12) with $S = (i,j)$.

In this subgame $\Gamma^{(i,j)}(\xi^{N-(ij)})$, we require that the dividends resulting from $\xi^{(ij)}(\in \xi)$, that is

(14) $$\sigma_i^{(ij)} = f_i(\xi^{(ij)}, \xi^{N-(ij)}), \quad \sigma_j^{(ij)} = f_j(\xi^{(ij)}, \xi^{N-(ij)})$$

constitute the Nash bargaining solution, when we take the point

(15) $$d^{(ij)} \begin{cases} d_i^{(ij)} \equiv \Delta_i^{(i)} = f_i(\xi^{(i)}, \xi^{N-(i)}) , \\ d_j^{(ij)} \equiv \Delta_j^{(j)} = f_j(\xi^{(j)}, \xi^{N-(j)}) \end{cases}$$

as the claim point of this bargaining subgame. I should like to remark that here and in the following, we will use the definition of the Nash solution in the generalized sense which is given and the existence of which is proved in Harsanyi's paper [3].

Let

(16) $$a_\mu^{(ij)}(\xi^{N-(ij)}) = \left. \frac{\partial H(x_i, x_j; \xi^{N-(ij)})}{\partial x_\mu} \right|_{x = \sigma^{(ij)}, \ \mu = i,j} .$$

Then, the above condition will be expressed by

(17) $$H(\sigma_i^{(ij)}, \sigma_j^{(ij)}; \xi^{N-(ij)}) = 0 ,$$

and

(18) $$a_i^{(ij)}(\xi^{N-(ij)})\Delta_i^{(ij)} = a_j^{(ij)}(\xi^{N-(ij)})\Delta_j^{(ij)} ,$$

for all 2-person coalitions (i,j) , where $\Delta_i^{(ij)}$ and $\Delta_j^{(ij)}$ are defined by (3) .

The situation may occur in which the Nash solution could not be expressed by the conditions such as (18) and the ones which will appear later. However, I should like to remark that in proving the existence of an equilibrium strategy set, we will not rely upon the conditions expressed in (18), but just upon the fact that there exists at least one Nash solution in the subgame as proved in Harsanyi's paper [3]. But in section VI below, in our treatment of the optimal strategy, we will consider the cases in which the Nash solution of the subgame could be characterized by the analytical expressions, like (17) and (18).

(ii) Next, we consider 3-person subgames $\Gamma^{(ijk)}(\xi^{N-(ijk)})$, for all 3-person coalitions (ijk) in the game Γ, under the condition that the strategy chosen by the complementary coalition $N-(ijk)$ is $\xi^{N-(ijk)}$. In this subgame among players i,j and k, the domain of joint strategies $Z^{(ijk)}$ is $\Theta^{(ijk)}$, the resulting payoffs to each player i,j and k are given by (11), and the outcome space is given by (12), with $S = (ijk)$.

In this subgame $\Gamma^{(ijk)}(\xi^{N-(ijk)})$, we require that the dividends resulting from $\xi^{(ijk)}(\epsilon\ \xi)$, that is

(19) $$\sigma_i^S = f_i(\xi^S,\xi^{\overline{S}}),\ \sigma_j^S = f_j(\xi^S,\xi^{\overline{S}}),\ \sigma_k^S = f_k(\xi^S,\xi^{\overline{S}}),$$

where $S = (ijk)$, constitute the Nash solution, when we take the point:

(20) $$d^{(ijk)}\begin{cases} d_i^{(ijk)} \equiv \triangle_i^{(i)} + \triangle_i^{(ij)} + \triangle_i^{(ik)} \\ d_j^{(ijk)} \equiv \triangle_j^{(j)} + \triangle_j^{(ij)} + \triangle_j^{(jk)} \\ d_k^{(ijk)} \equiv \triangle_k^{(k)} + \triangle_k^{(ik)} + \triangle_k^{(jk)} \end{cases}$$

as the claim point of this bargaining subgame, where all $\triangle_i^{(i)}$, $\triangle_i^{(ij)}$ etc., are defined in the stage (i).

Let

(21) $$a_\mu^{(ijk)}(\xi^{N-(ijk)}) = \left.\frac{\partial H(x_i,x_j,x_k;\xi^{N-(ijk)})}{\partial x_\mu}\right|_{x = \sigma^{(ijk)}},\quad \mu = i,j,k .$$

Then, the above condition will be expressed by

(22) $$H(\sigma_i^S,\ \sigma_j^S,\ \sigma_k^S\ ;\ \xi^{\overline{S}}) = 0,$$

and

(23) $$a_i^S(\xi^{\overline{S}})\triangle_i^S = a_j^S(\xi^{\overline{S}})\triangle_j^S = a_k^S(\xi^{\overline{S}})\triangle_k^S,$$

for all 3-person coalitions $S = (ijk)$, where $\triangle_\mu^S$, $\mu = i,j,k$ are defined by (19), (20) and the relation (7).

(iii) In this way, assuming that $\triangle_i^R$, $i \in R$, which satisfy conditions such that (17), (18); (22), (23); and so on, are defined through the given strategy ξ for all proper subsets R of $S \subseteq N$, we consider the s-person subgame $\Gamma^S(\xi^{\bar{S}})$, under the condition that the strategy chosen by the complementary coalition $\bar{S}$ is $\xi^{\bar{S}}$. In this subgame $\Gamma^S(\xi^{\bar{S}})$ among the members in the coalition S, the domain of the joint strategies Z^S is Θ^S, the payoff to each member $i \in S$ is given by (11), and the outcome space $P^S(\xi^{\bar{S}})$ is given by (12). We require that the dividends of this subgame to each member μ of the coalition S through the strategy $\xi^S(\in \xi)$, that is

$$(24) \qquad \sigma_\mu^S = f_\mu(\xi^S, \xi^{\bar{S}}), \quad \mu = i,j,\ldots,k; \; S = (i,j,\ldots,k),$$

constitute the Nash bargaining solution in this subgame $\Gamma^S(\xi^{\bar{S}})$, when we take the point

$$(25) \qquad d^S : \; d_\mu^S = \sum_{\substack{R \ni \mu \\ R \subset S}} \Delta_\mu^R, \quad \mu = i,j,\ldots,k; \quad S = (i,j,\ldots,k),$$

as the claim point of this game.

Let

$$(26) \qquad a_\mu^S(\xi^{\bar{S}}) = \left. \frac{\partial H(x_i, x_j, \ldots, x_k; \xi^{\bar{S}})}{\partial x_\mu} \right|_{x = \sigma^S}, \quad \mu = i,j,\ldots,k; \; S = (i,j,\ldots,k),$$

and we define Δ_μ^S by the equation

$$(27) \qquad \Delta_\mu^S = \sigma_\mu^S - \sum_{\substack{R \ni \mu \\ R \subset S}} \Delta_\mu^R,$$

that is,

$$(28) \qquad \sigma_\mu^S = f_\mu(\xi^S, \xi^{\bar{S}}) = \sum_{\substack{R \ni \mu \\ R \subseteq S}} \Delta_\mu^R,$$

using the net incremental payoffs Δ_μ^R, for the proper subsets R of S, which are defined in the preceding subgames through the Nash solutions.

