Issued Under Office of Naval Research
Contract No. Nonr- 1858(16)

THE COST-OF-LIVING INDEX:

Combinatorial Theory

S. N. Afriat

Econometric Research Program
Research Memorandum No. 27
10 April 1961

Princeton University
Econometric Research Program
92-A Nassau Street
Princeton, N. J.

PREFACE

The problem considered is the evaluation of the range for a cost-of-living index which is compatible with given expenditure data.

This problem, as it is to be considered, has been discussed in a previous memorandum (No. 24), and the method developed here for treating it involves concepts and results given in various other memoranda (Nos. 7, 11,13,18 and 21).

Our method for solution of the index-number problem has a scope much beyond this application. It gives the means of establishing the range of any characteristics of a normal preference scale admitted by given expenditure data; for example, an equilibrium on a balance constraint, an elasticity, properties such as substitutability, complementarity, decomposability, and so forth, with every concept which depends on preferences in consumer theory. In this way it should be possible to give all such concepts a precise meaning in terms of a finite scheme of expenditure data. With that done, fitting instruments would be available for the empirical investigation of consumer behaviour, so far as that behaviour can be grasped systematically in direct terms of the classical framework.

It is always recognized that preferences change, and therefore will fail to be systematically consistent; and it is possible to identify many forces controlling preferences, which can enter into an explanation of such change and inconsistency. But so far there has been an absence of appropriate analytical machinery for investigating the drift of an entire preference structure under the influence of specific forces for change. An empirical preference structure must be somewhat loosely characterized, in view of the fragmentary character of the empirical data. The present method seems entirely appropriate in that requirement; it precisely represents the very vagueness of preferences.

Most important is the problem, which remains to be treated, of how inconsistent data should be analyzed. This calls for definition of what is to be considered most nearly a solution of those inequalities which are fundamental to the method for consistent data, but which have a solution if and only if the data are consistent. The same definition will determine a best solution even when there is a variety of solutions, that is, in the case of consistency, and thus give a principle by which the range of indeterminacy on consistent data can be narrowed in a significant way.

I wish to express my thanks to Professor T. C. Koopmans for drawing my attention to some difficulties in the question of convex representation, which is fundamental to the method, and for informing me of the work of de Finetti[1] and Fenchel[2] relating to it. Here I give a simple proof for the convex representation of a continuously twice-differentiable function with convex levels on a compact domain.

Also I have to acknowledge with thanks many useful discussions with Robert J. Aumann and John M. Danskin, who have contributed to some detailed aspects of this work, as will be fully recorded in the complete account.

The entire subject has its foundation in the revealed preference

---

[1] de Finetti, Bruno. Sulla stratificazioni convesse. Ann. Mat. Pura Appl. (4), 30(1949), 173-183.

[2] Fenchel, W. Convex Cones, Sets and Functions (from notes by D. W. Blackett of lectures at Princeton University, Spring Term, 1951; issued by the Department of Mathematics, September 1953).

method of Samuelson,[3] as amplified by Houthakker,[4] who proved the fundamental theorem on the numerical representation of preferences, and which has been investigated further by Uzawa.[5] Together with Houthakker's revealed preference axiom, the regularity condition, which appears in Houthakker's demonstration, and which has been treated further by Uzawa, seems to suffice and to be more appropriate for the needed concept of a normal expenditure system, rather than the over-restrictive uniformity condition which is used here. The appropriate modification of the normality concept can, however, be deferred for the moment, since it brings no modification in the method of empirical analysis, and the processes of calculation which belong to it, which are now the main object of investigation.

S. N. Afriat

Princeton, New Jersey
April 10, 1961.

---

[3] Samuelson, P. A. Consumption theory in terms of revealed preference. Economica 28 (1948), 243-253.

[4] Houthakker, H. S. Revealed preference and the utility function. Economica 17 (1950), 159-174.

[5] Uzawa, H. On the logical relation between preference and revealed preference. Technical Report No. 38, Department of Economics, Stanford University, 1956; or Mathematical Methods in the Social Sciences, Stanford Mathematical Studies in the Social Sciences V (Stanford, 1959).

TABLE OF CONTENTS

1. Expenditure systems and preference functions

Balance and composition spaces B, C are given by the positive orthants of Euclidean spaces of dimension n . An expenditure system ξ , with domain and range given by regions $B_o \subset B$, $C_o \subset C$, is defined by a mapping of balance vectors $u \in B_o$ into composition vectors $x \in C_o$ subject to the balance constraint $u'x = 1$:

$$\xi : B_o \to C_o \quad (u \to x ; u'x = 1) .$$

The composition belonging to balances u,v, ... are denoted by x,y, ...

$$x = \xi(u),\ y = \xi(v),\ \ldots ,$$

in which case

$$u'x = 1,\ v'y = 1,\ \ldots .$$

The vector pairs [u;x], [v;y], ... define the expenditure figures belonging to the system ξ .

