

SINGLE-MARKET DISEQUILIBRIUM MODELS:
ESTIMATION AND TESTING*

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Econometric Research Program
Research Memorandum No. 264

May 1980

* We are indebted to the participants of the Conference on Model Selection, Gainesville, Fla., April 18-21, 1980, for useful comments and to David Romer for computer programming. Financial support from NSF Grant No. SOC77/07680 is gratefully acknowledged.

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1. INTRODUCTION

The literature on disequilibrium econometric models has grown rapidly in recent years. While there is no universally accepted specification for such models, in the case of a single market one quite commonly used model has the following features: (1) quantities demanded and supplied are functions of both price and exogenous variables; (2) price does not clear the market and the quantity transacted is given as the minimum of demand and supply; and (3) price evolves according to an adjustment rule which makes the change in price a function of excess demand, and perhaps, a stochastic disturbance.

The aim of the present paper is to investigate issues of estimation and hypothesis testing in the context of the disequilibrium model just described. We shall attempt to shed light on the following four questions: (1) what are the small-sample properties of the maximum likelihood estimator in various disequilibrium models; (2) how can one test the hypothesis of equilibrium vs. disequilibrium; (3) can one reasonably estimate the unobservable demand and supply quantities from observable data; and (4) what are the consequences of using an equilibrium model instead of a disequilibrium one, or of using a misspecified disequilibrium model. Each of these questions will be examined with the aid of sampling experiments.

Section 2 outlines the basic structure of two types of disequilibrium models. Section 3 considers the estimation of the unobservable demand and supply and Section 4 discusses various possible tests of the hypothesis that data were generated by an equilibrium structure as against various alternative disequilibrium structures. The sampling experiments are described in Section 5, while Section 6 contains some brief concluding remarks.

2. THE BASIC MODEL

Two simple linear versions of the disequilibrium model are given in Eqns. (1) to (4):

$$D_t = X_{dt}\beta_1 + \alpha_1 P_t + u_{1t} \tag{1}$$

$$S_t = X_{st}\beta_2 + \alpha_2 P_t + u_{2t} \tag{2}$$

$$Q_t = \min(D_t, S_t) \tag{3}$$

$$P_t - P_{t-1} = \begin{cases} \gamma (D_t - S_t) \\ \text{or} \\ \gamma (D_t - S_t) + u_{3t} \end{cases} \tag{4a}$$

$$\tag{4b}$$

where X_{dt} , X_{st} are vectors of explanatory variables, the vector (u_{1t}, u_{2t}, u_{3t}) is iid $N(0, \Sigma)$, D_t and S_t are unobserved, and Q_t and P_t are observed. The two versions differ in whether the price equation is assumed to be nonstochastic as in (4a) or stochastic as in (4b). In what follows we shall refer to these as the nonstochastic price (NSP) and stochastic price (SP) models, respectively.

Various estimating methods have been proposed for these models. For both NSP and SP models, maximum likelihood methods are available. In addition, for NSP models a variety of two-stage, three-stage, and instrumental-variables estimators are possible (Amemiya (1974), Ito and Ueda (1979), Laffont and Monfort (1979)). Both to keep things manageable and because we wish to contrast the NSP and SP models, we shall restrict attention to maximum likelihood methods in the present paper. The likelihood functions for both models are well-known.

For the NSP model it has the form

$$L_{NSP} = \prod_{\Delta P_t < 0} f_1(Q_t, P_t) \prod_{\Delta P_t > 0} f_2(Q_t, P_t) \quad (5)$$

where $f_1(Q_t, P_t)$ is the joint density of Q_t and P_t when $Q_t = D_t$ and $f_2(Q_t, P_t)$ is the corresponding density when $Q_t = S_t$. Given the joint normality of u_{1t} and u_{2t} , an explicit algebraic expression for (5) can be readily computed (see, for example, Maddala and Nelson (1974)).

Under the SP model, a precise a priori separation of the sample into demand and supply points is not possible. As is well-known, the relevant likelihood function has the following additive structure.

$$L_{SP} = \prod \left[\int_Q^\infty g(D, Q, P) dD + \int_Q^\infty g(Q, S, P) dS \right] \quad (6)$$

where $g(D, S, P)$ is the trivariate normal induced by the normality assumption for the u_{it} . Once again, an explicit form for (6) is readily available (see Appendix).

Various properties of the maximum likelihood estimator have been investigated in the literature. The question of consistency is considered in Hartley and Mallela (1977) and Amemiya and Sen (1977). The problem of the unboundedness of (6) and certain computational difficulties created by covariances were addressed in Goldfeld and Quandt (1979). In the present paper these latter issues need not concern us. However, as we shall see below, computational problems of a different sort will be encountered. The source of these can be explained and indeed sheds some light on the relationship between L_{NSP} and L_{SP} . In particular, as shown in the Appendix, the following limiting property holds: $\lim_{\sigma_3^2 \rightarrow 0} L_{SP} = L_{NSP}$ where $\sigma_3^2 = \text{var}(u_{3t})$. That is, the SP

likelihood function approaches the NSP function as the variance of the disturbance in the price adjustment equation tends to zero. What this portends is that attempts to estimate the SP model when the underlying model is actually the NSP model may well encounter computational difficulties. (See also Section 5.)

3. ESTIMATING THE UNOBSERVABLE DEMAND AND SUPPLY

As mentioned at the outset, one of the purposes of this paper is to investigate various measures of the unobservables, D_t and S_t , which can be estimated from data. These estimates are typically of considerable interest since they can be used quantitatively to characterize periods of excess demand or excess supply. In some instances there may be extraneous information on this, of either a qualitative or quantitative sort, so one may get a rough check on the reasonableness of the underlying model.¹

The most straightforward method of characterizing the unobservable random variables, D_t and S_t , is to calculate their expectations. Depending upon what one assumes about the relevant information set, there are a variety of ways to compute these expectations. To illustrate this point we first make the following simplifying assumptions: the parameters of the demand and supply functions ($\alpha_1, \alpha_2, \beta_1, \beta_2$ in Eqns. (1) and (2)) are known as is the variance-covariance matrix of the disturbances; and price can be regarded as an exogenous variable. We are thus, for the moment, dealing with the following simplified disequilibrium model:

¹ See Rosen and Quandt (1978) and Romer (forthcoming) for examples of this. One might also examine whether excess demands are "large" and use this as an informal indication of whether a disequilibrium or an equilibrium model is appropriate.

