MEASUREMENT ERROR, BUDGET IDENTITIES AND THE ESTIMATION OF FINANCIAL MODELS

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1. INTRODUCTION

In an influential paper, Brainard and Tobin [1968] pointed out the implications of an economic agent's wealth constraint for the specification of models of financial behavior. Because wealth must equal the sum of its components, once the level of wealth and the specification of demand functions for all but one of the assets in the agent's portfolio are given, a unique specification for the last asset's demand function is implied. In flow terms, the budget identity is that net acquisition of financial assets minus net acquisition of financial liabilities must equal net financial investment. Any budget restriction which must be satisfied by a set of demand equations leads to cross-equation restrictions on the parameters characterizing the demand functions.

Several authors (Bachus and Purvis [1980], Hendershott [1971, 1977], Hendershott and Lemmon [1975], Motley [1970], Saito [1977] and Wachtel [1972]) have utilized the Brainard and Tobin approach to estimate models of the financial behavior of various sectors in the economy. These models have been estimated with data from the Federal Reserve Board's Flow of Funds Accounts and have typically not dealt explicitly with certain econometric problems which arise in attempting to implement the Brainard and Tobin framework. For example, the Flow of Funds Accounts are often criticized for being relatively inaccurate, particularly in allocating financial assets among the sectors of the economy. This measurement error problem shows up in the sizable discrepancies that appear between gross saving and gross investment in some of the sector statements of saving and investment. If the Flow of Funds Accounts are used as a data source for explanatory variables in a regression model of financial behavior, the standard errors in variables analysis implies that the resulting ordinary least squares estimators are biased and inconsistent. In addition, the presence of measurement error means that the data for some sectors do not satisfy the budget identities which are at the heart of the Brainard and Tobin approach.

In this paper a method is developed for estimating models containing an adding-up requirement due to a budget restriction when the observed data fail to satisfy the budget restrictions because of measurement error. The proposed estimator has a simple interpretation as an instrumental variable estimator, and, when the same set of explanatory variables appears in each equation describing the sector's behavior, as in Brainard and Tobin's original specification, parameter estimates satisfy the cross-equation constraints due to the budget identity even though each equation is estimated separately.

In section 2, the basic model is specified and estimation methods for the case of no measurement error are reviewed. The implications of measurement error are examined in section 3 and estimators are developed first for the case in which each equation contains the same explanatory variables and then for the case in which not all equations contain the same variables. Section 4 considers the problems that arise when the disturbance terms are autocorrelated, while section 5 provides a summary of the paper.

2. FINANCIAL SECTOR MODELS WITHOUT MEASUREMENT ERROR

Suppose we have a set of K equations describing the allocation of an exogenously determined constraint variable y amongst K different categories s_i, $i=1,\ldots,K$;

$$s_{it} = \beta_i y_t + x_{it}\gamma_i + \varepsilon_{it} \quad . \tag{2.1}$$

x_{it} is a $1xH_i$ vector of explanatory variables with coefficient vector γ_i, and it is assumed that all variables are written as deviations from their sample means. Let $(\gamma_{j1},\ldots,\gamma_{jK})$ be the vector of coefficients in all K equations of the j^{th} explanatory variable where $j=1,\ldots,H$ and H is the total number of explanatory variables other than y .

By definition

$$\sum_{i=1}^{K} s_{it} = y_t$$

which implies that

(2.2) $$\sum_i \beta_i = 1 \; ; \quad \sum_i \gamma_{ji} = 0 \quad \text{for} \quad j = 1,\ldots,H \; ; \quad \sum_i \varepsilon_{it} = 0 \quad .$$

We make the following set of assumptions which will be maintained throughout:

(2.3a) $$\text{plim} \frac{1}{T} \sum_t y_t \varepsilon_{it} = 0 \qquad \text{for all } i \; ;$$

(2.3b) $$\text{plim} \frac{1}{T} \sum_t x_{jt} \varepsilon_{it} = 0 \qquad \text{for all } i,j \; ;$$

(2.3c) $$\varepsilon_t' = (\varepsilon_{1t},\ldots,\varepsilon_{Kt}) \sim N(0,\Sigma_\varepsilon) \quad .$$

The basic error terms are taken to be normally distributed with mean zero and covariance matrix Σ_ε while y_t and x_{it} are assumed to be asymptotically uncorrelated with the ε_{it}'s . It will also be assumed that ε_t is distributed independently over time. This last assumption is relaxed in section 4.

Consider first the case in which each of the K equations contains the same set of explanatory variables; $x_{it} = x_t$ for all i . Let z be the $T\text{x}(H+1)$ matrix of observations on $z_t = (y_t x_t)$. We can write the i^{th} equation for all T observations as

(2.4) $$s_i = z\delta_i + \varepsilon_i$$

where $\delta_i = \binom{\beta_i}{\gamma_i}$. Let $\Delta = (\delta_1,\ldots,\delta_K)$ be the $(H+1)\text{x}K$ matrix of unknown coefficients. If ι is a $K\times 1$ vector of 1's, the constraints on the coefficient matrix Δ given by (2.2) can be written as $\Delta\iota = \binom{1}{0}$.

