ESTIMATION AND OPTIMAL CONTROL OF ECONOMETRIC MODELS UNDER RATIONAL EXPECTATIONS

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Abstract: This paper extends the author's recent (1979) paper on the estimation of rational expectations models in two directions. First, two players are introduced instead of only one, and the estimation of a model of dynamic games is studied under the assumption of a dominant player or a noncooperative Nash equilibrium. Second, with the second player (government) treated as the dominant player, we consider policy evaluation and optimization by the government under the assumption of rational expectations.

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1. INTRODUCTION

This paper is concerned with further developments of Chow (1979), entitled "Estimation of Rational Expectations Models," where I have proposed two methods for the estimation of the parameters of a linear model

$$y_t = Ay_{t-1} + Cx_t + b_t + u_t \tag{1}$$

which describes the environment of a set of economic decision makers, and the parameters of a quadratic objective function

$$-E_o \sum_{t=1}^{T} (y_t - a_t)' K_t (y_t - a_t) \tag{2}$$

which the decision makers are assumed to maximize. Resulting from this maximization is a linear behavioral equation (feedback control equation) for the decision makers who control x_t, written as

$$x_t = G_t y_{t-1} + g_t \tag{3}$$

The parameters G_t and g_t in (3) are derived from the parameters of (1) and (2). The econometrician observes the data on x_t and y_t, and wishes to estimate the parameters of (1) and (2). The two methods proposed in Chow (1979) are maximum likelihood and a consistent method corresponding to two-stage least squares. Detailed knowledge of these methods is not required for the reader of this paper, who is asked only to keep in mind that the methods exist for the estimation problem just described.

The present paper is concerned with two extensions of the above estimation problem. First, there are two sets of economic decision makers, so that the model becomes

$$y_t = Ay_{t-1} + C_1x_{1t} + C_2x_{2t} + b_t + u_t \tag{4}$$

Each set i of decision makers chooses its control variables x_{it} to maximize an objective function

$$-E_o \sum_{t=1}^{T} (y_t - a_{it})' K_{it}(y_t - a_{it}) \qquad (i = 1,2) \tag{5}$$

and derives its optimal behavioral equation

$$x_{it} = G_{it}y_{t-1} + g_{it} \qquad (i = 1,2). \tag{6}$$

The econometric problem is to estimate the parameters of (4) and (5). Second, when one decision maker is the government, we are concerned with the evaluation of the effects of government policy changes and the choice of an optimum policy for the government.

To illustrate the application of this model, let x_{1t} be the variables subject to the control of some group of decision makers of the private sector and x_{2t} be the variables subject to the control of the government. If the government adheres to a policy rule, i.e., if G_2 and g_{2t} are given, the environment facing the private decision makers is

$$y_t = (A+C_2G_2)y_{t-1} + C_1x_{1t} + (b_t+C_2g_{2t}) + u_t$$

$$\equiv A_{12}y_{t-1} + C_1x_{1t} + b_{12,t} + u_t . \tag{7}$$

They would maximize their objective function to derive their behavioral equation. As Lucas (1976) has stressed, if the policy rule of the government changes, the behavioral equation of the private decision makers will also change. Therefore, an econometrician should not rely on a stable relation (3) to evaluate the effects of government policy. A correct procedure is to estimate (1) and (2), rather than (1) and (3), and then derive the changes in (3) due to changes in (1). Lucas (1976, p. 20) reminded the reader that this point had been made by the proponents of structural estimation for simultaneous-equation models, and cited Marschak (1953) for having pointed out the change in the reduced-form equations due to a policy change. Another manifestation of this problem occurs when the behavioral equations of the private sector contain expectations variables which are explained by some distributed lag relationships. As government policy changes, the model (1) or (7) will change, and these expectations will also change under rational expectations, thus making the historical distributed lag relationships unstable. The solution again is to rederive the expectations using the new structure (1) or (7), but this topic will not be treated in the present paper as the estimation and control problems associated with it are discussed in Taylor (1979), Wallis (1979), and Chow (1980).