$$D_t = X_{dt}\beta_1 + u_{1t} \quad (7)$$

$$S_t = X_{st}\beta_2 + u_{2t} \quad (8)$$

$$Q_t = \min (D_t, S_t) \quad (9)$$

In this context there are two ways we can compute expectations. The first just conditions on the exogenous variables. From Eqns. (7) and (8) we have

$$E(D_t | X_{dt}) = X_{dt}\beta_1 \quad (10)$$

$$E(S_t | X_{st}) = X_{st}\beta_2 \quad (11)$$

These expressions, however, ignore the information contained in the observable variable, Q_t . Since $E(u_{it} | Q_t) \neq 0$, in estimating D_t and S_t we should be able to do better in a mean-squared-error sense than Eqns. (10) or (11) by conditioning expectations on Q_t . Suppressing the time subscript, let $g(D,S)$ denote the joint density of D and S , $\mu_{1.2}$ be the conditional mean of D given S , and $\sigma_{1.2}^2$ the conditional variance.² Further, let $f(Q)$ be the p.d.f. for the observed variable Q , i.e.

$$f(Q) = \int_Q^\infty g(Q,S) dS + \int_Q^\infty g(D,Q) dD \equiv f_1(Q) + f_2(Q) \quad (12)$$

It can then be shown (Hartley (1977)) that

$$E(D|Q) = Q \cdot \frac{f_1(Q)}{f(Q)} + \mu_{1.2} \frac{f_2(Q)}{f(Q)} + \sigma_{1.2}^2 \frac{g(Q,Q)}{f(Q)} \quad (13)$$

² Under normality we have $\mu_{1.2} = X_d\beta_1 + \rho \frac{\sigma_1}{\sigma_2} (Q - X_s\beta_2)$ and $\sigma_{1.2}^2 = \sigma_1^2 (1 - \rho^2)$ where σ_1^2 and σ_2^2 are the unconditional variances and ρ is the correlation coefficient.

The expression for $E(S|Q)$ can be obtained in corresponding fashion.

For the more general NSP model with a price equation, consisting of Eqns. (1) - (3) and (4a), we can again use Eqns. (10) and (13). We require a reduced form expression for D analogous to Eqn. (7). This can be obtained by substituting the price equation, (4a), into Eqns. (1) and (2) and solving for D and S as functions of the exogenous variables, X_d and X_s (and lagged price). These reduced forms can then be used to obtain the natural generalizations of (10) and (13), i.e., to yield $E(D|X_d, X_s)$ and $E(D|Q, X_d, X_s)$. Basically the same approach works for the SP model except that one uses Eqn. (4b) to derive the reduced form. Since $E(u_3|X_d, X_s) = 0$, the calculation of unconditional expectations is, in fact, identical to the NSP model.

Since price is an endogenous variable in both the SP and NSP models, one can go one step further than Eqn. (13) and condition expectations on price as well. For the SP model, the relevant formula for $E(D|P, Q)$ looks much like Eqn. (13), except that univariate and bivariate densities in (13) are replaced with the appropriate bivariate and trivariate densities.³ For the NSP model, matters are somewhat simpler. We have $\Delta P_t = \gamma(D_t - S_t)$, so that observing P_t (and thus ΔP_t) tells us whether D_t exceeds S_t or not. Given the min condition we then know whether we have observed demand or supply. Thus,⁴

$$E(D|P, Q) = \begin{cases} Q & \text{if } \Delta P \leq 0 \\ Q + \frac{\Delta P}{\gamma} & \text{if } \Delta P > 0 \end{cases} \quad (14)$$

and

$$E(S|P, Q) = \begin{cases} Q - \frac{\Delta P}{\gamma} & \text{if } \Delta P \leq 0 \\ Q & \text{if } \Delta P > 0 \end{cases} \quad (15)$$

Footnotes 3 and 4 on following page.

We thus see, in principle at least, that for both the NSP and SP models it is relatively straightforward to calculate expectations of D_t and S_t , conditional on varying amounts of information. In practice, of course, the structural parameters of the underlying models are not known so these conditional expectations can only be estimated. The obvious procedure is to replace the unknown parameters with consistent estimates of these parameters. In the context of sampling experiments, it is possible to compare these estimated measures of demand and supply with the generally unobservable D_t and S_t .⁵ Various comparisons of this sort will be reported below.

4. TESTING OF EQUILIBRIUM VS. DISEQUILIBRIUM

We have thus far considered specification and estimation of disequilibrium models. We turn now to the issue of hypothesis testing. In contrast to the disequilibrium model of Eqns. (1)-(4), the null hypothesis of equilibrium can be represented by the equations

³ Bowden (1978) reports a closely related conditional expectation, $E(P^* | P, Q)$, where P^* is the hypothetical equilibrium price at which demand equals supply.

⁴ Eqns. (14) and (15) imply $\Delta P = \gamma [E(D|P, Q) - E(S|P, Q)]$ which is what one gets by taking expectations of equation (4a).

⁵ It may also be of interest to compare the sample estimates of the various expectational measures with their theoretical counterparts. While we did compute the latter for some experiments, the results reported below focus on the comparison between the estimated expectations and D_t and S_t .

$$D_t = X_{dt} \beta_1 + \alpha_1 P_t^* + u_{1t}$$

$$S_t = X_{st} \beta_2 + \alpha_2 P_t^* + u_{2t}$$

$$D_t = S_t = Q_t$$

where we employ P_t^* to denote the equilibrium price in order to distinguish it from the solution to Eqns. (1) to (4). We have

$$P_t^* = \frac{X_{dt} \beta_1 - X_{st} \beta_2}{\alpha_2 - \alpha_1} + \frac{(u_{1t} - u_{2t})}{\alpha_2 - \alpha_1} \quad (16)$$

$$Q_t = \frac{\alpha_2 X_{dt} \beta_1 - \alpha_1 X_{st} \beta_2}{\alpha_2 - \alpha_1} + \frac{\alpha_2 u_{1t} - \alpha_1 u_{2t}}{\alpha_2 - \alpha_1} \quad (17)$$

In some intuitive sense, the equilibrium model ought to be "close" to a disequilibrium model in which prices adjust rapidly to excess demands, i.e., in which γ in Eqns. (4a) or (4b) is large. That this is indeed the case, can be seen in a variety of ways. For example, if we solve for the reduced form of P_t for the NSP model we obtain

$$P_t = \frac{1}{1 + \gamma(\alpha_2 - \alpha_1)} [X_{dt} \beta_1 - X_{st} \beta_2 + u_{1t} - u_{2t}] + \frac{P_{t-1}}{1 - \gamma(\alpha_1 - \alpha_2)} \quad (18)$$

It follows that, $\lim_{\gamma \rightarrow \infty} P_t = P_t^*$. Similarly, one may verify from the

reduced forms for D_t and S_t that $\lim_{\gamma \rightarrow \infty} D_t = \lim_{\gamma \rightarrow \infty} S_t = Q_t$ where Q_t is given

by (17). Finally, if we denote the equilibrium likelihood function by L_e ,

Quandt (1978) has shown that $\lim_{\gamma \rightarrow \infty} L_{SP} = L_e$, while Gourieroux, Laffont and

Monfort (1978) have demonstrated that $\lim_{\gamma \rightarrow \infty} L_{NSP} = L_e$.