An instrumental variable estimator of δ_i in (2.4) would be given by

(2.5) $$\hat{\delta}_i = (w'z)^{-1} w' s_i$$

where w is a $Tx(H+1)$ matrix of instrumental variables with the properties

$$\text{plim}\,\frac{1}{T}\,w'\varepsilon_i = 0\ , \qquad\qquad i = 1,\ldots,K$$

$$\text{plim}\,\frac{1}{T}\,w'z = Q$$

where ε_i is the $Tx1$ vector of disturbances from the i^{th} equation and Q is a finite matrix of rank $H+1$. Letting $s = (s_1,\ldots,s_K)$,

$$\hat{\Delta} = (w'z)^{-1}w's\ .$$

$\hat{\Delta}$ is a consistent estimator of Δ and

$$\hat{\Delta}\iota = (w'z)^{-1}\,w's\iota$$

$$= (w'z)^{-1}\,w'y$$

$$= \binom{1}{0}\ .$$

Hence, the parameters obtained by estimating each equation separately satisfy the constraints on the true coefficients.[1]

Because the same set of explanatory variables appears in each equation, there is no gain in jointly estimating all K equations, and, given assumptions (2.3a) and (2.3b), efficient estimates are given by OLSQ where $w = z$. Each equation can be estimated separately to produce efficient estimates which satisfy the cross-equation coefficient restrictions implied by the requirement that $s\iota = y$.

Now consider the case in which the same explanatory variables do not appear in each equation (i.e., some elements of Δ are known to be zero). This produces two complications. First, estimating each equation by OLSQ results in inefficient estimates which will not satisfy the constraints in (2.2). Second, since $\varepsilon_t'\,\iota = 0$, Σ_ε is a singular, so Zellner's method for seemingly unrelated equations cannot be applied to the K equations in the model without some modification.

Suppose we write all K equations in stacked form:

(2.6) $$S = \begin{pmatrix} s_1 \\ \vdots \\ s_K \end{pmatrix} = \begin{pmatrix} z_1 & & 0 \\ & \ddots & \\ 0 & & z_K \end{pmatrix} \begin{pmatrix} \delta_1 \\ \\ \delta_K \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_K \end{pmatrix} = Z\delta + \varepsilon$$

where $z_i = (y, x_i)$ is a $T \times H_i + 1$ matrix of the explanatory variables which do appear in the i^{th} equation, $\delta_i = \binom{\beta_i}{\gamma_i}$ is a $1+H_i \times 1$ vector of coefficients, Z is $TK \times \Sigma H_i + K$ and $E(\varepsilon\varepsilon') = (\Sigma_\varepsilon \otimes I)$. In the general case for which rank $\Sigma_\varepsilon = K' < K$, Theil [1971, pp. 274-289] develops the appropriate generalized least squares estimator. Let Σ_ε^- be the generalized inverse of Σ_ε and let G be the $K \times K-K'$ matrix whose columns are the characteristic vectors of Σ_ε corresponding to its zero roots. By construction, $\Sigma_\varepsilon G = 0$. Premultiplying (2.6) by $(G' \otimes I)$ yields

(2.7) $$(G' \otimes I)S = (G' \otimes I)Z\delta + (G' \otimes I)\varepsilon \quad .$$

However, $E[G' \otimes I)\varepsilon\varepsilon'(G \otimes I)] = (G' \otimes I)(\Sigma_\varepsilon \otimes I)(G \otimes I) = 0$ so that $(G' \otimes I)\varepsilon \equiv 0$. Hence, (2.7) reduces to

(2.8) $$(G' \otimes I)S = (G' \otimes I)Z\delta \quad .$$

Unless $(G' \otimes I)Z = 0$, (2.8) implies the existence of cross-equation constraints on the elements of δ. With the additional assumption that rank $(F' \otimes I)Z = \Sigma H_i + K$ where F is a $K \times K'$ matrix with columns equal to the characteristic vectors of Σ_ε corresponding to its nonzero roots, the efficient estimator of δ (Theil, p. 285) is given by

(2.9) $$\hat{\delta}^* = \hat{\delta} + CZ'(GJ' \otimes I)[(JG' \otimes I)ZCZ'(GJ' \otimes I)]^{-1}(JG' \otimes I)(S - Z\hat{\delta})$$

where $$\hat{\delta} = CZ'(\Sigma_\varepsilon^- \otimes I)S$$

$$C = [Z'(\Sigma_\varepsilon^- \otimes I)Z]^{-1}$$

and J is a (rank $(G' \otimes I)Z) \times K-K'$ matrix of full row rank equal to a submatrix of the $K-K' \times K-K'$ unit matrix.

The constraint that $\varepsilon_t' \iota = 0$ given in (2.2) implies that $\sum_i E(\varepsilon_{it}\varepsilon_{jt}) = 0$ so that the rows of Σ_ε sum to zero: $\Sigma_\varepsilon \iota = 0$. Assuming this is the only linear dependency involving the rows of Σ_ε , rank $\Sigma_\varepsilon = K-1$, and the matrix G is given by $\iota/\sqrt{K}$. In addition,

(2.10) $$\Sigma_\varepsilon^- = (\Sigma_\varepsilon + \iota\iota'/K)^{-1} - \iota\iota'/K .$$ [2]

Thus, an efficient estimate of δ in (2.6) is given by $\hat{\delta}^*$ in (2.9) with Σ_ε^- given by (2.10), $G = \iota/\sqrt{K}$ and $J = 1$.