Then, the condition for $\sigma^S = \{\sigma_\mu^S\}$ to be the Nash solution of the subgame $\Gamma^S(\xi^{\bar{S}})$ will be expressed as follows:

$$(29) \qquad H(\sigma_i^S, \sigma_j^S, \ldots, \sigma_k^S; \xi^{\bar{S}}) = 0,$$

$$(30) \qquad a_i^S(\xi^{\bar{S}})\Delta_i^S = a_j^S(\xi^{\bar{S}})\Delta_j^S = \cdots = a_k^S(\xi^{\bar{S}})\Delta_k^S,$$

where $S = (i,j,\ldots,k) \subseteq N$.

Definition 1. If in a strategy set $\xi = \{\xi^S, S \subseteq N\}$, $\xi^S \in \Theta^S$, each strategy $\xi^S(\in \xi)$, $s \geq 2$, $S \subseteq N$, where s is the number of the members in S , gives the Nash solution for each subgame $\Gamma^S(\xi^{\bar{S}})$ among the members of the coalition S , taking the point d^S as defined by (25) as the claim point of this game, then we call ξ an equilibrium strategy set, and we call the set of the net incremental payoffs $\Delta = \{\Delta_\mu^S ; \mu \in S , S \subseteq N\}$ an equilibrium incremental payoff set, and the resulting set of the final payoffs $u = (u_1, u_2, \ldots, u_n)$, where

$$u_i = \sum_{\substack{R \ni i \\ R \subseteq N}} \Delta_i^R , \tag{31}$$

an equilibrium payoff.

V. The existence of an equilibrium strategy set

In order to prove the existence of an equilibrium strategy set, we assume our n-person game Γ satisfies the following additional conditions:

Assumption 1. In each subgame $\Gamma^S(Z^{\bar{S}})$, $S \subseteq N$, for the given $Z^{\bar{S}}(\in \Theta^{\bar{S}})$, the outcome space $P^S(Z^{\bar{S}})$ is a compact and convex subset of s-dimensional Euclidean space E^s .

Definition 2. If $f_S(Z^S, Z^{\bar{S}}) \in H^S(Z^{\bar{S}})$, then we call Z^S good for $Z^{\bar{S}}$. If Z^S is good for $Z^{\bar{S}}$, and $Z^{\bar{S}}$ is good for Z^S , then Z^S and $Z^{\bar{S}}$ are mutually good.

Assumption 2. If Z^S is good for some $Z^{\bar{S}}$, then Z^S is good for all $Z'^{\bar{S}} \in \Theta^{\bar{S}}$. (We should like to remark that in some cases we can do without this strong assumption as will be shown later on.)

Assumption 3. If Z^S and Z'^S are both good for some $Z^{\bar{S}}$, then

$$H^{\bar{S}}(Z^S) = H^{\bar{S}}(Z'^S) .$$

<u>Assumption 4.</u> If Z^S is good for some $Z^{\bar{S}}$, and $H^S(Z^{\bar{S}}) = H^S(Z'^{\bar{S}})$ for some $Z^{\bar{S}}$ and $Z'^{\bar{S}}$ ($\neq Z^{\bar{S}}$), then

$$f_S(Z^S, Z^{\bar{S}}) = f_S(Z^S, Z'^{\bar{S}}) .$$

<u>Theorem 1.</u> Under the assumptions $1 \approx 4$, there exists at least one equilibrium strategy set $\xi = \{\xi^S,\ S \subseteq N\}$ for the n-person bargaining game Γ.

<u>Proof:</u>

(i) At first we will prove that, for any two complementary coalitions S and $\bar{S}$, there exists a pair of mutually good strategies $y^S (\in \Theta^S)$ and $y^{\bar{S}} (\in \Theta^{\bar{S}})$.

Now we choose arbitrarily a pair of strategies $Z^S (\in \Theta^S)$ and $Z^{\bar{S}} (\in \Theta^{\bar{S}})$. Then, there exists $y^S (\in \Theta^S)$ such that

$$f_S(y^S, Z^{\bar{S}}) \in H^S(Z^{\bar{S}}) , \tag{32}$$

and there exists $y^{\bar{S}} (\in \Theta^{\bar{S}})$ such that

$$f_{\bar{S}}(y^S, y^{\bar{S}}) \in H^{\bar{S}}(y^S) , \tag{33}$$

by the definition of $H^S(Z^{\bar{S}})$ and $H^{\bar{S}}(y^S)$.

Since (32) means y^S is good for $Z^{\bar{S}}$, by Assumption 2, y^S is also good for $y^{\bar{S}}$, that is

$$f_S(y^S, y^{\bar{S}}) \in H^S(y^{\bar{S}}) . \tag{34}$$

Accordingly, (33) and (34) show that y^S and $y^{\bar{S}}$ are mutually good strategies (see Fig. 1). In this way, we obtain a strategy set $y = \{y^S,\ S \subseteq N\}$

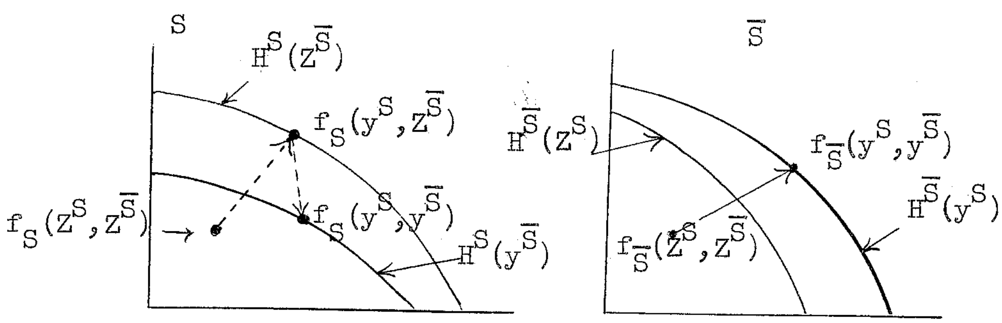

Figure 1.

in which each pair y^S and $y^{\bar{S}}$ for two complementary coalitions S and $\bar{S}$ form a pair of mutually good strategies. In this strategy set y, y^N is an arbitrary element in Θ^N.

Note: For example, in the following case, we can prove the existence of a pair of mutually good strategies y^S and $y^{\bar{S}}$ without relying on Assumption 2. We assume Θ^S and $\Theta^{\bar{S}}$ are compact and convex subsets in E^s and E^{n-s} respectively. For any $Z^S \in \Theta^S$, let $\varphi_1(Z^S)$ be a set of all elements $Z^{\bar{S}} \in \Theta^{\bar{S}}$ such that $Z^{\bar{S}}$ is good for Z^S, that is

$$\varphi_1(Z^S) = \{Z^{\bar{S}} \mid f_{\bar{S}}(Z^S, Z^{\bar{S}}) \in H^{\bar{S}}(Z^S)\},$$

and for any $Z^{\bar{S}} \in \Theta^{\bar{S}}$, let $\varphi_2(Z^{\bar{S}})$ be a set of all $Z^S \in \Theta^S$ such that Z^S is good for $Z^{\bar{S}}$, that is

$$\varphi_2(Z^{\bar{S}}) = \{Z^S \mid f_S(Z^S, Z^{\bar{S}}) \in H^S(Z^{\bar{S}})\}.$$

Now we define a mapping ψ from $\Theta^S \times \Theta^{\bar{S}}$ to subsets of $\Theta^S \times \Theta^{\bar{S}}$ by

$$\psi : (Z^S, Z^{\bar{S}}) \rightarrow (\varphi_2(Z^{\bar{S}}), \varphi_1(Z^S)).$$

If φ_1 and φ_2 are upper semicontinuous, and $\varphi_1(Z^S)$ and $\varphi_2(Z^{\bar{S}})$ are convex, then by Kakutani's fixed point theorem, there exist y^S and $y^{\bar{S}}$ such that

$$(y^S, y^{\bar{S}}) \in (\varphi_2(y^{\bar{S}}), \varphi_1(y^S)).$$

That is, there exists a pair of mutually good strategies y^S and $y^{\bar{S}}$.