The above observations suggest that the size of γ provides a basis for a

test of the equilibrium vs. disequilibrium hypothesis. However, since it is awkward to test whether γ is "large," the following reformulations of the price equation -- due to Bowden (1978) -- are useful.

$$P_t = \mu P_{t-1} + (1-\mu)P_t^* \tag{4a'}$$

$$P_t = \mu P_{t-1} + (1-\mu)P_t^* + \mu u_{3t} \tag{4b'}$$

where P_t^* is given by Eqn. (16). In this reformulation, equilibrium corresponds to $\mu=0$. Although perhaps not immediately apparent, the Bowden formulation is equivalent to reparameterizing Eqns. (4a) and (4b). This can be seen most easily by substituting for P_t^* , from (16), into (4b'). The result is Eqn. (18) with $\mu = 1/(1-\gamma(\alpha_1-\alpha_2))$. Thus, (4a') and (4b') are directly analogous to (4a) and (4b) and the equilibrium condition that $\gamma=\infty$ is equivalent to $\mu=0$. From the point of view of hypothesis testing, the Bowden formulation is obviously of considerable convenience.

The problem of testing for disequilibrium thus reduces to a test of $\mu=0$.⁶ There are, in fact, several general ways in which this could be carried out.

- (1) The disequilibrium model could be estimated and the estimate for μ compared

⁶ Note that neither (4a,b) nor (4a',b') is useful if either the demand or supply function is perfectly horizontal. Even if such a case is ruled out a priori, some ambiguity remains in a test of μ , since it is a joint test of γ , α_1 and α_2 . On the other hand, if the original formulation of the problem had used (4a') or (4b'), then (4a) or (4b) would have been regarded as a reparameterization and its coefficient an amalgam of μ , α_1 and α_2 . There is complete symmetry between these views and one's concern for what the "natural" test is will be influenced by whether (4a,b) or (4a',b') are taken as the "natural" structural relations. In the light of this, it may be preferable to test $1/\gamma$ rather than μ , although see also footnote 21. For a still different approach see Hwang (1980).

with its asymptotic standard error. (2) Under maximum likelihood, a likelihood ratio test is also available via a comparison of L_{NSP} or L_{SP} with L_e . (3) A simpler approach is to estimate the reduced-form price equation, (18) by OLS. This involves regressing P_t on X_{dt} , X_{st} , and P_{t-1} . Since the coefficient of P_{t-1} provides an estimate of μ , this can be tested directly. It should be emphasized that this test only works when X_{dt} and X_{st} do not include a lagged price variable.

Several points are worth noting about these tests. The likelihood ratio statistic for the NSP model, $-2\log(L_e/L_{NSP})$, involves one restriction, implying that its asymptotic distribution is $\chi^2(1)$. The SP model contains two more parameters than the equilibrium model -- μ and σ_3^2 . While this naively suggests that $\chi^2(2)$ is appropriate, there is a problem in that, with $\mu=0$, σ_3^2 is not identified. In view of this ambiguity, we empirically examine both $\chi^2(1)$ and $\chi^2(2)$ for the SP model. There may, in fact, be a deeper problem involved since this observation raises the question of whether the likelihood ratio test based on the χ^2 distribution is strictly applicable in this instance at all.

A somewhat different but related point has recently been raised by Gourieroux and Monfort (1980). They argue that in testing whether $\mu=0$ one can obtain a more powerful test by imposing the restriction that $\mu \geq 0$.⁷ The problem then becomes one of testing whether μ is on the boundary of the constrained interval. The statistical methodology for carrying out such a test has been developed in Gourieroux, Holly and Monfort (1979). The details of their proposed procedure depend on precisely which test is being utilized.

⁷ As Gourieroux and Monfort seem to suggest, one might go further and impose $0 < \mu < 1$. The test they actually propose, however, only involves the one-sided inequality.

Consider first the case of the simple test of regressing P_t on X_{dt} , X_{st} , and P_{t-1} . Denoting by $\hat{\mu}$ the OLS estimator of μ , the constrained estimator is $\hat{\mu}$ if $\hat{\mu} \geq 0$ and zero otherwise. The relevant test statistic \tilde{t} , is the conventional t-statistic if $\hat{\mu} > 0$ and zero otherwise. Gourieroux and Monfort (1979) show that if a critical value, c , is chosen so that $\text{Prob}(\tilde{t} > c | H_0) = \alpha$, then c is identical to the critical value for the unconstrained test at level 2α . A similar procedure can be applied to the likelihood ratio test. One first computes the constrained maximum likelihood estimator and uses this to calculate the likelihood ratio statistic, $-2 \log \lambda$. The asymptotic critical region at level α is $-2 \log \lambda > c$ where c is chosen so that $\text{Prob}[\chi^2(1) > c] = 2\alpha$. For the NSP model this procedure is a relatively straightforward alternative to the unconstrained approach. In the SP model, the identification problem noted above appears to be a potential problem for the constrained approach as well.

5. SOME SAMPLING EXPERIMENTS

In this section we describe the results of some limited computer experiments designed to shed light on the issues of testing and estimation raised earlier. In essence we seek to improve on and extend the results obtained by Quandt (1978). These earlier results suffered from some shortcomings: a high failure rate in the computations⁸ and extremely high type-I errors. In addition,

⁸ Quandt (1978) estimated covariances which, as noted above, can be a source of computational failures. In the present experiments zero covariances are assumed. In addition, the use of analytic derivatives substantially improved the ease with which estimates can be computed.