Replacing G by $\iota/\sqrt{K}$ in (2.8) yields T equations of the form

(2.11) $$\sum_i s_{it} = \sum_i \beta_i y_t + \sum_i x_{it}\gamma_i ,$$

but since $\sum_i s_{it} = y_t$, the constraints on δ implied by (2.8) are that $\Sigma\beta_i = 1$ and $\sum_i x_{it}\gamma_i = 0$. This last condition is satisfied if $\sum_i \gamma_{ji} = 0$.
It is also satisfied if
γ_{ji} is the same for all i and $\sum_i x_{it} = 0$. Hendershott [1971], for example, estimates a budget constrained model by specifying his explanatory variables so that $\sum_i x_{it} = 0$ and $\gamma_i = \gamma$ for all i . That is, each explanatory variable sums to zero across all K equations and has the same coefficient in each equation. Similar treatment of y_t produces a system in which $(G' \times I)Z = 0$ so that (2.8) places no constraints on δ and the efficient estimator is simply $\hat{\delta}$.[3]

Malinvaud [1970, p. 167] proposes estimating a model such as (2.6) with a singular variance matrix by minimizing

$$L_M = \varepsilon'[(\Sigma_\varepsilon + GG') \otimes I]^{-1} \varepsilon$$

subject to $(G' \otimes I)\varepsilon = 0$. Theil's procedure leading to (2.9) is equivalent to minimizing

$$L_T = \varepsilon'[\Sigma_\varepsilon^- \otimes I]^{-1}\varepsilon = \varepsilon'[((\Sigma_\varepsilon+GG')^{-1} - GG') \otimes I]\varepsilon$$

also subject to $(G' \otimes I)\varepsilon = 0$. However,

$$L_T = \varepsilon'[(\Sigma_\varepsilon + GG')^{-1} \otimes I]\varepsilon - \varepsilon'(GG' \otimes I)\varepsilon$$

$$= L_M - \varepsilon'(GG' \otimes I)\varepsilon \quad ,$$

and $\varepsilon'(GG' \otimes I)\varepsilon = \varepsilon'(G \otimes I)(G' \otimes I)\varepsilon$. Since both L_T and L_M are minimized subject to $(G' \otimes I)\varepsilon = 0$, both procedures will yield the same estimators.

In practice, Σ_ε would also have to be estimated. This could be done by using the residuals from estimating (2.6) by OLSQ subject to the constraints on the coefficients:

$$(2.11) \quad \hat{\delta}_{LS} = (Z'Z)^{-1}Z'S$$

$$+ (Z'Z)^{-1}Z'(G\otimes I)[(G'\otimes I)(Z'(Z'Z)^{-1}Z)(G\otimes I)]^{-1}(G'\otimes I)(I-Z(Z'Z)^{-1}Z)S \ .$$

The resulting estimated covariance matrix, $\hat{\Sigma}_\varepsilon = (S-Z\hat{\delta}_{LS})(S-Z\hat{\delta}_{LS})'/T$ will be singular but its generalized inverse can be calculated from (2.10). Final estimates of δ could then be obtained by replacing $\hat{\Sigma}_\varepsilon^-$ in (2.9).

The complications introduced by the singularity of Σ_ε could be eliminated by dropping one equation and using generalized least squares to estimate the remaining K-1 equations. Unless the eliminated equation happened to have contained y_t plus all the explanatory variables which appear in any of the K-1 other equations, the new system will still be subject to cross equation restrictions so that the estimator will still be of the form (2.9).

Maximum likelihood estimators for a system such as (2.1) have been studied by Barten (1969) who shows that such estimators can be derived from the maximum likelihood estimation of the K-1 equation obtained by dropping one equation. As long as those cross equation restrictions which remain when the deleted equation does not contain all the explanatory variables are utilized, the maximum likelihood estimators are invariant with respect to the particular equation to be deleted.

3. FINANCIAL SECTOR MODELS WITH MEASUREMENT ERROR

So far we have assumed that all the variables appearing in our model can be measured without error. If this is not the case, then the observed values of the variables may not satisfy the budget identities which we know must hold amongst the true variables. For example, the Flow of Funds Accounts report a statistical discrepancy for some sectors that measures the difference between, in the notation of section 2, the observed values of y_t and Σs_{it}, quantities which should be equal. If we attribute this statistical discrepancy to measurement error, the structure of the measurement error can be utilized to develop estimators which would be applicable to financial models of some of the sectors of the Flow of Funds Accounts. The estimators reported below were developed specifically for the household sector.