The first extension of this paper is to allow for two sets of decision makers whose actions affect the environment of each other. In the above example, while the government policy rule $x_{2t} = G_2 y_{t-1} + g_{2t}$ affects the optimal policy of the private sector, the latter's optimal behavioral relation $x_{1t} = G_1 y_{t-1} + g_{1t}$ will also affect the policy rule of the government if it is also assumed to maximize its objective function. We will study this dynamic game model in this paper. Section 2 deals with the estimation of the parameters of this model under the assumption that Player 2 (the government) is the dominant player. Section 3 treats the estimation problem when the two players are assumed to be in a noncooperative Nash equilibrium. Section 4 is concerned with the related topic of government policy evaluation and optimization under the assumption that the government is the dominant player. In this paper, we assume that the optimal reaction coefficient G_{it} in (6) for both players will reach a steady-state G_i, that is, the rational expectations equilibrium. Otherwise, no stable relationships can be estimated.

2. ESTIMATION OF DYNAMIC GAME MODEL WITH A DOMINANT PLAYER

When x_{2t} in (4) represents the policy instruments of the government and the government is treated as the dominant player, we will study the estimation problem in two stages. First, assuming that the government adheres to a policy rule $x_{2t} = G_2 y_{t-1} + g_{2t}$, which is decided upon by whatever means, we will consider the estimation of the parameters of (4) and (5) for $i = 1$ under the assumption that the private sector behaves optimally. Second, from the above framework we take the next step by assuming that the government is also trying to maximize (5) for $i=2$ and consider the estimation of the parameters of its objective function as well.

For the first problem, the stochastic environment facing the private sector consists of two equations, (4) and

$$x_{2t} = G_2 y_{t-1} + g_{2t} \tag{6.2}$$

These two equations comprise the model (1) in the framework of Chow (1979). In that paper, two methods were provided to estimate the parameters of (1), now consisting of (4) and (6.2), and of (2), now represented by (5) with i=1. The methods are maximum likelihood and a consistent method analogous to two-stage least squares. The latter method requires consistent estimates of the parameters of (1) and (3), and, using them, solves for the parameters of (2) in the second stage of two-stage least squares.

We now incorporate the assumption that the government also maximizes to obtain its behavioral equation (6.2). If we are not interested in estimating the objective function of the government, and are willing to assume that the parameters of (4) and (5) remained unchanged for the sample observations, then (6.2) is a stable equation and the methods of Chow (1979) would suffice, as pointed out in the last paragraph. The new problem is to estimate the objective function of the government as well. From the viewpoint of the maximizing government, the stochastic environment consists of (4) and (5) with i=1, which, together with its own policy (G_2, g_{2t}), determine G_1 and g_{1t} in (6) as a result of the private sector's maximizing behavior.

Maximum-likelihood estimation of the parameters of (4) and (5) under the assumption that Player 2 (the government) is the dominant player can proceed as follows. Adding a residual v_{it} to (6) and assuming a joint normal distribution of u_t, v_{1t} and v_{2t}, one can easily write down the likelihood function which has the parameters of (4) and (6) as arguments. As a first step, we postpone the estimation of K_{2t} and a_{2t}, and assume some given values for G_2 and g_{2t} (which could be the coefficients of a least-squares regression of x_{2t} on y_{t-1} and appropriate trend terms). Given G_2 and g_{2t}, we can express G_1 and g_{1t} as functions of the parameters of (4) and K_{1t} and a_{1t} in (5) through the maximization of the private sector. K_{1t} and a_{1t} thus replace G_1 and g_{1t} as arguments in the likelihood function. To reduce the number of parameters, we assume here as in Chow (1979) that

$K_{1t} = \beta_1^t K_{10}$ and $a_{1t} = \phi_1^t a_{10}$, β_1 being the discount factor for the private sector and ϕ_1 being a diagonal matrix with some elements known to be one if the targets in a_t are constant through time. Given G_2 and g_{2t}, then, we can maximize the likelihood function with respect to the parameters of (1) and K_{10}, β_1, a_{10}, and ϕ_1. This problem was solved in Chow (1979).

As a second step of our estimation method, we relax the assumption that G_2 and g_{2t} are given and try to estimate the parameters $K_{2t} = \beta_2^t K_{20}$ and $a_{2t} = \phi_2^t a_{20}$. Note first that consistent estimates of K_{20}, β_2, a_{20} and ϕ_2 can be obtained by solving the well-known equations for $G_{2t} = G_2$ (in a steady state) under the assumption of a maximizing government and using consistent estimates of the parameters C_2 and A_2:

$$G_2 = -(C_2'H_2C_2)^{-1}C_2'H_2A_2$$

$$H_2 = K_2 + \beta_2 A_2'H_2(A_2+C_2G_2)$$

which can be found in Chow (1975, pp. 178-179) and where the subscript 2 denotes the parameters in the relevant equations for Player 2 (the government). This problem was solved in Section 4 of Chow (1979) by a method analogous to two-stage least squares. Thus our problem is solved if we are willing to accept these consistent estimates of the parameters of the government's objective function.