Quandt (1978) employed $1/\gamma$ as the test statistic rather than the potentially more useful μ and also did not examine the NS model. As we shall see below, this latter is both the easiest and also a quite reliable source of tests of equilibrium vs. disequilibrium.

Design of experiments

The basic disequilibrium model utilized had the following form.

$$D_t = \beta_1 + \beta_2 x_{1t} + \beta_3 x_{2t} + \alpha_1 P_t + u_{1t}$$

$$S_t = \beta_4 + \beta_5 x_{3t} + \beta_6 x_{4t} + \alpha_2 P_t + u_{2t}$$

$$\Delta P_t = \gamma(D_t - S_t) + u_{3t}$$

$$Q_t = \min(D_t, S_t)$$

In the standard experiments the true values of the parameters were $\beta_1 = 18.0$; $\beta_2 = -.37$, $\beta_3 = 1.6$, $\beta_4 = 0.0$, $\beta_5 = -.54$, $\beta_6 = .2$, $\alpha_1 = -13.0$, $\alpha_2 = 3.5$, $\sigma_1^2 = 18.0$, and $\sigma_2^2 = 7.6$. The exogenous variables were identical in repeated samples and were generated in most cases from the uniform distribution over the ranges (20,90) for x_1 , (40, 120) for x_2 , (1,7) for x_3 , and (135,400) for x_4 . In addition to this "standard set" of exogenous variables, several other sets of exogenous variables were used. The general nature of these variants is given in Table 1, which documents the salient characteristics of the experiments⁹ Aside

⁹ As should be evident from the price-adjustment equation, the experiments also require an observation on P_t at time 0 . In essence, we selected P_0 by computing the equilibrium price corresponding to the mean values of the exogenous variables and the disturbances. Given symmetry, this tended to produce disequilibrium data with roughly half demand and half supply points.

from the exogenous variables, the experiments differed from one another in terms of sample size, error variances, and value of γ (or μ). This latter parameter is a critical one, since it determines the extent to which disequilibrium behavior is present in the experimental data. The other parameter of critical interest is σ_3^2 , the variance of the error in the price-adjustment equation. For $\sigma_3^2 = 0$ the NSP model is the correct one, while as σ_3^2 gets larger, the NSP model should involve an increasing element of misspecification.

An individual experiment consisted of 50 replications of generating data according to one of the three possible truths -- the equilibrium model, the NSP disequilibrium model, and the SP disequilibrium model -- and then estimating all three models. The one exception was that in those experiments where the underlying truth was the equilibrium model, we estimated the SP model in only two of the four experiments. The reason for this was the relatively high expense of computing SP estimates with equilibrium data.

Some Computational Details

Numerical optimization was generally performed by the Davidon-Fletcher-Powell (DFP) algorithm. DFP was used with numerical first derivatives when estimating the equilibrium model and the NSP model, and with analytic first derivatives when estimating the SP model. In all cases, however, the asymptotic variance-covariance matrix was calculated directly from the matrix of second partials and not from the so-called H matrix. In the few instances in which we estimated the SP model from underlying equilibrium data we made use of the quadratic hill-climbing algorithm.¹⁰

¹⁰ Considerable use was made of the quadratic hill-climbing algorithm in preliminary experiments, both to check on the reliability of DFP and to calibrate various control parameters for DFP.

Our computational experience was generally quite good, with no problems of unboundedness. In instances where we estimated the correct model, there were virtually no computational failures.¹¹ While the number of such failures was quite limited, in those cases where we estimated the incorrect model, two sorts of computational problems did arise. Almost invariably these arose in instances in which we estimated a more general model than the true underlying model. Specifically, when estimating the SP model with both equilibrium and NSP data, σ_3^2 tended to be driven to zero about 36% of the time.

The fact that σ_3^2 was sometimes driven to zero when estimating the SP model where it was inappropriate should hardly be surprising. Recalling that the limit of the SP likelihood function as $\sigma_3^2 \rightarrow 0$ is the NSP function, the data are simply trying to tell us that the NSP model is more appropriate. This is clearly sensible when the NSP model is the true one, and should not be particularly surprising when the underlying model is one of equilibrium. As a consequence, in those replications where σ_3^2 is driven to less than $.8 \times 10^{-5}$ (typically associated with computational indigestion of one sort or another), we did not discard the replication. Rather, we accepted the results at face value and took the SP estimates to be identical to the NSP estimates.¹²

The second computational problem arose when estimating the NSP model with equilibrium data. In about one-half of the replications, the optimized value of the NSP likelihood function did not exceed the optimized equilibrium likelihood

¹¹ Out of 750 replications there were two instances -- both with sample size 25 -- when the SP model failed to converge in the allowable number of iterations. These were the only failures when the correct model was estimated. See also the next footnote.

¹² In experiment 14 where the data come from an equilibrium-like version of the SP model, σ_3^2 was also driven to zero in 36% of the cases. For the other SP experiments, this happened 1% of the time (only in experiments 10 and 13).

function, although the differences were typically quite small. Since we know that the NSP function value can be made arbitrarily close to the equilibrium value (by choosing the equilibrium parameter estimates and letting $\mu \rightarrow 0$), the most plausible interpretation of this situation is that we have only achieved a local maximum and that the global maximum for these replications occurs at the boundary where $\mu=0$. As a consequence, for purposes of computing type-I errors when the equilibrium hypothesis was indeed true, we treated these cases as ones in which the hypothesis of equilibrium was accepted.¹³

Properties of the parameter estimates

Given all the raw data generated by the sampling experiments, there are many ways to compare the various parameter estimates. For simplicity, we shall focus on a few representative summary statistics, which are given in the upper parts of Tables 2-4. Each table corresponds to a particular underlying model with, for example, Table 2 containing the results for cases (1)-(4) in which the equilibrium model was used to generate the data.

¹³ A similar issue arose in estimating the SP model from equilibrium data where roughly 25% of the SP likelihood function values were less than the corresponding equilibrium values.

The equilibrium model has 10 parameters in common with the NSP and SP models while the two disequilibrium models have 11 parameters in common. For each table corresponding to a particular underlying model, the first two rows report the number of instances that the mean absolute deviation (MAD) is smaller for the two incorrect models. The next two rows contain the medians of the ratios obtained by dividing for each parameter the MADs of the two incorrect models by the corresponding MAD for the correctly specified model. This statistic provides a rough measure of the effects of misspecification on the parameter estimates.