Consider first the case of common explanatory variables in each equation; in this situation, there is no loss of generality if we assume that y_t is the only explanatory variable. The model is thus

(3.1) $$s^*_{it} = \beta_i y^*_t = \varepsilon_{it} ; \qquad i=1,\ldots,K$$

where the * denotes the true value of the variable and, again, $\sum_i s^*_{it} = y^*_t$ so that $\sum_i \beta_i = 1$ and $\sum_i \varepsilon_{it} = 0$. Suppose that instead of observing s^*_i and y^* we observe

(3.2a) $$s_{it} = s^*_{it} + u_{it} ; \qquad i=1,\ldots,K$$

(3.2b) $$y_t = y^*_t + v_t$$

where u_{it} and v_t are random measurement errors assumed to be normally distributed with mean zero and covariance matrix given, if $u'_t = (u_{1t},\ldots,u_{Kt})$, by

(3.3) $$E\begin{bmatrix} u_t \\ v_t \end{bmatrix} [u'_t \; v_t] = \begin{bmatrix} \Sigma_u & \sigma_{uv} \\ \sigma'_{uv} & \sigma_{vv} \end{bmatrix} = \Omega \; .$$

Assume also that u_t and v_t are independent of ε_t and are independently distributed over time. Substituting (3.2a) into (3.1) yields

$$s_{it} = \beta_i y_t^* + (\varepsilon_{it} + u_{it}) \ . \tag{3.4}$$

In order to derive the maximum likelihood estimator of the β_i's in (3.4) it is necessary to assume that y_t^* is normally distributed with mean zero and variance σ_{yy}^*. As pointed out by Hsiao [1976], this is a very strong assumption and certainly unlikely to be satisfied in a time series setting unless we are dealing with detrended, seasonally adjusted variables. The maximum likelihood estimator of (3.1) will be shown, however, to have a simple interpretation as an instrumental variable estimator. Consequently, it will continue to have desirable properties even when y^* is not normally distributed. It will also be assumed that y^* and the random measurement errors are asymptotically uncorrelated.

Consider (3.2b) and (3.4). In this form, the framework is one of multiple indicators of the unobservable y_t^* where the covariance matrix of the observable indicators is

$$E\begin{bmatrix} s_t \\ y_t \end{bmatrix} [s_t' \ y_t] = \sigma_{yy}^* \begin{bmatrix} \beta\beta' & \beta \\ \beta' & 1 \end{bmatrix} + \Omega + \begin{bmatrix} \Sigma_\varepsilon & 0 \\ 0 & 0 \end{bmatrix} = \Theta \ . \tag{3.5}$$

where $\beta' = (\beta_1, \ldots, \beta_K)$. Goldberger [1974] discusses models of this type under the assumption that Ω is diagonal and develops maximum likelihood methods of estimation for $K > 2$ (if $K \leq 2$ the system is unidentified). In the present case, Ω is not assumed to be diagonal so without further restrictions the system is unidentified for all K .

We will make the following assumption: $\sigma_{u_i v} = 0$ for $i=1,\ldots,K$ so that

$$\Omega = \begin{bmatrix} \Sigma_u & 0 \\ 0 & \sigma_{vv} \end{bmatrix} \ . \tag{3.6}$$

That is, the measurement errors contained in the variables s_{it} are independent of any measurement error in y_t . To justify this assumption, suppose we are dealing with a model of the financial behavior of the household sector. In this case, y_t could be interpreted as total household saving, defined as disposable income minus consumer expenditures, while $s_{1t},\ldots,s_{Kt}$ would be net acquisitions of various categories of financial assets or, with negative signs, liabilities. Since y would normally be obtained from the National Income and Product Accounts while s_i would come from the Flow of Funds Accounts, it would seem reasonable to assume $\sigma_{u_i v} = 0$. This is particularly so for the household sector since each s_{it} for that sector is calculated as a residual, equal to the difference between total investment in the i^{th} asset and the investment by the non-household sectors in that asset.

This assumption of independence between the measurement error in the constraint variable and the measurement errors in the component variables cannot be made for all sectors in the Flow of Funds Accounts. For example, in the farm business sector, y_t is defined as the sum of the s_{it}'s . In this case, $v_t = \sum_i u_{it}$ and $\sigma_{u_i v}$ is obviously not zero for all i . Thus, the applicability of any estimation method based upon the assumption that $\sigma_{u_i v} = 0$ will depend crucially upon the sector of the economy being studied.

From (3.5) and (3.6) it is clear that we will not be able to separately identify Σ_ε and Σ_u since they enter (3.5) only in the form $\Sigma_\varepsilon + \Sigma_u$. To simplify the form of the equations to follow, Σ_ε , will be dropped. In the remainder of the paper then, Σ_u can be interpreted as the covariance matrix of $\varepsilon_t + u_t$.

Letting $P_t' = (s_{1t},\ldots,s_{Kt},y_t)$, the likelihood function of our sample of T observations on P_t is given, apart from a constant, by

$$(3.7) \qquad L = |\Theta|^{\frac{-T}{2}} \exp[-\tfrac{1}{2} \sum_t P_t' \Theta^{-1} P_t]$$

$$= |\Theta|^{\frac{-T}{2}} \exp[-\tfrac{1}{2} T \operatorname{tr}(\Theta^{-1} M)]$$

where $M = \frac{1}{T}\Sigma P_t P_t'$ is the matrix of sample variances and covariances among the observable variables. The maximization of L with respect to the unknown parameters of the model must be carried out subject to two types of constraints. First, we have the relationship between the reduced form parameters in Θ and the structural parameters consisting of β , Σ_u , σ_{vv} , and σ^*_{yy} . This relationship is given by (3.5) and (3.6) (with Σ_ε subsumed in Σ_u). We can rewrite these equations as

$$(3.8) \qquad \sigma^*_{yy}\begin{pmatrix} \beta\beta' & \beta \\ \beta & 1 \end{pmatrix} + \begin{pmatrix} \Sigma_u & 0 \\ 0 & \sigma_{vv} \end{pmatrix} = \Theta \ .$$