To obtain maximum-likelihood estimates of K_{20}, β_2, a_{20} and ϕ_2 in the second step, we use the consistent estimates of the last paragraph as the starting point. Given $\hat{K}_{20}$, $\hat{\beta}_2$, $\hat{a}_{20}$ and $\hat{\phi}_2$, say, we use standard optimal control to calculate the government's optimal policy for $\hat{G}_2$ and $\hat{g}_{2t}$ under the assumptions that the government is maximizing <u>and that</u> its environment consists of (4) and (G_1, g_{1t}) as estimated above. However, these estimates

$\hat{G}_2$ and $\hat{g}_{2t}$ ignore the fact that the estimates of G_1 and g_{1t} used in the above calculations will in turn depend on them. Iterations between the estimates for (G_2, g_{2t}) and for (G_1, g_{1t}) will be required at this stage, yielding the solution (G_2^*, g_{2t}^*) and (G_1^*, g_{1t}^*), say, for the given $\hat{K}_{20}$, $\hat{\beta}_2$, $\hat{a}_{20}$ and $\hat{\phi}_2$. Using (G_2^*, g_{2t}^*) and (G_1^*, g_{1t}^*) as coefficients of (6), we can evaluate the likelihood function for (4) and (6). Accordingly, we can apply a gradient method to maximize the likelihood function with respect to the parameters K_{20}, β_2, a_{20} and ϕ_2. These are the primary parameters in the following sense. Given their values, we have in effect maximized the likelihood function with respect to all other parameters by the maximum-likelihood method of Chow (1979) together with iterations between $(\hat{G}_2, \hat{g}_{2t})$ and $(\hat{G}_1, \hat{g}_{1t})$ as just described. The second step in this step-wise maximization procedure is to maximize the likelihood function with respect to them. Note that our iterative method to obtain (G_2^*, g_{2t}^*) and (G_1^*, g_{1t}^*) differs from the solution proposed by Kydland (1975, p. 330) for the dominant player problem since Kydland assumes a given value x_{2t} as the (open-loop) strategy for the dominant Player 2 as conceived by Player 1, while we assume a feedback strategy (G_2, g_{2t}) for Player 2. Our solution is more appropriate when a feedback control policy for the government is being examined, as it is in this paper.

3. ESTIMATION OF DYNAMIC GAME MODEL UNDER NASH EQUILIBRIUM

Having solved the estimation problem for a dynamic game model with a dominant player, we can apply similar iterative techniques to estimate the model under a Nash equilibrium. Again we consider the estimation problem in two stages. First, assuming tentatively that the government adheres to a policy rule (G_2, g_{2t}), we will consider the estimation of the parameters of (4) and $K_{1t} = \beta_1^t K_{10}$ and $a_{1t} = \phi_1^t a_{10}$ under the assumption that the private sector behaves optimally. Our estimation procedure assumes optimal behavior (G_1, g_{1t}) of the private sector, with (G_2, g_{2t}) taken as given. Second, assuming that the private sector adheres to the policy (G_2, g_{2t}) as determined above, we consider the estimation of the parameters of (4) and $K_{2t} = \beta_2^t K_{20}$ and $a_{2t} = \phi_2^t a_{20}$ under the assumption that the government behaves optimally. Similarly, this estimation procedure assumes optimal behavior (G_2, g_{2t}) of the government, with (G_1, g_{1t}) taken as given. We now go back to step one, and iterate back and forth until convergence.

As pointed out previously, given (G_2, g_{2t}), the methods of Chow (1979) can be used to estimate the parameters of (4), K_{10}, β_1, a_{10}, ϕ_1 and, accordingly, G_1 and g_{1t}. Similarly, given (G_1, g_{1t}), the same methods can be used to estimate the parameters of (4), K_{20}, β_2, a_{20}, ϕ_2 and, accordingly, G_2 and g_{2t}. If the method of maximum likelihood is used, we start with some consistent estimates of G_2 and g_{2t} (as obtained by regressing x_{2t} on y_{t-1} and appropriate trends), and maximize the likelihood function with respect to the parameters of (4), K_{10}, β_1, a_{10} and ϕ_1, yielding maximum likelihood estimates of G_1 and g_{1t} as well. Using these estimates of G_1 and g_{1t}, we again maximize the likelihood function with respect to the parameters of (4), K_{20}, β_2, a_{20} and ϕ_2, and so forth until convergence. This procedure amounts to maximizing the likelihood function with respect to two sets of parameters iteratively, i.e., to one set while holding the other set fixed and alternatively.