(ii) Starting from a strategy set $y = \{y^S, S \subseteq N\}$ constructed in stage (i), we will attain an equilibrium strategy set ξ through several steps of the modification of strategies.

1) We consider all two-person bargaining subgames $\Gamma^{(ij)}(y^{N-(ij)})$, for all two-person coalitions $(ij) \subseteq N$. In this subgame, the strategy space is $\Theta^{(ij)}$ and the outcome space is $P^{(ij)}(y^{N-(ij)})$ with the upper-boundary $H^{(ij)}(y^{N-(ij)})$. We take the point

$$d^{(ij)} = (f_i(y^{(i)}, y^{N-(i)}), f_j(y^{(j)}, y^{N-(j)})) \tag{35}$$

as the claim point of this bargaining subgame $\Gamma^{(ij)}(y^{N-(ij)})$, and we define

(36) $$\begin{cases} \sigma_i^{(i)} \equiv \Delta_i^{(i)} = f_i(y^{(i)}, y^{N-(i)}) , \\ \sigma_j^{(j)} \equiv \Delta_j^{(j)} = f_j(y^{(j)}, y^{N-(j)}) . \end{cases}$$

Let $\sigma^{(ij)} = (\sigma_i^{(ij)}, \sigma_j^{(ij)})$ be the Nash solution of this bargaining subgame. Then, by the definition of the Nash solution,

(37) $$\sigma^{(ij)} \in H^{(ij)}(y^{N-(ij)}) ,$$

and there exists $\xi^{(ij)}(\in \Theta^{(ij)})$ such that

(38) $$f_{(ij)}(\xi^{(ij)}, y^{N-(ij)}) = \sigma^{(ij)} .$$

Then, we define $\Delta_i^{(ij)}$ and $\Delta_j^{(ij)}$ through the equations

(39) $$\sigma_i^{(ij)} = \Delta_i^{(i)} + \Delta_i^{(ij)} , \quad \text{and} \quad \sigma_j^{(ij)} = \Delta_j^{(j)} + \Delta_j^{(ij)} ,$$

where $\Delta_i^{(i)}$ and $\Delta_j^{(j)}$ are defined by (36). Moreover, we define $\xi^{(i)}$ as

(40) $$\xi^{(i)} = y^{(i)}, \quad i = 1,2,\ldots,n .$$

Now we construct a new strategy set $y' = \{y'^S; S \subseteq N\}$ from a given strategy set $y = \{y^S; S \subseteq N\}$, by defining

$$y'^{(ij)} = \xi^{(ij)} , \text{ for all two-person coalitions } (ij),$$
$$y'^S = y^S , \text{ for all other coalitions } S ,$$

where $\xi^{(i)}$ and $\xi^{(ij)}$ are defined by (40) and (38) respectively.

Now, all the terms which may be affected by this modification from y to y' are $f_{(ij)}(y^{(ij)}, y^{N-(ij)})$, $f_{N-(ij)}(y^{(ij)}, y^{N-(ij)})$ and $H^{N-(ij)}(y^{(ij)})$. But, $f_{(ij)}(y^{(ij)}, y^{N-(ij)})$ is changed so as to give the Nash solution to the subgame $\Gamma^{(ij)}(y'^{N-(ij)})$, by our definition of y'. From Assumption 3 and the definition of $y'^{(ij)}$, it is clear that

(41) $$H^{N-(ij)}(y^{(ij)}) = H^{N-(ij)}(y'^{(ij)}) .$$

Also we have

(42) $$f_{N-(ij)}(y^{(ij)}, y^{N-(ij)}) \in H^{N-(ij)}(y^{(ij)})$$

by the property of y. That is, $y^{N-(ij)}$ is good for $y^{(ij)}$. Accordingly, by Assumption 2, $y^{N-(ij)}$ is good for $y^{(ij)}$ and $y'^{(ij)}$, and we have (41). Accordingly, from Assumption 4, we have

(43) $$f_{N-(ij)}(y'^{(ij)}, y^{N-(ij)}) = f_{N-(ij)}(y^{(ij)}, y^{N-(ij)}) .$$

Consequently, from (41) and (43), we see that, by changing from y to y', we just obtain the Nash solution of the subgame $\Gamma^{(ij)}(y^{N-(ij)})$, without giving any variation to other terms.

2) Now, with respect to the new strategy set y', we consider 3-person subgames $\Gamma^{(ijk)}(y'^{N-(ijk)})$ for all 3-person coalitions (ijk). In this subgame $\Gamma^{(ijk)}(y'^{N-(ijk)})$, the strategy space is $\Theta^{(ijk)}$, and the outcome space is $P^{(ijk)}(y'^{N-(ijk)})$ with the upper boundary $H^{(ijk)}(y'^{N-(ijk)})$. Taking the point $d^{(ijk)}$ defined by the relation (20) as the claim point of this game $\Gamma^{(ijk)}(y'^{N-(ijk)})$, we can find a strategy $\xi^{(ijk)}$ which gives the Nash solution $\sigma^{(ijk)}$ to this subgame. Then, we construct a new strategy set $y'' = \{y''^S;\ S \subseteq N\}$ by the following rule:

$$y''^{(ijk)} = \xi^{(ijk)}, \text{ for all 3-person coalitions } (ijk),$$
$$y''^S = y'^S, \text{ for all other coalitions } S \subseteq N .$$

It can be easily seen, as in step 1), that by changing from y' to y'', we just obtain the Nash solution of the subgame $\Gamma^{(ijk)}$, without giving any variation to other terms.

3) Proceeding in this way, we can attain a strategy set $\xi = \{\xi^S;\ S \subseteq N\}$ in which each strategy ξ^S gives the Nash solution to the subgame $\Gamma^S(\xi^{\overline{S}})$, when we take the point $d^S = \{d_i^S,\ i \in S\}$ defined by

$$d_i^S = \sum_{\substack{R \ni i \\ R \subset S}} \Delta_i^R , \quad i \in S$$

as the claim point of this subgame. Accordingly, the resulting strategy set $\xi = \{\xi^S; S \subseteq N\}$ is an equilibrium strategy set of the n-person bargaining game Γ. This proves the existence of an equilibrium strategy set ξ as we intended.

VI. The optimal strategy set and the solution of the game.

In the preceding section, we defined the equilibrium strategy set of the n-person bargaining game Γ and proved the existence of such strategy sets. But, usually, there may be many such strategy sets, as will be seen later in Example 3. So, in this section we will try to define the optimal strategy set among the equilibrium strategy sets and to prove its existence.