In addition,

$$(3.9) \qquad \sum_i \beta_i = 1 \ .$$

Equation (3.8) expresses the $\frac{1}{2}(K+1)(K+2)$ elements of Θ in terms of σ^*_{yy} , β , σ_{vv} , and the $\frac{1}{2}K(K+1)$ elements of Σ_u . Equation (3.9) implies that β contains only K-1 free parameters, so the total number of free structural parameters is $1+(K-1)+1+\frac{1}{2}K(K+1) = \frac{1}{2}(K+1)(K+2)$. The model is just identified;[4] (3.8) and (3.9) place no restrictions on Θ so that Θ can be estimated by the value that maximizes (3.7). The maximum likelihood estimator of Θ is therefore given by

$$(3.10) \qquad \hat{\Theta} = M \ .$$

The maximum likelihood estimators of σ^*_{yy} , β , σ_{vv} , and Σ_u can then be obtained as the solutions to (3.8) and (3.9) with $\hat{\Theta}$ replacing Θ .

If M_{xy} is the sample covariance between variables x and y ,

$$(3.11) \qquad \begin{pmatrix} M_{ss} & M_{sy} \\ M'_{sy} & M_{yy} \end{pmatrix} = \hat{\Theta} = \hat{\sigma}^*_{yy}\begin{pmatrix} \hat{\beta}\hat{\beta}' & \hat{\beta} \\ \hat{\beta}' & 1 \end{pmatrix} + \begin{pmatrix} \hat{\Sigma}_u & 0 \\ 0 & \hat{\sigma}_{vv} \end{pmatrix} .$$

Hence we have that $M'_{sy} = \hat{\sigma}^*_{yy}\hat{\beta}'$. Post multiplying both sides by ι yields

$$M'_{sy}\iota = \hat{\sigma}^*_{yy}\hat{\beta}'\ \iota = \hat{\sigma}^*_{yy}$$

since $\hat{\beta}\iota = 1$. Therefore

$$\hat{\beta}' = M'_{sy}/\hat{\sigma}^*_{yy} = M'_{sy}/M'_{sy}\ \iota = M'_{sy}/\sum_j M_{s_j y}$$

and

$$(3.12)\qquad \hat{\beta}_i = M_{s_i y}/\sum_j M_{s_j y}$$

$$= M_{s_i y}/M_{\tilde{y}y}$$

where $\tilde{y}_t = \sum_j s_{jt}$. The maximum likelihood estimator of β_i is thus equal to the instrumental variable estimator from the regression of s_{it} on $\tilde{y}_t$ with y_t used as the instrumental variable.

To see why this is the case, note that (3.1) can be written as

$$(3.13)\qquad s_{it} = \beta_i\tilde{y}_t + \varepsilon_{it} + u_{it} - \beta_i\sum_i u_{it}$$

since $\tilde{y}_t = \sum_i s_{it} = \sum_i (s^*_{it} + u_{it}) = y^*_t + \sum_i u_{it}$. Using OLSQ to estimate (3.13) will result in estimates for the β_i's which satisfy the adding-up requirement. This procedure of using the sum of the dependent variables as the constraint variable is the normal way of treating the problem of Σs_{it} not equalling y_t . However, $\tilde{y}_t$ is clearly correlated with the error term in each equation so that OLSQ estimators are biased and inconsistent.

Because $E(u_{it}y_t) = E(u_{it}v_t) = 0$, and $E(\tilde{y}_t y_t) = \sigma^*_{yy} \neq 0$, y_t can be used as an instrumental variable for $\tilde{y}_t$. As shown in section 2, any instrumental variable estimator of (3.13) will produce estimates which satisfy the cross-equation constraint on the coefficients, and using y_t as the instrumental variable produces the maximum likelihood estimator.

In terms of the structure of the Flow of Funds Accounts,

$$y_t - \tilde{y}_t = (y_t^* + v_t) - (y_t^* + \sum_i u_{it}) = v_t - \sum_i u_{it}$$

is equal to the statistical discrepancy reported in the Accounts.