To propose a simpler and yet consistent method, we start with consistent estimates of the parameters of (4), and of (G_2, g_{2t}) and (G_1, g_{1t}), by the method of least squares, for instance. The parameters of (4) and (G_i, g_{it}) can be employed to solve for K_{i0}, β_i, a_{i0} and ϕ_i for $i = 1,2$ by the method analogous to two-stage least squares as given in Section 4 of Chow (1979). Given the parameters of (4) and K_{i0}, β_i, a_{i0} and ϕ_i $(i = 1,2)$, one can then find the Nash equilibrium solution for (G_1, g_{1t}) and (G_2, g_{2t}) iteratively, to improve upon the initial, consistent estimates of these parameters.[1] The situation is exactly analogous to the estimation of the reduced-form parameters Π in linear simultaneous stochastic equations. Consistent estimate $\hat{\Pi}$ of Π by least squares can be used to estimate the parameters $(B \;\; \Gamma)$ of the structure using the method of two-stage least squares. Given these estimates of $(B \;\; \Gamma)$, denoted by $(\hat{B} \;\; \hat{\Gamma})$, we can obtain a new estimate of Π as $\hat{B}^{-1}\hat{\Gamma}$, to improve upon the initial estimate $\hat{\Pi}$.

This section has treated the estimation of rational expectations models under the assumption of Nash equilibrium. If Player 2 represents the government, the solution concept of having a dominant player as expounded in Section 2 may be more appropriate. Given their likelihoods, these two models can be tested, but this topic is not pursued here.

4. POLICY EVALUATION AND OPTIMIZATION UNDER RATIONAL EXPECTATIONS

The critique by Lucas (1976) of econometric policy evaluation is essentially that when the policy of Player 2 (the government) is being evaluated, the econometrician should not take the behavioral equation $x_{1t} = G_1 y_{t-1} + g_{1t}$ for the private sector as given. In Section 2, when we treat the government as the dominant player, we have indicated, as a by-product of our estimation procedure, how the consequences of any government policy rule (G_2, g_{2t}) can be evaluated after proper account is taken of the optimizing reaction of the private sector. To recapitulate, the environment facing the private sector consists of (4) and $x_{2t} = G_2 y_{t-1} + g_{2t}$, and the private sector derives its optimum behavioral equation $x_{1t} = G_1 y_{t-1} + g_{1t}$ by maximizing its objective function subject to the above environment. Linear-quadratic optimal control theory as found in Chow (1975) can be used to find this optimal feedback control equation. The problem of policy evaluation is thus solved.

Turning to policy optimization by the government, once we know how to compute G_1 and g_{1t} for given G_2 and g_{2t}, we can derive the following procedure to find the government's optimal policy, assuming the parameters of (4) to be known and given a quadratic objective function for the government. Using (4) and (G_1, g_{1t}), we write the model facing the government as

$$y_t = (A+C_1G_1)y_{t-1} + C_2x_2 + (b_t+C_1g_{1t}) + u_t$$

$$\equiv A_{21}y_{t-1} + C_2x_{2t} + b_{21,t} + u_t. \quad (8)$$

Denote the expectations of y_t and x_{2t} respectively by $\bar{y}_t$ and $\bar{x}_{2t}$. Given any deterministic or mean path $\bar{x}_{2t}$, the mean path $\bar{y}_t$ can be obtained by taking the expectation of (8), yielding

$$\bar{y}_t = A_{21}\bar{y}_{t-1} + C_2\bar{x}_{2t} + b_{21,t} \quad (9)$$

The deviation $y_t^* = y_t - \bar{y}_t$ of y_t from its mean path obeys, by subtraction of (9) from (8),

$$y_t^* = A_{21}y_{t-1}^* + C_2x_{2t}^* + u_t \quad (10)$$

where the setting of the control variable x_{2t} by the government is also decomposed into the mean or deterministic part $\bar{x}_{2t}$ and the stochastic part x_{2t}^* which will depend on the stochastic disturbance u_t and thus y_t as yet to occur in the future (i.e., for $t > 0$).