The results in the first four rows of Tables 2-4 are as expected and have the following features. (1) When the truth is equilibrium, the equilibrium model almost invariably yields smaller MADs than either disequilibrium model. (2) With either disequilibrium truth, the equilibrium model uniformly yields larger MADs, except for a few instances in cases (13) and (14). These cases have both a small sample size and a large value of γ which tends to bring the disequilibrium model closer to an equilibrium one. (3) When the truth is equilibrium, the median MADs for the disequilibrium models are 6 to 18% larger than the equilibrium MADs.¹⁴ (4) Conversely, with either disequilibrium truth, the median equilibrium MADs are substantially higher than the disequilibrium MADs, again with the exception of cases (13) and (14). As seen in Table 3 and the first three columns of Table 4, the misspecification inherent in the equilibrium model

¹⁴ These percentages may overstate the consequences of misspecifying the equilibrium model since, in computing the summary statistics in Table 2, we did not make any adjustments for those cases where the equilibrium function values exceeded the disequilibrium value.

declines as γ increases.¹⁵ Nevertheless, there is an asymmetry in that it is considerably more costly to use an equilibrium model when a disequilibrium model is appropriate than vice versa. (5). To a lesser extent, a similar asymmetry exists between the two disequilibrium models in that the SP model works better for NSP data than vice versa. As anticipated, the two disequilibrium models behave more similarly the smaller is σ_3^2 and the larger is γ .

We now turn briefly to some other features of the results. One issue we examined was the consequence of correlated exogenous variables. In case (15) we generated normally distributed exogenous variables which are equicorrelated with a coefficient of .7. In all other respects case (15) is identical to case (9). The primary effect of the correlation was a mild increase in the variance of the sampling distribution of the parameter estimates.¹⁶ We also examined, via the Shapiro-Wilk test, the normality of the sampling distribution of the estimated parameters. Normality was almost invariably accepted for the equilibrium model, regardless of the true underlying model. With the exception of $\gamma(\mu)$ and σ_3^2 , the same was true for two disequilibrium models. Finally, for each parameter estimated, we examined the ratio of the average asymptotic standard deviation to the corresponding RMSE. For consistent estimates this ratio should be close to unity. Table 5 summarizes the frequency distribution of this statistic where we

¹⁵ Although not shown, the levels of the MADs tended to decrease as γ increased.

¹⁶ The coefficient-by-coefficient ratios of MADs for the SP model in case (15) to MADs in case (9) have a median of 1.22, suggesting a 22% increase in MADs due to multicollinearity in this 'median' sense. The median RMSE increase was 15%.

have aggregated over the various experiments of a given type. The distributions are clearly tightest when the correct model is used for estimation. As in earlier results, there are several notable asymmetries. In particular, the equilibrium distribution is quite poor when the underlying model is a disequilibrium one while the reverse is not true. Similarly, the NSP distribution is substantially more dispersed when the SP model is appropriate than vice versa.

Estimating D_t and S_t

In Section 3 we outlined several ways of estimating the expected values of D_t and S_t conditional on varying amounts of information. In particular, for both the SP and NSP models we suggested three measures of $E(D_t|I_t)$ and $E(S_t|I_t)$, where the information set, I_t , consisted of (X_{dt}, X_{st}) or (X_{dt}, X_{st}, Q_t) or $(X_{dt}, X_{st}, Q_t, P_t)$. As these calculations yield an estimate for each observation, they produce a vast quantity of data which must be severely compressed for summary purposes. Table 6 presents by experiment the average RMSE for each method of estimation. The numbers in this table were obtained by computing the RMSE of $E(D_t|I_t)$ around D_t for a given replication and then averaging over replications. Table 7 presents corresponding estimates for the average bias for each case, although to preserve space we have only reported results for the largest information set. Finally, Table 8 gives the average correlation between the estimates of $E(D_t|I_t) - E(S_t|I_t)$ and $(D_t - S_t)$. This table gives some indication of the usefulness of the various estimates in characterizing periods of excess demand or excess supply.¹⁷

The following summary observations apply to experiments (5)-(7) for which

¹⁷ Although not reported, we have also computed RMSES and the sign concordance for the excess demand measures. These additional statistics tell the same story as those we have reported.

the NSP model is the underlying truth. (1) All the average bias statistics are quite small.¹⁸ (2) The average RMSEs indicate a clear gain from conditioning on more information, both for the NSP model and for the SP model as well. (3) For a given information set, the correct NSP model yields systematically smaller RMSEs than the SP model which involves estimating an additional parameter.¹⁹ As compared with the next set of experiments, however, the differences are not particularly large. (4) All measures of estimated excess demand are highly correlated with actual excess demand.

We turn next to the cases in which the SP model was the underlying true one. Given the greater diversity of the characteristics of the experiments, the results are harder to summarize but the following generalizations seem appropriate. (1) The average bias statistics are small for the SP model but more substantial for the misspecified NSP model. This is particularly apparent in cases (11) and (13) which have relatively large values of σ_3^2 and case (8) which has the smallest value of γ (i.e., in which disequilibrium behavior is most pronounced). (2) For the SP model, conditioning on more information improves both the

¹⁸ To put these in perspective, we note that the mean values of demand and supply in the various cases ranged from 70-75, except in cases (13) and (14) where the range was 50-55.

¹⁹ As noted earlier, expected demand conditional on (X_d, X_s) should be identical for the NSP and SP models. With estimated parameter values, however, they need not be the same. In cases (5), (6), and (7), the NSP and SP parameter values are close enough to yield virtually identical estimates of expected demand. This is not true in the remaining cases.

average RMSEs and the average correlation coefficients.²⁰ The improvement in the correlation coefficients is most pronounced when σ_3^2 is large (cases (11) and (13)). For the misspecified NSP model the RMSEs and correlation coefficients do not systematically improve with increased information. Furthermore, for a given information set, there is frequently a substantial cost to using the misspecified NSP model. Not surprisingly, this effect is particularly pronounced for large values of σ_3^2 or small values of γ . This suggests it may be of considerable interest to be able to distinguish between the SP and NSP models. Fortunately, as we shall now see, we are able to do this quite well when it matters most.

Hypothesis Testing

In Section 4 we outlined tests of the hypothesis of equilibrium vs. disequilibrium based on μ and on the likelihood ratio statistic. The lower portions of Tables 2-4 contain the relevant results for these tests.