Let $\xi = \{\xi^S; S \subseteq N\}$ be an equilibrium strategy set of the n-person bargaining game Γ. Then, we define the quantities W^S, U^S and v_i as follows:

(44) $$W^S = \sum_{\mu \in S} \sum_{\substack{R \ni \mu \\ R \subseteq S}} a_\mu^R(\xi^{\overline{R}}) \Delta_\mu^R ,$$

(45) $$U^S = \sum_{\mu \in S} a_\mu^S(\xi^{\overline{S}}) \Delta_\mu^S ,$$

(46) $$v_i = \sum_{\substack{R \ni i \\ R \subseteq N}} a_i^R(\xi^{\overline{R}}) \Delta_i^R , \quad i = 1,2,\ldots,n ,$$

where $a_\mu^R(\xi^{\overline{R}})$ and Δ_μ^R are defined by (26) and (27) respectively, and in the case where R is one-person coalition $R = (i)$, then we define $a_i^{(i)}(\xi^{N-(i)}) = 1$ for $i = 1,2,\ldots,n$. Then, from these definitions, it is clear that

(47) $$W^S = \sum_{R \subseteq S} U^R , \text{ for each } S \subseteq N .$$

From these equations, solving for U^R, we have

(48) $$U^R = \sum_{S \subseteq R} (-1)^{r-s} W^S ,$$

where r and s are the numbers of the members in R and S respectively.

In this section we treat the case where the analytical characterization of the Nash solution of each subgame considered in Section III is given by conditions (29) and (30) (see Harsanyi [3] and [4]). Then, from (30) we have

$$a_\mu^S(\xi^{\bar{S}})\Delta_\mu^S = \frac{1}{s}U^S \text{ , for each } \mu \in S \text{ , and for each } S \subseteq N \text{ .} \tag{49}$$

Accordingly, from (46), (48) and (49), we have

$$v_i = \sum_{\substack{R \ni i \\ R \subseteq N}} \frac{1}{r} U^R = \sum_{\substack{R \ni i \\ R \subseteq N}} \sum_{S \subseteq R} \frac{1}{r}(-1)^{r-s} W^S \text{ .} \tag{50}$$

So, following the same reasoning as Harsanyi [9], we have

$$v_i = \sum_{\substack{S \ni i \\ S \subseteq N}} \frac{(s-1)!(n-s)!}{n!} (W^S - W^{\bar{S}}), \ i = 1,2,\ldots,n \text{ .} \tag{51}$$

Now, as was shown in (31), the final payoff u_i to each player i , resulting from the strategy set $\xi = \{\xi^S; S \subseteq N\}$, has the structure of the sum of the net incremental payoffs Δ_i^S , that is

$$u_i = \sigma_i^N = \sum_{\substack{S \ni i \\ S \subseteq N}} \Delta_i^S \text{ , } i = 1,2,\ldots,n \text{ .} \tag{52}$$

Accordingly, the quantity v_i defined by (46) as the weighted sum of Δ_i^S is not the final payoff itself, but we can give v_i the following interpretation. As far as we follow the idea of the Nash solution in each subgame $\Gamma^S(\xi^{\bar{S}})$, we do not consider the incremental payoffs Δ_i^S themselves, but rather the modified incremental payoffs $a_i^S(\xi^{\bar{S}})\Delta_i^S$, for $i \in S$; and we try to equate these modified payoffs among the members of the coalition S . Therefore, $a_i^S(\xi^{\bar{S}})$ has the meaning of a deflator in comparing incremental payoffs among the members of the coalition S .

Accordingly, for each player i who wishes to obtain the relative advantage of his final payoff among the players of the game Γ, it is natural to search for a strategy set which maximizes the sum of the modified incremental payoffs, v_i. In this sense, v_i may be called a deflated final payoff to player i, $i = 1,2,\ldots,n$.

Now, from the expression of v_i in (51), it is natural to assume that each member of a coalition S will try to maximize $(W^S - W^{\bar{S}})$, and each member of a complementary coalition $\bar{S}$ will try to maximize $(W^{\bar{S}} - W^S)$, that is, to minimize $(W^S - W^{\bar{S}})$. So that, roughly speaking, an equilibrium strategy set which gives the maximin value of $(W^S - W^{\bar{S}})$ for each pair of the coalitions S and $\bar{S}$ may be defined as the optimal strategy set of the n-person bargaining game Γ.

Now let $\xi = \{\xi^S; S \subseteq N\}$ be an arbitrary equilibrium strategy set of the game Γ, and let us consider any coalition S, and the difference $(W^S - W^{\bar{S}})$. If all strategies $\xi^R (\in \xi)$ other than ξ^S, $\xi^{\bar{S}}$, and ξ^N are fixed, then it can be easily seen from (28) and (44) that $(W^S - W^{\bar{S}})$ is a function of ξ^S and $\xi^{\bar{S}}$.

Let this function be written

(53) $W^S - W^{\bar{S}} = g_{S,\bar{S}}(\xi^S, \xi^{\bar{S}}$; all other strategies ξ^R, $R \neq S, \bar{S}$, $R \subset N$ being kept constant).

Then, in this case, each member of the coalition S will wish to use a joint strategy ξ^S which maximizes the value of the function $g_{S,\bar{S}}$ and each member of the complementary coalition $\bar{S}$ will wish to use a joint strategy $\xi^{\bar{S}}$ which minimizes the value of the function $g_{S,\bar{S}}$. For this function $g_{S,\bar{S}}$, we consider just the following kind of variations of ξ^S and $\xi^{\bar{S}}$. That is, ξ^S and $\xi^{\bar{S}}$ are allowed to change to ξ'^S and $\xi'^{\bar{S}}$ such that, by choosing the appropriate ξ'^N, the whole set of strategies ξ'^S, $\xi'^{\bar{S}}$; ξ^R, $R \neq S, \bar{S}$, $R \subset N$ and ξ'^N constitute an equilibrium strategy set.

Definition 3. If an equilibrium strategy set $\xi_o = \{\xi_o^S ; S \subseteq N\}$ satisfies the following conditions (54) for all $S \subset N$, then ξ_o is called an optimal strategy set of the n-person bargaining game Γ, and the resulting final payoff set $u_o = (u_1^o, u_2^o, \ldots, u_n^o)$ is called the solution of the game Γ.

$$(54) \quad g_{S,\bar{S}}(\xi_o^S, \xi_o^{\bar{S}} ; \text{all other strategies } \xi_o^R, R \neq S, \bar{S}, R \subset N \text{ being fixed})$$
$$= \max_{\xi^S} \min_{\xi^{\bar{S}}} g_{S,\bar{S}}(\xi^S, \xi^{\bar{S}} ; \text{all other strategies } \xi_o^R, R \neq S, \bar{S}, R \subset N \text{ being fixed}),$$

where the function $g_{S,\bar{S}}$ is defined by (53) and the maximin is taken with respect to ξ^S and $\xi^{\bar{S}}$ which will constitute an equilibrium strategy set when taken together with all other strategies ξ_o^R, $R \neq S, \bar{S}, R \subset N$, and with an appropriate strategy ξ^N.