Suppose we now consider the case in which not all the explanatory variables appear in every equation so that the model takes the form of (2.1), rewritten here in terms of the true variables:

$$(3.14) \qquad s_{it}^* = \beta_i y_t^* + x_{it}\gamma_i + \varepsilon_{it} ; \qquad i=1,\ldots,K .$$

It is assumed that the variables in the x_{it} vectors are observed without error. Substituting $s_{it} - u_{it}$ for s_{it}^* and $\tilde{y}_t - \sum_i u_{it}$ for y_t^*, we have, in terms of the observable variables,

$$(3.15) \qquad s_{it} = \beta_i \tilde{y}_t + x_{it}\gamma_i + \varepsilon_{it} + u_{it} - \beta_i \sum_i u_{it} ; \qquad i=1,\ldots,K .$$

Let the error term in (3.15) be denoted by ϕ_{it} and let ϕ_t be the $K\times 1$ vector of error terms for the t^{th} observation.

$$\phi_t = \varepsilon_t + (I - \beta\iota')u_t .$$

Define $\Sigma_\phi = E(\phi_t \phi_t')$. The rank of Σ_ϕ is only $K-1$ since $\iota'\beta = 1$ and $\iota'\varepsilon_t = 0$ implies that

$$\iota'\phi_t = \iota'\varepsilon_t + \iota'(I-\beta\iota')u_t = 0 .$$

The K equations in (3.14) could be estimated by using a procedure which combines the generalized least squares estimator in the presence of a singular covariance matrix that was given in (2.9) together with the use of y_t as an instrumental variable for $\tilde{y}_t$. Using the notation of (2.6) for the system of K equations written in stacked form with $\tilde{z}_i = (\tilde{y} \;\; x_i)$, $z_i = (y \;\; x_i)$, $\tilde{Z} = \text{diag}(\tilde{z}_1,\ldots,\tilde{z}_K)$, and $Z = \text{diag}(z_1,\ldots,z_K)$, we have

$$(3.16) \qquad s = \tilde{Z}\delta + \phi$$

and

$$(3.17)\qquad \hat{\hat{\delta}}^* = \hat{\hat{\delta}} + C\tilde{Z}'(G \otimes I)[(G' \otimes I)\tilde{Z}C\tilde{Z}'(G \otimes I)]^{-1}(G' \otimes I)(S - \tilde{Z}\hat{\hat{\delta}})$$

where

$$\hat{\hat{\delta}} = CZ'(\Sigma_{\phi}^{-} \otimes I)S$$

$$C = [Z'(\Sigma_{\phi}^{-} \otimes I)\tilde{Z}]^{-1}$$

$$\Sigma_{\phi}^{-} = [\Sigma_{\phi} + GG']^{-1} - GG'$$

and $$G = \imath/\sqrt{K} \quad .$$

Since the values taken on by the exogenous variables are arbitrary in the sense that we wish to place no constraints on the values they can take, it will be more convenient to rewrite the restrictions contained in equation (2.8) and utilized in (3.17) in a form that more explicitly shows the restrictions being placed on δ . We can define an $(H+1) \times (\Sigma H_i + K)$ matrix R (where H is the total number of explanatory variables other than y_t^* appearing anywhere in the model) consisting of zeros and ones such that

$$(3.18)\qquad R\delta = \begin{pmatrix} \Sigma\beta_i \\ \Sigma\gamma_{1i} \\ \vdots \\ \Sigma\gamma_{Hi} \end{pmatrix}$$

The cross-equation constraints can now be expressed as

$$(3.19)\qquad R\delta = \binom{1}{0} = r \ .$$

The estimator $\hat{\hat{\delta}}^*$ can be written then as

$$(3.17)\qquad \hat{\hat{\delta}}^* = \hat{\hat{\delta}} + CR'[RCR']^{-1}(r - R\hat{\hat{\delta}}) \ .$$

Given the budget identity and K-1 equations the K^{th} equation provides no additional information. We could drop any one equation and apply our instrumental variable-generalized least squares method to the remaining K-1 . The covariance matrix of the error terms for the K-1 equations, which we can write as $\underline{\Sigma_\phi} \otimes I$, will be nonsingular. Let a bar beneath a variable denote that the terms involving the K^{th} equation have been deleted. Let $\underline{R}$ differ from R in that $\underline{R}$ represents only the constraints on the coefficients of those explanatory variables which do not appear in the K^{th} equation. Since y_t^* is assumed to appear in all K equations, the constraints on $\underline{\delta}$ are that $\underline{R\delta} = 0$. Then,

$$(3.20) \qquad \underline{\hat{\hat{\delta}}*} = \underline{\hat{\hat{\delta}}} - \underline{CR'}[\underline{RCR'}]^{-1}\underline{R\hat{\hat{\delta}}}$$

where $\underline{\hat{\hat{\delta}}} = \underline{CZ'}(\underline{\Sigma_\phi}^{-1} \otimes I)\underline{S}$

$$\underline{C} = [Z'(\underline{\Sigma_\phi}^{-1} \otimes I)\underline{\tilde{Z}}]^{-1} \; .$$

$\underline{\Sigma_\phi}$ will normally not be known, but it can be estimated from the residuals obtained by estimating each equation separately using y_t as an instrumental variable for $\tilde{y}_t$. Since plim $\frac{1}{T}\underline{Z'}\phi = 0$, this will yield a consistent estimator, $\underline{\hat{\Sigma}_\phi}$ of $\underline{\Sigma_\phi}$. Substituting $\underline{\hat{\Sigma}_\phi}^{-1}$ for $\underline{\Sigma_\phi}^{-1}$ in (3.20) results in essentially a three stage least squares estimator subject to cross equation coefficient restrictions. Under the assumptions we have made, $\underline{\hat{\hat{\delta}}*}$ will be a consistent estimator of $\underline{\delta}$ and $\sqrt{T}(\underline{\hat{\hat{\delta}}*}-\underline{\delta})$ will have a normal limiting distribution with mean zero. To find the covariance matrix of this limiting distribution, we define the following:

$$(3.21) \qquad Q = \text{plim}[\tfrac{1}{T}\underline{Z'}(\underline{\Sigma_\phi}^{-1} \otimes I)\underline{\tilde{Z}}]^{-1} = \text{plim}[\tfrac{1}{T}\underline{Z^{*\prime}}(\underline{\Sigma_\phi}^{-1} \otimes I)\underline{Z^*}]^{-1}$$

where Z* differs from Z $(\tilde{Z})$ only in that y* appears rather than y $(\tilde{y})$. Let $P_i' = (v_i \; 0)$ be the $T \times (H_i+1)$ matrix of measurement errors in Z_i (by assumption, only y is measured with error). $\underline{P} = \text{diag}(p_1,\ldots,p_{K-1})$. Define

$$Q_v = \text{plim}[\tfrac{1}{T}\underline{P'}(\underline{\Sigma_\phi}^{-1} \otimes I)\underline{P}] \; .$$

Finally, let $B = Q - QR'(RQR')^{-1}RQ$. The covariance matrix of the limiting distribution of $\sqrt{T}(\hat{\hat{\delta}}^*-\delta)$ is given by

$$V = B + BQ_vB \quad . \qquad (3.22)$$

If we were able to observe y^* , the asymptotic covariance matrix for the generalized least squares estimator of δ would just be B . When we observe y and $\tilde{y}$ but not y^* , the covariance matrix is given by (3.22) which exceeds B by a positive semidefinite matrix, BQ_vB .

4. ESTIMATION WITH AUTOCORRELATED DISTURBANCES

So far we have assumed that the random disturbance terms, ε_{it} , u_{it} , and v_t , are distributed independently over time. With aggregate economic models, however this assumption is often invalid. For example, if the lagged dependent variable, $s^*_{i\,t-1}$, appears as an explanatory variable, replacing it with $s_{i\,t-1}$, will result in the lagged value of the measurement error, u_{it-1} , appearing in the error term. Also, the statistical discrepancy for the household sector in the Flow of Funds Accounts is highly serially correlated. Since this discrepancy is equal to $v_t - \iota' u_t$ either v_t or u_t , or both, must be serially correlated. If it is only v_t which is serially correlated,[5] then the error term in (3.15) will still be serially independent (in the absence of lagged dependent variables on the right hand side), and the estimation methods discussed in section 3 can be used without modification. Since it is unlikely, however, that the structure of the disturbance terms would take such a fortuitous form it is necessary to consider the estimation of (3.16) when the structure disturbances u_t and ε_t are serially correlated.

Under the assumption that u_t and ε_t are covariance stationary linearly nondeterministic stochastic processes, the multivariate generalization of Wold's decomposition theorem implies that we can express u_t and ε_t as multivariate moving average processes,

$$\begin{pmatrix} u_t \\ \varepsilon_t \end{pmatrix} = \begin{pmatrix} A(L) & 0 \\ 0 & B(L) \end{pmatrix} \eta_t$$

where $A(L)$ and $B(L)$ are $K \times K$ matrices of polynomials in the lag operator L and η_t is a $2k \times 1$ vector white noise process with the properties that $E(\eta_t) = 0$, $E(\eta_t \eta_s') = 0$ for $t \neq s$, and $E(\eta_t \eta_t')$ is a diagonal matrix. The error term for the K equations in (3.15),

$$\phi_t = \begin{pmatrix} I-\beta\iota' & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} u_t \\ \varepsilon_t \end{pmatrix} = \begin{pmatrix} I-\beta\iota' & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A(L) & 0 \\ 0 & B(L) \end{pmatrix} \eta_t ,$$

is a composite disturbance term (see Pagan [1973]), and can also be expressed as a multivariate moving average process (Granger and Morris [1976]). Since $\iota'\phi_t = 0$, we can delete the K^{th} equation and write $\underline{\phi}_t' = (\phi_{1t}, \ldots, \phi_{K-1,t})$,

$$\underline{\phi}_t = H(L)\Psi_t \tag{4.1}$$

where $H(L)$ is a $(K-1) \times (K-1)$ matrix of polynomials in L and Ψ_t is a $(K-1) \times 1$ vector white noise process, $E(\Psi_t) = 0$, $E(\Psi_t \Psi_s') = 0$ for $t \neq s$, and $E(\Psi_t \Psi_t') = \Sigma_\Psi$ is a diagonal matrix. It will be assumed that the roots of $|H(z)|$ all lie outside the unit circle and that $H(z)$ has full rank for all z in or on the unit circle. $H(L)$ can thus be taken to have an inverse, assumed to be of finite order and denoted by $C(L)$. We then have the autoregressive representation

$$C(L)\underline{\phi}_t = \Psi_t . \tag{4.2}$$

Writing the first $K-1$ equations for the s_i variables in stacked form as in section 3 yields

$$\underline{S} = \underline{\tilde{z}}\,\delta + \underline{\phi} \tag{4.3}$$

where we now have $(C(L) \otimes I)\underline{\phi} = \Psi$. Premultiplying both sides of (4.3) by $(C(L) \otimes I)$ produces

$$(4.4) \qquad (C(L) \otimes I)\underline{S} = (C(L) \otimes I)\underline{Z}\delta + \Psi .$$

The equation system (4.4) has an error term which is serially uncorrelated. In addition, the disturbance terms are uncorrelated across equations.