The government's quadratic loss function is decomposed accordingly,

$$E_o\sum_{t=1}^{T}(y_t-a_{2t})'K_{2t}(y_t-a_{2t}) = E_o\sum_{t=1}^{T}y_t^{*\prime}K_{2t}y_t^* + \sum_{t=1}^{T}(\bar{y}_t-a_{2t})'K_{2t}(y_t-a_{2t}) \quad (11)$$

$$\equiv W_s + W_d$$

First, we choose a feedback rule $x^*_{2t} = G_2 y^*_{t-1}$ to minimize the first part W_s of (11), given the model (10). Second, we choose a deterministic policy $\bar{x}_{2t}$ to minimize the second part W_d of (11) subject to the constraint of (9). If the intercept $b_{21,t}$ in (9) were zero, and the targets a_{2t} were also zero, then the solution for the second part would be $\bar{x}_{2t} = G_2\bar{y}_{t-1}$, the coefficient G_2 being the same as in the first, stochastic part. This assertion results from the simple observation that the optimal linear feedback solution for the first problem (minimization of W_s) applies to the second problem (minimization of W_d with $a_t = 0$ and $b_{21,t} = 0$) as the latter is a special case of the former when the covariance matrix of u_t is reduced to a null matrix. When a_{2t} and $b_{21,t}$ are not zero, a deterministic intercept term g_{2t} in the optimal feedback control equation can be used to adjust for these purely additive deterministic terms in the linear model, since $\bar{y}_t - a_{2t}$ can be written as

$$(A_{21}\bar{y}_{t-1} + C_2\bar{x}_{2t}) + (b_{2t} - a_{2t}),$$

where the second term in parentheses is purely additive and the control for it is independent of the known state $\bar{y}_{t-1}$, requiring no feedback.

We can find G_2 to minimize W_s by a gradient method, noting that the coefficient matrix $A_{21} = (A+C_1G_1)$ depends on G_2 as we have pointed out in discussing the problem of policy evaluation. Given G_2, we can evaluate the corresponding optimizing G_1, A_{21} and hence the expectation

$$W_s = E_o \sum_{t=1}^{T} y^{*\prime}_t K_{2t} y^*_t = \sum_{t=1}^{T} \mathrm{tr}\{K_{2t} E_o y^*_t y^{*\prime}_t\} .$$

In this evaluation, we use the relation based on (10) that

$$E_o y^*_t y^{*\prime}_t = (A_{21}+C_2G_2)(Ey^*_{t-1}y^{*\prime}_{t-1})(A_{21}+C_2G_2)' + Eu_t u_t' \tag{12}$$

with initial condition $y_0^* = 0$ and $Ey_1^* y_1^{*'} = Eu_t u_t'$. Therefore, a gradient method can be applied to minimize W_s with respect to G_2. Once G_2 is found, we can evaluate

$$W_d = \sum_{t=1}^{T} (\bar{y}_t - a_{2t})' K_{2t} (y_t - a_{2t})$$

for each deterministic path g_{2t} using (9) and $\bar{x}_{2t} = G_2 \bar{y}_{t-1} + g_{2t}$. Also, a gradient method can be applied to minimize W_d with respect to the path g_{2t}. Our problem of finding an optimal feedback rule for the government as a dominant player under rational expectations is thus solved.

In the above solution algorithm, we have separated the loss function into W_s (due to stochastic disturbances in the model) and W_d (deterministic) and have found the optimal G_2 and g_{2t} separately. Without exploiting this separation, one could minimize the entire expression (11) with respect to G_2 and g_{2t} by some gradient algorithm since the function (11) can be evaluated once G_2 and g_{2t} are known. The above decomposition algorithm will probably lead to more efficient computations as it treats the variables G_2 and g_{2t} separately and maximizes with respect to the former without having to deal with the latter simultaneously.

Footnote

[1]An algebraic expression for the Nash equilibrium can be found in Kydland (1975, pp. 323-326) for instance, but here we need only a numerical solution by iterations, i.e., by solving a standard optimal control problem to get G_1 and g_{1t} for the first player, given G_2 and g_{2t}, and then solving a standard optimal control problem to find G_2 and g_{2t} given the above G_1 and g_{1t}, and so forth. For more references on dynamic games, see Cruz (1975).

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