Now, for the optimal strategy set ξ_o as defined by Definition 3, let us look at the pair of strategies ξ_o^S and $\xi_o^{\bar{S}}$, where S is neither a one-player coalition nor an (n-1)-person coalition. Then, ξ_o satisfies the condition (54) with respect to the coalitions S and $\bar{S}$. Let ξ^S and $\xi^{\bar{S}}$ be any strategies of the coalitions S and $\bar{S}$, respectively, which constitute an equilibrium strategy set when taken together with ξ_o^R, $R \neq S, \bar{S}, R \subset N$ and with an appropriate strategy ξ^N. Let this equilibrium strategy set be ξ^*. Then, applying the definition of an equilibrium strategy set to ξ^*, we know that ξ^S gives the Nash solution to the subgame $\Gamma^S(\xi^{\bar{S}})$. Let this solution be

$$(55) \qquad \sigma^{*S} = f_S(\xi^S, \xi^{\bar{S}}),$$

where

$$(56) \qquad \sigma_i^{*S} = \sum_{\substack{R \ni i \\ R \subset S}} \Delta_i^R + \Delta_i^{*S}, \quad \text{for } i \in S ;$$

since Δ_i^R, for $R \subset S$, is determined by ξ_o^R and $\xi_o^{\bar{R}}$, they are not affected by ξ^S and $\xi^{\overline{S}}$.

Now, let k be any player who is not a member of the coalition S . In the subgame $\Gamma^{S+(k)}(\xi_o^{\overline{S+(k)}})$, $\xi_o^{S+(k)}$ gives the Nash solution

(57) $$\sigma^{S+(k)} = f_{S+(k)}(\xi_o^{S+(k)}, \xi_o^{\overline{S+(k)}}) ,$$

since $\xi_o^{S+(k)}$ and $\xi_o^{\overline{S+(k)}}$ are elements of an equilibrium strategy set ξ_o . Of course, $\xi_o^{S+(k)}$ and $\xi_o^{\overline{S+(k)}}$ may also be considered elements of an equilibrium strategy set ξ^* .

If we consider $\xi_o^{S+(k)}$ as an element of ξ_o , then the claim point of the subgame $\Gamma^{S+(k)}(\xi_o^{\overline{S+(k)}})$ is the point $d^{S+(k)}$, given by

(58) $$d_i^{S+(k)} = \sum_{\substack{R \ni i \\ R \subset [S+(k)] \\ R \neq S}} \Delta_i^R + \Delta_i^S , \text{ for } i \in S ,$$

and

(59) $$d_k^{S+(k)} = \sum_{\substack{R \ni k \\ R \subset [S+(k)]}} \Delta_k^R .$$

On the other hand, if we consider $\xi_o^{S+(k)}$ as an element of ξ^* , then the claim point of the subgame $\Gamma^{S+(k)}(\xi_o^{\overline{S+(k)}})$ is the point $d^{*S+(k)}$, given by

(60) $$d_i^{*S+(k)} = \sum_{\substack{R \ni i \\ R \subset [S+(k)] \\ R \neq S}} \Delta_i^R + \Delta_i^{*S} , \text{ for } i \in S ,$$

and

(61) $$d_k^{*S+(k)} = \sum_{\substack{R \ni k \\ R \subset [S+(k)]}} \Delta_k^R ,$$

where Δ_i^{*S} is defined by (56) . That is, in the subgame $\Gamma^{S+(k)}(\xi_o^{\overline{S+(k)}})$, $\sigma^{S+(k)}$ as defined by (57) is the Nash solution whether we take $d^{S+(k)}$, defined by (58) and (59), as the claim point, or $d^{*S+(k)}$, defined by (60) and (61), as the claim point of the game.

Accordingly, from the conditions for the Nash solution, in the (s+1)-dimensional Euclidean space, the three points $\sigma^{S+(k)}$, $d^{S+(k)}$ and $d*^{S+(k)}$ must lie on the same straight line. But, as can be seen from (59) and (61), the two points $d^{S+(k)}$ and $d*^{S+(k)}$ have the same $(s+1)^{\underline{th}}$ coordinate

(62) $$d_k^{S+(k)} = d*_k^{S+(k)} .$$

Therefore, in order to meet the condition mentioned above, the two points $d^{S+(k)}$ and $d*^{S+(k)}$ must coincide, except in the case where $\sigma^{S+(k)}$ has the same $(s+1)^{\underline{th}}$ coordinate (59) as $d^{S+(k)}$ and $d*^{S+(k)}$, that is, the case where $\Delta_k^{S+(k)} = 0$.

We assume that in the case we are treating here, this kind of singularity never occurs.

Accordingly, we have

(63) $$d_j^{S+(k)} = d*_j^{S+(k)} , \text{ for all } j \in S+(k) .$$

From (58), (60) and (63), we have

(64) $$\Delta_i^S = \Delta*_i^S , \text{ for } i \in S .$$

So that we have

(65) $$\sigma_i^S = \sigma*_i^S , \text{ for } i \in S ,$$

where $\sigma*_i^S$ is defined by (55) and

(66) $$\sigma_i^S = f_i(\xi_o^S, \xi_o^{\bar{S}}) , \text{ for } i \in S .$$

Therefore

(67) $$f_S(\xi_o^S, \xi_o^{\bar{S}}) = f_S(\xi^S, \xi^{\bar{S}}) ,$$

and, similarly

(68) $$f_{\bar{S}}(\xi_o^S, \xi_o^{\bar{S}}) = f_{\bar{S}}(\xi^S, \xi^{\bar{S}}) .$$

Accordingly, we have shown that, for a coalition S , which is different from a one-person coalition and an (n-1)-person coalition, in order for ξ^S to be an element of an equilibrium strategy set $\xi*$, it must be equivalent to ξ_o^S as a strategy. So that for any equilibrium strategy set ξ_o, its components ξ_o^S and $\xi_o^{\overline{S}}$ satisfy the maximin condition of (54). This is true, because any ξ^S and $\xi^{\overline{S}}$ which satisfy the conditions for taking the maximin value of the function $g_{S,\overline{S}}$ in (54) must be equivalent to ξ_o^S and $\xi_o^{\overline{S}}$ respectively.

Next, we consider the case where S is a one-person coalition (i) and $\overline{S}$ is an (n-1)-person coalition $N-(i)$, i.e., the case where an equilibrium strategy set ξ^* is given as follows:

$$\xi^* = \{\xi^{(i)},\ \xi^{N-(i)};\ \xi_o^R \text{ for } R \subset N,\ R \neq (i),\ N-(i);\ \text{and } \xi^N\}, \tag{69}$$

where ξ^N is to be chosen so that ξ^* constitutes an equilibrium strategy set.