(1) The underlying truth is equilibrium (Table 2). There are three possible test statistics based on μ depending on whether we use SP, NSP, or OLS estimates. Given a 5% significance level, the fractions in Table 2 should be of the order of .05. For the unconstrained parameter estimates the relevant critical value is a "t-statistic" of 1.96 which yields an

²⁰ The RMSEs for the SP model also tend to decline with increasing γ . This is consistent with the earlier observation that larger γ makes for more precise parameter estimates.

average type I error of 9% for μ_{SP} , 6.5% for μ_{NSP} and 1.5% for μ_{OLS} .²¹ Since all of the estimated values of SP and NSP are strictly positive, the unconstrained and constrained estimates are identical. As a consequence, following the Gourieroux and Monfort (1980) suggestion yields average type I errors of 18% and 12.5% for SP and NSP respectively. For μ_{OLS} , their suggestion is right on the mark as using a critical value of 1.64 yields an average type I error of precisely 5%.

Table 2 also gives results for the various likelihood ratio tests based on the χ^2 distribution. For the SP model, as anticipated, $\chi^2(2)$ works best yielding an average type I error of 7%, while for the NSP model $\chi^2(1)$ yields an average type I error of 9%.²² While it is probably desirable to increase the number of replications to pin these percentages down more accurately, these limited experiments suggest that tests based on μ and on the likelihood ratio behave quite reasonably when the null hypothesis of equilibrium is indeed true.

²¹ It should be noted that estimation was carried out using the parameterization based on γ rather than μ . The implied value of μ was then computed along with its approximate standard error. In some preliminary experiments analogous to case (1) we also used the μ -parameterization and found the direct and indirect estimates of μ were identical and that the standard errors of μ were virtually identical. We have a slight suspicion that the accuracy of the standard errors may have deteriorated a bit in case (2) which has larger structural variances. This may well account for the relatively high type-I errors in case (2). In retrospect, it also suggests that the μ -parameterization may be preferable for estimation.

²² As with tests based on μ_{SP} and μ_{NSP} , the Gourieroux-Monfort suggestion gives substantially larger type-I errors.

(2) Underlying truth is disequilibrium (Tables 3 and 4). In cases (5) to (11) all of the tests are able to reject the hypothesis of equilibrium 100% of the time. In particular, for these cases tests based on the SP or the NSP models work equally well, regardless of which type of disequilibrium model is the underlying truth. The remaining three cases were designed especially to make it harder to detect disequilibrium. Case (12) is identical to case (9) except X_{1t} was chosen to be highly correlated with P_{t-1} (the average correlation coefficient is roughly .99). As anticipated, this sharply reduces the power of the reduced-form test based on μ OLS but the remaining tests work as well as before.

Cases (13) and (14), which have a sample size of 25 and the largest value of γ we tried, are in many ways the most interesting. The smaller sample size and the greater equilibrium character of the data (from the larger γ) should reduce our ability to detect disequilibrium and this is borne out in Table 4. In case (13), both tests based on the SP model yield a power of 82%. Tests based on the NSP model and OLS price regression have still lower power. In case (14), which differs from (13) only in that σ_3^2 is smaller, the superiority of tests based on the correct SP model remains but, as expected, the differences in power are narrower. What this and the earlier results on parameter estimates suggest is that it is desirable to be able to distinguish between the NSP and SP disequilibrium models.

This question is addressed in the last two rows of Tables 3 and 4 where we have reported a likelihood ratio test for the NSP vs. the SP model. Heuristically, the likelihood ratio test involves testing the restriction that $\sigma_3^2 = 0$. Since, however, σ_3^2 is bounded from below, appeal to the Gourieroux-Monfort view suggests that the use of a critical value based on $\chi_{.10}^2(1)$ might be appropriate. This seems to be borne out in the last row of Table 3,

where the NSP model is true, as the average type-I error is 6%. Using $\chi^2_{.05}(1)$ yields an average type I error of 2% which is a bit too low. In Table 4, where the SP model is true, the power of the test varies quite a bit. It is generally excellent when γ is small so that disequilibrium behavior is pronounced and when σ_3^2 is large -- cases (8) and (11). Looking at the median MAD ratios in row 4, we see that these are precisely the cases where it is important to be able to distinguish between the two models.

6. CONCLUDING REMARKS

The results of the previous section suggest that maximum likelihood methods can be quite successfully applied to estimate the types of disequilibrium models we have considered. Such methods yield parameter estimates which have quite acceptable small sample properties. Furthermore, these parameter estimates can easily be used to compute reasonable estimates of the expected values of the unobservable demand and supply variables.

As for hypothesis testing, both tests based on μ and on the likelihood ratio statistic work quite well in discriminating between equilibrium and disequilibrium models. A likelihood ratio test is also satisfactory for distinguishing between the NSP and SP models. As a practical matter, as long as lagged price does not appear in the demand or supply functions, the reduced-form OLS test on μ would appear to be a natural first step in testing for equilibrium. As we have seen, however, there are a variety of circumstances in which this test has relatively low power. The NSP model, which is relatively easy to compute offers a somewhat more reliable and effective test. There is no escaping the fact, however, that if σ_3^2 is non-zero, the most reliable tests require the estimation of the SP model.

TABLE 1

CHARACTERISTICS OF EXPERIMENTS

Case	True Model	Sample Size	$\gamma(\mu)$	σ_3^2	Other Features
1	Equilibrium	50	$\infty(0)$	0	
2	Equilibrium	50	$\infty(0)$	0	$4\sigma_1^2, 4\sigma_2^2$
3	Equilibrium	100	$\infty(0)$	0	
4	Equilibrium	50	$\infty(0)$	0	All X ranges $\div 2$
5	NSP	50	.045 (.576)	0	
6	NSP	50	.150 (.288)	0	
7	NSP	50	.242 (.200)	0	
8	SP	50	.045 (.576)	.34	
9	SP	50	.150 (.288)	.34	
10	SP	50	.242 (.200)	.34	
11	SP	50	.150 (.288)	2.72	
12	SP	50	.150 (.288)	.34	X_{1t} correlated with P_{t-1}
13	SP	25	.360 (.144)	2.72	All X ranges $\div 2$
14	SP	25	.360 (.144)	.34	All X ranges $\div 2$
15	SP	50	.150 (.288)	.34	X's normally distributed and equicorrelated ($r=.7$)