Estimation when $C(L)$ is known is straightforward. Letting $\underline{S}^* = (C(L) \otimes I)\underline{S}$, $\underline{\tilde{Z}}^* = (C(L) \otimes I)\underline{\tilde{Z}}$, and $\underline{Z}^* = (C(L) \otimes I)\underline{Z}$ where $\underline{Z}$ differs from $\underline{\tilde{Z}}$ in that y appears in place of $\tilde{y}$, the instrumental variable estimator of δ is given by

$$(4.5) \qquad \delta^{IV} = (\underline{Z}^{*\prime}\underline{\tilde{Z}}^*)^{-1}\underline{Z}^{*\prime}\underline{S}^* .$$

There is no gain in efficiency if all equations are estimated jointly. However, (4.5) ignores the cross-equation restrictions placed on $\underline{\delta}$ by the budget identity if not all the model's explanatory variables appear in the K^{th} equation. If there are such restrictions, estimation should proceed by using (3.20) with $\underline{Z}^*$, $\underline{\tilde{Z}}^*$ and $\underline{S}^*$ replacing $\underline{Z}$, $\underline{\tilde{Z}}$, and $\underline{S}$.

In the general case $C(L)$ is, of course, not known and must be estimated. In this situation the methods discussed in Fair [1972] can be modified to apply to the estimation of (4.4). First, estimate (4.3), ignoring the serial correlation in $\underline{\phi}$, using y as an instrumental variable for $\tilde{y}$. Instrumental variables will also be needed if any lagged values of s_i appear as explanatory variables in $\tilde{Z}$. The calculated residuals, $\hat{\underline{\phi}}$, from such a regression will be consistent estimators of $\underline{\phi}$. Normalizing the coefficient on ϕ_{it} in the i^{th} equation to equal one, we can write the typical equation in (4.2) as

$$(4.6) \qquad \phi_{it} = \sum_{j=1} c_{ij} \phi_{it-j} + \sum_{m \neq i} \sum_{j=0} c_{imj}\phi_{mt-j} + \Psi_{it} ; \qquad i=1,\ldots,K-1 .$$

Since Ψ_{it} is serially uncorrelated and $E\,\Psi_{st}\Psi_{it} = 0$, each of the K-1 equations of the form (4.6) can be estimated by OLSQ with $\hat{\phi}_{it}$ in place of ϕ_{it} . In addition to providing estimates of $C(L)$, this procedure also produces an estimate of Σ_{ψ} , which can be used if cross-equation restrictions require the joint estimation of all K-1 equations. The matrices $\underline{Z}^*$, $\tilde{\underline{Z}}^*$, and $\underline{S}^*$ can be calculated using the estimated $C(L)$, and then δ can be estimated using equation (4.5) or, in the case of cross-equation restrictions, (3.20) with the appropriate substitutions.

5. SUMMARY

The Flow of Funds Accounts provide data on the net acquisition of various assets and liabilities by the different sectors of the economy. Each sector is subject to a budget constraint, $\Sigma s^*_{it} = y^*_t$, which implies cross-equation restrictions on the sector's asset demand equations. In specifying equations for the s^*_{it}'s , y^*_t will appear as an explanatory variable. For some sectors, however, we have, due to measurement error, two alternative measures of y^*_t : Σs_{it} or y_t . The common practice is to use Σs_{it} as a measure of y^*_t since this choice ensures, if the same variable appear in each equation, that OLSQ applied to each equation yields coefficient estimates which satisfy the restrictions implied by the budget identity. This procedure neglects the information contained in y_t and will produce biased and inconsistent estimators. It was shown that a simple solution to this problem is to use Σs_{it} as a proxy for y^*_t and y_t as an instrumental variable for Σs_{it} . For the case of identical explanatory variables in each equation, this instrumental variable estimator is equal to the maximum likelihood estimator and produces estimates which satisfy the cross-equation restrictions on the coefficients.

FOOTNOTES

[1]This result is well-known. See Denton [1978].

[2]See Theil [1971, p. 274].

[3]Hendershott [1971] ignores the nonsingularity of Σ_ε, assumes it is diagonal, and obtains estimates of the diagonal elements from the OLSQ residuals from (2.6). Powell [1969] also considers a model similar to (2.6) in which x_{it} is partitioned as $[x_t, \bar{x}_{it}]$ where x_t is a vector of variables common to all equations and $\bar{x}_{it}$ are variables in equation i that do not appear in all K equations. He then requires that $\sum_i \bar{x}_{it} = 0$ and $\bar{\gamma}_i = \gamma$ for all i where $\bar{\gamma}_i$ is the vector of coefficients of $\bar{x}_{it}$.

[4]Again, this is with the proviso that only $\Sigma_\varepsilon + \Sigma_u$ can be identified.

[5]This could be tested since from data on y and the statistical discrepancy we can obtain the sample autocorrelations of v which could be compared with those of the statistical discrepancy. If the u_i's are serially independent, these would be the same.

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