Let

$$\Delta^{*(i)}_i = f_i(\xi^{(i)},\ \xi^{N-(i)}). \tag{70}$$

Then, in the subgame $\Gamma^{(ik)}(\xi_o^{N-(ik)})$, the strategy $\xi_o^{(ik)}$ gives the Nash solution $\sigma^{(ij)}$, whether we take the point $(\Delta_i^{(i)},\ \Delta_k^{(k)})$, where

$$\Delta_i^{(i)} = f_i(\xi_o^{(i)},\ \xi_o^{N-(i)}),$$
$$\Delta_k^{(k)} = f_k(\xi_o^{(k)},\ \xi_o^{N-(k)}), \tag{71}$$

as the claim point, or the point $(\Delta^{*(i)}_i,\ \Delta^{*(k)}_k)$ as the claim point, where $\Delta^{*(i)}_i$ is given by (70) and

$$\Delta^{*(k)}_k = f_k(\xi_o^{(k)},\ \xi_o^{N-(k)}) = \Delta_k^{(k)}. \tag{72}$$

Following the same reasoning as above, we conclude that

$$\Delta^{*(i)}_i = \Delta_i^{(i)}. \tag{73}$$

Now, it is clear that in the subgames $\Gamma^{N-(i)}(\xi_o^{(i)})$ which result from ξ_o and $\Gamma^{N-(i)}(\xi^{(i)})$ which result from ξ^*, as defined by (69), the claim points are the same point in both cases. However, the outcome spaces

$P^{N-(i)}(\xi_o^{(i)})$ and $P^{N-(i)}(\xi^{(i)})$ of these games may differ, and consequently, the Nash solutions of $\Gamma^{N-(i)}(\xi_o^{(i)})$ and $\Gamma^{N-(i)}(\xi^{(i)})$ may differ. This difference gives rise to the difference in the value of $(W^{(i)} - W^{N-(i)})$ according to whether ξ_o is used or ξ^* is used.

From the above reasoning we can state the following theorem:

Theorem 2. The optimal strategy set ξ_o which is defined by Definition 3 is equivalent to the equilibrium strategy set ξ_o which satisfies the following conditions for $i = 1,2,\ldots,n$:

(74) $$g_{(i),N-(i)}(\xi_o^{(i)}, \xi_o^{N-(i)}; \text{ all other strategies } \xi_o^R \text{ being fixed})$$
$$= \max_{\xi^{(i)}} \min_{\xi^{N-(i)}} g_{(i),N-(i)}(\xi^{(i)}, \xi^{N-(i)}; \text{ all other strategies}$$
$$\xi_o^R,\ R \neq (i),\ N-(i),\ R \subset N \text{ being fixed}) ,$$

where the maximin is taken with respect to $\xi^{(i)}$ and $\xi^{N-(i)}$ which will constitute an equilibrium strategy set when taken together with all other strategies ξ_o^R, $R \neq (i)$, $N-(i)$, $R \subset N$, and with an appropriate strategy ξ^N .

Example 1. If each hyper-surface $H^S(\xi^{\bar{S}})$ is an (s-1)-dimensional hyper-plane, with direction numbers (1,1,...,1), for all $S \subseteq N$, then it is clear that

(75) $$a_\mu^S(\xi^{\bar{S}}) = 1 \text{ , for all } \mu \in S \text{ and all } S \subseteq N .$$

Accordingly, in this case, we have

(76) $$W^S = \sum_{\mu \in S} \sum_{\substack{R \ni \mu \\ R \subseteq S}} \Delta_\mu^R$$
$$= \sum_{\mu \in S} \sigma_\mu^S = \sum_{\mu \in S} f_\mu(\xi^S, \xi^{\bar{S}}) .$$

So that, W^S is a function only of ξ^S and $\xi^{\bar{S}}$; that is, we can write

$$W^S = h_S(\xi^S, \xi^{\bar{S}}) .$$

Similarly we have

$$W^{\overline{S}} = \sum_{\upsilon \in \overline{S}} \sigma_{\upsilon}^{\overline{S}} = \sum_{\upsilon \in \overline{S}} f_{\upsilon}(\xi^{S}, \xi^{\overline{S}}) , \tag{77}$$

that is $W^{\overline{S}}$ is also a function only of ξ^S and $\xi^{\overline{S}}$, so we can write

$$W^{\overline{S}} = h_{\overline{S}}(\xi^S, \xi^{\overline{S}}) .$$

Accordingly, we have

$$\begin{aligned} W^S - W^{\overline{S}} &= h_S(\xi^S, \xi^{\overline{S}}) - h_{\overline{S}}(\xi^S, \xi^{\overline{S}}) \\ &= g_{S,\overline{S}}(\xi^S, \xi^{\overline{S}}) , \end{aligned} \tag{78}$$

that is, $W^S - W^{\overline{S}}$ is a function only of ξ^S and $\xi^{\overline{S}}$.

Consequently, in this case, there will exist—under the relevant conditions on the strategy spaces Θ^S —the maximin strategies ξ_o^S and $\xi_o^{\overline{S}}$ of the function $g_{S,\overline{S}}(\xi^S, \xi^{\overline{S}})$, for all $S \subseteq N$. That is, there will exist an optimal strategy ξ_o of the game. We note that in this case we have

$$v_i = \sum_{\substack{S \ni i \\ S \subseteq N}} \Delta_i^S = u_i , \quad i = 1,2,\ldots,n , \tag{79}$$

that is, v_i is equal to the final payoff u_i .

Example 2. If we have

$$a_{\mu}^S(\xi^{\overline{S}}) = a_{\mu} , \text{ for all } S \ni \mu , \text{ and all } S \subseteq N ,$$

then, as in Example 1, it will readily be seen that the difference $(W^S - W^{\overline{S}})$ is a function only of ξ^S and $\xi^{\overline{S}}$. So that, there will exist a maximin strategy ξ_o in this case. Moreover, we remark that, in this case, we have

$$v_i = \sum_{\substack{S \ni i \\ S \subseteq N}} a_i \Delta_i^S = a_i u_i , \tag{80}$$

and this corresponds to the case Harsanyi treated in his paper [4].

Example 3. Let us consider the three-person bargaining game Γ with $N = (1,2,3)$ defined as follows:

The strategy sets $\Theta^{(i)}$, $i = 1,2,3$, of each player i is the set of real numbers such that,

$$\Theta^{(1)} = \{Z^1;\ 0 \leq Z^1 \leq 5\} ,$$
$$\Theta^{(2)} = \{Z^2;\ 0 \leq Z^2 \leq 9\} ,$$
$$\Theta^{(3)} = \{Z^3;\ 0 \leq Z^3 \leq 10\} .$$

The strategy set $\Theta^{(ij)}$ of joint strategies $Z^{(ij)}$ of a two-person coalition (ij) is the set of points in the (i,j)-plane with coordinates $(Z_i^{(ij)}, Z_j^{(ij)})$ such that

$$\Theta^{(12)} = \{Z^{(12)};\ 0 \leq Z_1^{(12)} + Z_2^{(12)} \leq 15,\ Z_1^{(12)} \geq 0,\ Z_2^{(12)} \geq 0\} ,$$
$$\Theta^{(13)} = \{Z^{(13)};\ 0 \leq Z_1^{(13)} + Z_3^{(13)} \leq 18,\ Z_1^{(13)} \geq 0,\ Z_3^{(13)} \geq 0\} ,$$
$$\Theta^{(23)} = \{Z^{(23)};\ 0 \leq Z_2^{(23)} + Z_3^{(23)} \leq 20,\ Z_2^{(23)} \geq 0,\ Z_3^{(23)} \geq 0\} .$$

The strategy set $\Theta^{(123)}$ of all player joint strategies $Z^{(123)}$ is the set of points in the three-dimensional Euclidean space E^3 with coordinates $(Z_1^{(123)}, Z_2^{(123)}, Z_3^{(123)})$ such that

$$\Theta^{(123)} = \{Z^{(123)};\ 0 \leq Z_1^{(123)} + Z_2^{(123)} + Z_3^{(123)} \leq 30 ,$$
$$Z_i^{(123)} \geq 0 ,\ i = 1,2,3\} .$$