TABLE 2

SUMMARY RESULTS FOR EQUILIBRIUM MODEL TRUTH

	Case 1	Case 2	Case 3	Case 4
No. of parameters with smaller MAD in SP model	0	1	-	-
No. of parameters with smaller MAD in NSP model	0	1	0	2
median (SP MAD ÷ EQ MAD)	1.08	1.13	-	-
median (NSP MAD ÷ EQ MAD)	1.08	1.16	1.18	1.06
Fraction μ SP significant (1.96)	0.06	0.12	-	-
Fraction μ SP significant (1.64)	0.14	0.22	-	-
Fraction μ NSP significant (1.96)	0.04	0.08	0.08	0.06
Fraction μ NSP significant (1.64)	0.16	0.16	0.10	0.08
Fraction μ OLS significant (1.96)	0.02	0.02	0.0	0.02
Fraction μ OLS significant (1.64)	0.06	0.06	0.02	0.06
Fraction - $2 \log (L_e/L_{SP})$				
$\geq \chi^2_{.05} (1)$	0.08	0.10	-	-
$\geq \chi^2_{.10} (1)$	0.14	0.22	-	-
$\geq \chi^2_{.05} (2)$	0.06	0.08	-	-
$\geq \chi^2_{.10} (2)$	0.06	0.10	-	-
Fraction - $2 \log (L_e/L_{NSP})$				
$\geq \chi^2_{.05} (1)$	0.06	0.08	0.18	0.04
$\geq \chi^2_{.10} (1)$	0.10	0.16	0.22	0.08

TABLE 3

SUMMARY RESULTS FOR NSP MODEL TRUTH

	Case 5	Case 6	Case 7
No. of parameters with smaller MAD in EQ model	0	0	0
No. of parameters with smaller MAD in SP model	0	4	4
median (EQ MAD ÷ NSP MAD)	13.51	4.42	3.13
median (SP MAD ÷ NSP MAD)	1.05	1.00	1.00
Fraction μ SP significant (1.96)	1.00	1.00	1.00
Fraction μ NSP significant (1.96)	1.00	1.00	1.00
Fraction μ OLS significant (1.96)	1.00	1.00	1.00
Fraction $-2 \log (L_e/L_{SP}) \geq \chi^2_{.05} (2)$	1.00	1.00	1.00
Fraction $-2 \log (L_e/L_{NSP}) \geq \chi^2_{.05} (1)$	1.00	1.00	1.00
Fraction $-2 \log (L_{NSP}/L_{SP}) \geq \chi^2_{.05} (1)$	0.04	0.02	0.0
Fraction $-2 \log (L_{NSP}/L_{SP}) \geq \chi^2_{.10} (1)$	0.04	0.08	0.06

TABLE 4

SUMMARY RESULTS FOR SP MODEL TRUTH

	Case 8	Case 9	Case 10	Case 11	Case 12	Case 13	Case 14
No. of Parameters with smaller MAD in EQ model	0	0	0	0	0	1	3
No. of parameters with smaller MAD in NSP model	0	2	5	0	0	2	10
median(EQ MAD ÷ SP MAD)	7.71	3.58	2.67	3.43	7.47	1.20	1.05
median(NSP MAD ÷ SP MAD)	1.81	1.19	1.07	1.89	1.30	1.11	0.92
Fraction JSP significant (1.96)	1.00	1.00	1.00	1.00	1.00	0.82	0.94
Fraction JNSP significant (1.96)	1.00	1.00	1.00	1.00	1.00	0.70	0.90
Fraction JOLS significant (1.96)	1.00	1.00	1.00	1.00	0.18	0.54	0.72
Fraction JOLS significant (1.64)	1.00	1.00	1.00	1.00	0.24	0.74	0.80
Fraction -2 log (L _e /L _{SP}) ≥ χ _{0.05} ² (2)	1.00	1.00	1.00	1.00	1.00	0.82	0.90
Fraction -2 log (L _e /L _{NSP}) ≥ χ _{0.05} ² (1)	1.00	1.00	1.00	1.00	1.00	0.66	0.84
Fraction -2 log (L _{NSP} /L _{SP}) ≥ χ _{0.05} ² (1)	1.00	0.84	0.30	1.00	0.84	0.62	0.20
Fraction -2 log (L _{NSP} /L _{SP}) ≥ χ _{0.10} ² (1)	1.00	0.90	0.48	1.00	0.90	0.74	0.22

TABLE 5

FREQUENCY DISTRIBUTION OF RATIO OF AVERAGE
ASYMPTOTIC STANDARD DEVIATION TO RMSE

ESTIMATION METHOD	TRUTH	<.75	.75-.85	.85-.95	.95-1.05	1.05-1.15	1.15-1.25	>1.25
SP	SP	0.0	20.8	38.9	30.6	4.2	4.2	1.4
SP	NSP	0.0	3.7	37.0	40.7	14.8	3.7	0.0
SP	EQ	0.0	5.6	27.8	22.2	27.8	5.6	11.1
NSP	SP	9.7	12.5	38.9	25.0	5.6	5.6	2.8
NSP	NSP	0.0	3.7	37.0	44.4	7.4	7.4	0.0
NSP	EQ	0.0	5.6	33.3	30.6	16.7	5.6	8.3
EQ	SP	40.6	10.9	9.4	9.4	4.7	6.3	18.8
EQ	NSP	41.7	0.0	4.2	16.7	12.5	0.0	25.0
EQ	EQ	0.0	0.0	6.3	65.6	15.6	12.5	0.0

RMSES OF EXPECTED DEMAND AND SUPPLY
FOR VARYING INFORMATIONAL ASSUMPTIONS

TABLE 6

EXPERIMENT	NSP DEMAND CONDITIONED ON			SP DEMAND CONDITIONED ON			NSP SUPPLY CONDITIONED ON			SP SUPPLY CONDITIONED ON		
	X	X, Q	X, P, Q	X	X, Q	X, P, Q	X	X, Q	X, P, Q	X	X, Q	X, P, Q
5	2.97	1.90	.37	2.97	1.91	.48	2.49	1.69	.36	2.51	1.71	.53
6	2.42	1.07	.41	2.42	1.09	.51	2.38	1.10	.39	2.38	1.14	.56
7	2.35	0.82	.42	2.35	.88	.55	2.35	.85	.40	2.35	.93	.59
<u>SP-truth</u>												
8	5.88	4.73	8.09	5.27	4.11	3.16	3.76	4.46	8.05	2.89	2.34	2.26
9	3.29	2.09	2.59	3.25	2.25	2.09	2.57	2.23	2.75	2.51	1.84	1.82
10	2.79	1.50	1.68	2.78	1.62	1.54	2.44	1.61	1.77	2.43	1.52	1.50
11	7.00	5.30	6.75	6.53	5.11	2.99	3.88	5.86	7.13	3.04	2.57	2.27
12	3.37	2.20	2.69	3.31	2.26	2.14	2.60	2.30	2.81	2.54	1.84	1.82
13	3.83	2.76	3.00	3.71	3.05	2.40	2.80	2.71	2.94	2.63	2.22	2.16
14	2.51	1.23	1.28	2.51	1.48	1.40	2.36	1.24	1.30	2.37	1.39	1.42

TABLE 7

AVERAGE BIAS OF EXPECTED DEMAND AND SUPPLY

EXPERIMENT	NSP DEMAND	SP DEMAND	NSP SUPPLY	SP SUPPLY
NSP-truth				
5	- .04	- .04	- .04	- .04
6	- .03	- .01	- .03	- .01
7	- .03	- .01	- .03	- .00
SP-truth				
8	- 1.24	.00	- 1.08	- .03
9	- .34	.02	- .33	- .04
10	- .21	- .01	- .21	- .05
11	- 1.82	.04	- 1.78	- .01
12	- .42	- .09	- .42	- .12
13	- .99	.08	- .93	.23
14	- .25	.02	- .19	.04

TABLE 8

AVERAGE CORRELATION OF ESTIMATED
AND ACTUAL EXCESS DEMAND

EXPERIMENT	NSP EXCESS DEMAND CONDITIONED ON			SP EXCESS DEMAND CONDITIONED ON		
	NSP-truth	X	X,Q	X,P,Q	X	X,Q
5	.993	.998	1.00	.993	.995	1.00
6	.994	.995	1.00	.994	.994	.999
7	.995	.995	1.00	.995	.994	.998
SP-truth						
8	.965	.971	.877	.969	.982	.987
9	.974	.975	.960	.974	.979	.978
10	.976	.976	.970	.976	.977	.977
11	.859	.849	.738	.865	.926	.968
12	.978	.979	.967	.978	.983	.983
13	.699	.689	.607	.702	.751	.786
14	.934	.934	.930	.934	.918	.907

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APPENDIX

Assume for the sake of simplicity that the error terms u_{1t} , u_{2t} , u_{3t} in the SP model are normally distributed and independent of one another. We then prove the following

Theorem. $\lim_{\sigma_3^2 \rightarrow 0} L_{SP} = L_{NSP}$

Proof. It is sufficient to prove that the pdf of the endogenous variables in the SP model converges to that of the NSP model as $\sigma_3^2 \rightarrow 0$. It was shown in Quandt (1978) that the SP density may be written as

$$h(Q_t, P_t) = \frac{|1 + \gamma(\alpha_2 - \alpha_1)|}{2\pi\sigma_1\sigma_2\sigma_3} \left[\frac{e^{-B_{1t}/2}}{A_1^{1/2}} (1 - \Phi(\ell_{1t})) + \frac{e^{-B_{2t}/2}}{A_4^{1/2}} (1 - \Phi(\ell_{2t})) \right]$$

where

$$B_{1t} = (A_{3t}A_1 - A_{2t}^2)/A_1$$

$$B_{2t} = (A_{6t}A_4 - A_{5t}^2)/A_4$$

$$A_1 = 1/\sigma_1^2 + \gamma^2/\sigma_3^2$$

$$A_{2t} = -(\alpha_1 P_t + X_{dt} \beta_1)/\sigma_1^2 - \gamma(\gamma Q_t + \Delta P_t)/\sigma_3^2$$

$$A_{3t} = (\alpha_1 P_t + X_{dt} \beta_1)^2/\sigma_1^2 + (Q_t - \alpha_2 P_t - X_{st} \beta_2)^2/\sigma_2^2 + (\gamma Q_t + \Delta P_t)^2/\sigma_3^2$$

$$A_4 = 1/\sigma_2^2 + \gamma^2/\sigma_3^2$$

$$A_{5t} = -(\alpha_2 P_t + X_{st} \beta_2) / \sigma_2^2 + (\Delta P_t - \gamma Q_t) \gamma / \sigma_3^2$$

$$A_{6t} = (Q_t - \alpha_1 P_t - X_{dt} \beta_1)^2 / \sigma_1^2 + (\alpha_2 P_t + X_{st} \beta_2)^2 / \sigma_2^2 + (\Delta P_t - \gamma Q_t)^2 / \sigma_3^2$$

$$l_{1t} = A_1^{1/2} (Q_t + A_{2t} / A_1)$$

$$l_{2t} = A_4^{1/2} (Q_t + A_{5t} / A_4)$$

and where $\Phi(\)$ denotes cumulative standard normal distribution. The following are easy to verify:

$$(a) \lim_{\sigma_3 \rightarrow 0} \frac{|1 + \gamma(\alpha_2 - \alpha_1)|}{2\pi\sigma_1\sigma_2\sigma_3 A_i^{1/2}} = \frac{|1/\gamma + \alpha_2 - \alpha_1|}{2\pi\sigma_1\sigma_2} \quad \text{for } i=1,4$$

$$(b) \lim_{\sigma_3 \rightarrow 0} B_{1t} = \frac{(Q_t - \alpha_2 P_t - X_{st} \beta_2)^2}{\sigma_2^2} + \frac{(Q_t + \Delta P_t / \gamma - \alpha_1 P_t - X_{dt} \beta_1)^2}{\sigma_1^2}$$

$$\lim_{\sigma_3 \rightarrow 0} B_{2t} = \frac{(Q_t - \alpha_1 P_t - X_{dt} \beta_1)^2}{\sigma_1^2} + \frac{(Q_t - \Delta P_t / \gamma - \alpha_2 P_t - X_{st} \beta_2)^2}{\sigma_2^2}$$

$$(c) \lim_{\sigma_3 \rightarrow 0} l_{1t} = - \lim_{\sigma_3 \rightarrow 0} l_{2t} = \begin{cases} \infty & \text{if } \Delta P_t < 0 \\ -\infty & \text{if } \Delta P_t > 0 \\ 0 & \text{if } \Delta P_t = 0 \end{cases}$$

Substituting these limits in $h(Q_t, P_t)$ yields the pdf for the NSP model.