The payoff functions f are defined as follows (see Fig. 2):

$$(81) \qquad \begin{cases} f_1(Z^{(1)} , Z^{(23)}) = Z^1 \\ f_2(Z^{(1)} , Z^{(23)}) = (1 - \frac{Z^1}{20})\ Z_2^{(23)} \\ f_3(Z^{(1)} , Z^{(23)}) = (1 - \frac{Z^1}{20})\ Z_3^{(23)} \end{cases}$$

$$(82)\quad \begin{cases} f_2(Z^{(2)}, Z^{(13)}) = Z^2 \\ f_1(Z^{(2)}, Z^{(13)}) = (1 - \frac{Z^2}{18})\, Z_1^{(13)} \\ f_3(Z^{(2)}, Z^{(13)}) = (1 - \frac{Z^2}{18})\, Z_3^{(13)} \end{cases}$$

$$(83)\quad \begin{cases} f_3(Z^{(3)}, Z^{(12)}) = Z^3 \\ f_1(Z^{(3)}, Z^{(12)}) = (1 - \frac{Z^3}{15})\, Z_1^{(12)} \\ f_2(Z^{(3)}, Z^{(12)}) = (1 - \frac{Z^3}{15})\, Z_2^{(12)} \end{cases}$$

$$(84)\quad f_i(Z^{(123)}) = Z_i^{(123)}\ , \ i = 1,2,3 \ .$$

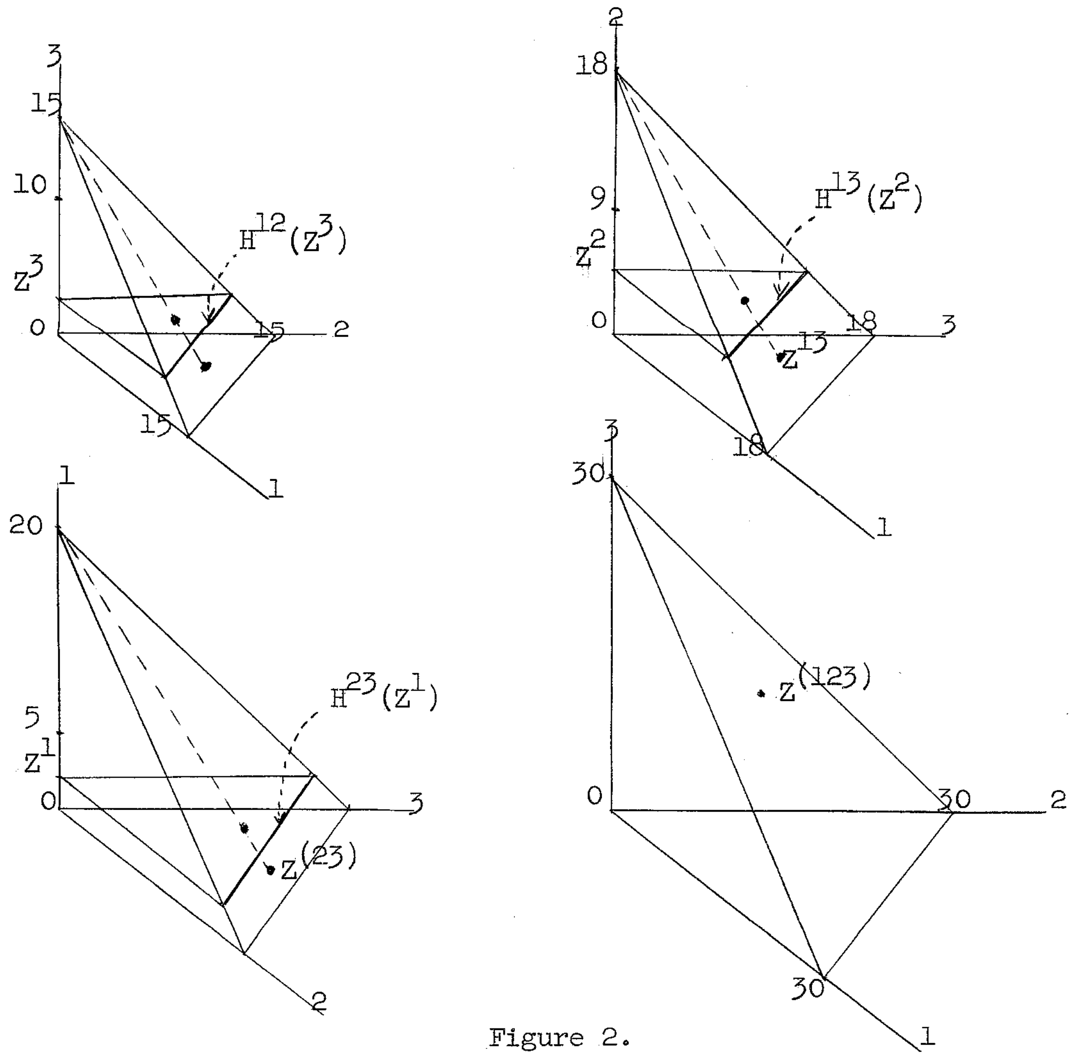

Figure 2.

In this game Γ, it is clear that all assumptions 1 through 4 hold.

At first we will find out an equilibrium strategy set of this game Γ. Let $Z^{(1)}$, $Z^{(2)}$, and $Z^{(3)}$ be arbitrarily chosen strategies of players 1,2, and 3 respectively. Then the Nash solution $(\sigma_1^{(12)}, \sigma_2^{(12)})$ of the subgame $\Gamma^{(12)}(Z^{(3)})$, taking the point $(Z^{(1)}, Z^{(2)})$ as the claim point, is obtained as follows. Let

(85)
$$Z^1 + \ell = \sigma_1^{(12)} \ , \ Z^2 + \ell = \sigma_2^{(12)} \ .$$

Then

$$Z^1 + Z^2 + 2\ell = \sigma_1^{(12)} + \sigma_2^{(12)} = 15 - Z^3 \ ,$$

since the point $(\sigma_1^{(12)}, \sigma_2^{(12)})$ is on $H^{(12)}(Z^3)$. Accordingly,

(86)
$$\ell = \frac{15}{2} - \frac{1}{2} Z^1 - \frac{1}{2} Z^2 - \frac{1}{2} Z^3 \ .$$

From (85) and (86), we have

(87)
$$\begin{cases} \sigma_1^{(12)} = \frac{15}{2} + \frac{1}{2}Z^1 - \frac{1}{2}Z^2 - \frac{1}{2}Z^3 \\ \sigma_2^{(12)} \quad \frac{15}{2} - \frac{1}{2}Z^1 + \frac{1}{2}Z^2 - \frac{1}{2}Z^3 \end{cases}$$

Similarly, the Nash solution $(\sigma_1^{(13)}, \sigma_3^{(13)})$ of the subgame $\Gamma^{(13)}(Z^{(2)})$, taking $(Z^{(1)}, Z^{(3)})$ as the claim point, is given by

(88)
$$\begin{cases} \sigma_1^{(13)} = 9 + \frac{1}{2}Z^1 - \frac{1}{2}Z^2 - \frac{1}{2}Z^3 \\ \sigma_3^{(13)} = 9 - \frac{1}{2}Z^1 - \frac{1}{2}Z^2 + \frac{1}{2}Z^3 \ . \end{cases}$$

The Nash solution $(\sigma^{(23)}, \sigma_3^{(23)})$ of the subgame $\Gamma^{(23)}(Z^{(1)})$, taking $(Z^{(2)}, Z^{(3)})$ as the claim point, is given by

(89)
$$\begin{cases} \sigma_2^{(23)} = 10 - \frac{1}{2}Z^1 + \frac{1}{2}Z^2 - \frac{1}{2}Z^3 \\ \sigma_3^{(23)} = 10 - \frac{1}{2}Z^1 - \frac{1}{2}Z^2 + \frac{1}{2}Z^3 \ . \end{cases}$$

Now, from (51), (52), and (53), we have

$$(90)\qquad \begin{cases} \sigma_1^1 = \Delta_1^1 = Z^1 , \\ \sigma_2^2 = \Delta_2^2 = Z^2 , \\ \sigma_3^3 = \Delta_3^3 = Z^3 . \end{cases}$$

Since $\Delta_1^{(12)} = \sigma_1^{(12)} - \Delta_1^{(1)}$, etc., we have, from (87), (88), (89), and (90)

$$(91)\qquad \begin{cases} \Delta_1^{(12)} = \Delta_2^{(12)} = \frac{15}{2} - \frac{1}{2}Z^1 - \frac{1}{2}Z^2 - \frac{1}{2}Z^3 , \\ \Delta_1^{(13)} = \Delta_3^{(13)} = 9 - \frac{1}{2}Z^1 - \frac{1}{2}Z^2 - \frac{1}{2}Z^3 , \\ \Delta_2^{(23)} = \Delta_3^{(23)} = 10 - \frac{1}{2}Z^1 - \frac{1}{2}Z^2 - \frac{1}{2}Z^3 . \end{cases}$$

Accordingly, in the final all player bargaining game, the claim point d^N is given by

$$(92)\qquad \begin{cases} d_1^N = \Delta_1^{(1)} + \Delta_1^{(12)} + \Delta_1^{(13)} = \frac{33}{2} - Z^2 - Z^3 , \\ d_2^N = \Delta_2^{(2)} + \Delta_2^{(12)} + \Delta_2^{(23)} = \frac{35}{2} - Z^1 - Z^3 , \\ d_3^N = \Delta_3^{(3)} + \Delta_3^{(13)} + \Delta_3^{(23)} = 19 - Z^1 - Z^2 , \end{cases}$$

and the upper-right boundary of the outcome space of this game is given by

$$u_1 + u_2 + u_3 = 30 .$$

So that, it can be easily seen that the Nash solution $u = (u_1, u_2, u_3)$ of this final all player bargaining game is given as follows:

$$(93)\qquad \begin{cases} u_1 = \frac{53}{6} + \frac{2}{3}Z^1 - \frac{1}{3}Z^2 - \frac{1}{3}Z^3 \\ u_2 = \frac{59}{6} + \frac{1}{3}Z^1 + \frac{2}{3}Z^2 - \frac{1}{3}Z^3 \\ u_3 = \frac{18}{6} - \frac{1}{3}Z^1 - \frac{1}{3}Z^2 + \frac{2}{3}Z^3 . \end{cases}$$

This is an equilibrium solution of this game Γ . Moreover, in this game we have

$$(94)\quad \begin{cases} W^{(1)} = \Delta_1^{(1)} = Z^1 , \; W^{(2)} = \Delta_2^{(2)} = Z^2 , \; W^{(3)} = \Delta_3^{(3)} = Z^3 , \\ W^{(12)} = \sigma_1^{(12)} + \sigma_2^{(12)} = 15 - Z^3 , \\ W^{(13)} = \sigma_1^{(13)} + \sigma_3^{(13)} = 18 - Z^2 , \\ W^{(23)} = \sigma_2^{(23)} + \sigma_3^{(23)} = 20 - Z^1 , \\ W^{(123)} = 30 . \end{cases}$$

From (93) and (94), we can check easily that, for example, u_1 is expressed as follows:

$$u_1 = \frac{2!}{3!}(W^{(1)} - W^{(23)}) + \frac{1!}{3!}[(W^{(12)} - W^{(3)}) + (W^{(13)} - W^{(2)})] + \frac{2!}{3!}W^{(123)}, \text{ etc.}$$

The optimal solution $u^o = (u_1^o, u_2^o, u_3^o)$ of this game is given by

$$(95)\quad \begin{aligned} u_1^o &= \frac{53}{6} + \frac{2}{3} \times 5 - \frac{1}{3} \times 9 - \frac{1}{3} \times 10 = \frac{35}{6} \fallingdotseq 5.83 , \\ u_2^o &= \frac{59}{6} - \frac{1}{3} \times 5 + \frac{2}{3} \times 9 - \frac{1}{3} \times 10 = \frac{65}{6} \fallingdotseq 10.83 , \\ u_3^o &= \frac{68}{6} - \frac{1}{3} \times 5 - \frac{1}{3} \times 9 + \frac{2}{3} \times 10 = \frac{80}{6} \fallingdotseq 13.33 . \end{aligned}$$

VII. The relation of this theory to previous theories.

The author of this paper is deeply indebted to Harsanyi's paper [4]. But in his paper, Harsanyi defines the set of equilibrium dividends in the following way:

For any coalition $S \subseteq N$, and for any two members i and j of S, he considers a two-person subgame Γ_{ij}^S between i and j. In defining equilibrium dividends, he requires σ_i^S and σ_j^S to be the Nash solution, taking the point

$$(96)\quad t_k^S = \sum_{\substack{R \ni k \\ R \neq S \\ R \subseteq N}} \Delta_k^R , \; k = i,j$$

as a claim point of this two-person subgame Γ_{ij}^S. But, if we were to consider Δ_i^S and Δ_j^S as the net incremental payoffs to players i and j respectively, which they will obtain as the result of advancing from (s-1)-person coalitions to the s-person coalition S , including i and j , then Δ_k^R , $R \ni k$, $S \subset R \subseteq N$ should not be taken into consideration at this moment. Accordingly, it does not seem natural to take the point (t_i^S, t_j^S) defined by (96) as the claim point of this subgame Γ_{ij}^S which is to determine incremental payoffs Δ_i^S and Δ_j^S . Moreover, it would seem natural to determine the incremental payoffs Δ_i^S , $i \in S$ to the members of the coalition S as the result of the bargaining among all s members of S , and not between only two members i,j of S , as Harsanyi does. Following this line of reasoning, we treated a subgame Γ^S among s-players in S , taking the point defined by (23) as the claim point of this subgame. Moreover, we made clear the dependence of the outcome space of this subgame Γ^S on the strategy taken by the complementary coalition $\bar{S}$. This consideration brought about a more complicated situation than Harsanyi's model [4].

If the dependency of the outcome space of the subgame Γ^S on the strategy chosen by the complementary coalition $\bar{S}$ could be neglected, then we would be able to follow a treatment similar to Isbell's [5], and a similar modified Shapley value similar to Harsanyi's [4] would come out.

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