An expenditure system is called responsive if different compositions correspond to different balances:

$$u \neq v \Rightarrow x \neq y .$$

In this case the expenditure system $x = \xi(u)$ mapping B_o onto C_o is a one-to-one correspondence, and defines an inverse expenditure system determining a unique balance $u = \xi^{-1}(x)$ belonging to any composition $x \in C_o$. Balances and compositions which belong to each other in a responsive expenditure system may be called reciprocals in that system.

The base-preference relation of a responsive expenditure system ξ is the relation Q_ξ between the elements of its range C_o defined by

$$xQ_\xi y \equiv u'y \leq 1 \wedge x \neq y .$$

The preference relation P_ξ of ξ is defined as the transitive closure of its base-preference relation:

$$P_\xi = \vec{Q}_\xi .$$

Thus P_ξ is transitive by construction, and is an order if it is irreflexive, in which case ξ is called <u>consistent</u>. This is the same as saying that the relation $\overset{o}{P}_\xi$, of <u>confusion</u> in ξ, defined by

$$x\overset{o}{P}y \equiv xP_\xi y \wedge yP_\xi x$$

is null. The relation $\tilde{P}_\xi$ of <u>irresolution</u> in ξ is the symmetrical relation defined between elements of C_o for which there is no relation of preference:

$$x\tilde{P}_\xi y \equiv \sim xP_\xi y \wedge \sim yP_\xi x \ .$$

If P_ξ is irreflexive, then $\tilde{P}_\xi$ is reflexive, and hence is an equivalence just if it is transitive.

Generally the relation $\tilde{P}_\xi$ of irresolution in a consistent system ξ is not an equivalence; but, if it is, then P_ξ will be that kind of order which is called a scale, which is asymmetric and whose negation $\overline{P}_\xi$ is transitive, which reduces to a complete order of the classes in this equivalence; and $\tilde{P}_\xi$ then defines the relation of <u>equivalence in the scale</u> P_ξ; and its classes define the <u>equivalence classes in the scale</u>.

An expenditure system is called <u>uniformly responsive</u> if there exists a number $\rho > 0$ such that

$$|x-y| > \rho|u-v| \ ,$$

in which case, to a displacement $u \to v$ of balance through a certain Euclidean distance $|u-v|$ there corresponds a displacement $x \to y$ of composition through a distance $|x-y|$ which is at least a fixed positive multiple ρ of that distance. Obviously a uniformly responsive expenditure is responsive, that is invertible; and the inverse system is continuous, giving $u = \xi^{-1}(x)$ as a continuous function of x.

For a uniformly responsive expenditure system which is consistent, this defining a <u>normal expenditure system</u>, the irresolution relation is an equivalence. Moreover, there exists a function φ, which is continuously differentiable, and has the property

$$\varphi(x) > \varphi(y) \Longleftrightarrow xP_{\xi}\,y$$

by which it represents P_{ξ}, and by which it is called a gauge of P_{ξ}. The equivalence classes determined by the relation $\tilde{P}_{\xi}$ are then the level surfaces of φ. Further, the function φ will necessarily be strictly increasing, and have strictly convex levels. Thus:

$$x \supset y \Rightarrow \varphi(x) > \varphi(y)$$

and

$$\varphi(x) = \varphi(y) \quad (x \neq y) . \Longrightarrow . \varphi(x\alpha + y\beta) > \varphi(x), \varphi(y) \qquad (\alpha, \beta > 0,\ \alpha + \beta = 1) .$$

A normal preference scale is defined as the preference relation of a normal expenditure system. The question arises as to whether it is possible to choose φ, from among all the possible gauges of P_{ξ}, to be a strictly convex function, satisfying

$$\varphi(x\alpha + y\beta) > \varphi(x)\alpha + \varphi(y)\beta \qquad (\alpha, \beta > 0,\ \alpha + \beta = 1) .$$

In other words, it is asked if every normal preference scale has a convex gauge.

This question has importance when ξ is specified only as being a normal expenditure configuration possessing a certain finite set of expenditure figures, forming an expenditure configuration $\mathfrak{F}$. It is required to characterize the totality of preference scales P_{ξ} such that $\mathfrak{F} \subset \xi$, with ξ normal. This is equivalent to the problem, which is going to be considered, of characterizing all the convex increasing continuously differentiable functions whose gradients at certain points, specified by the compositions in the figures of $\mathfrak{F}$, have certain directions, specified by the corresponding balances, just so long as it is known that every such scale has a convex gauge.

The matter is easily settled if the conditions for a normal expenditure system $\mathcal{E}$ are strengthened to include continuous differentiability, and the domain of the system is compact. In this case a gauge φ exists which is continuously twice differentiable, having a matrix $\varphi_{xx'}$ of second derivatives, which, existing and being continuous, must be symmetric. These conditions correspond exactly to the ones familiar in the theory of Slutzky, and which are necessary for the development of that theory, the essence of which is the symmetry condition. With these conditions, the possibility of the convexification of the preference function, by transformation to an equivalent one which is convex, has a simple and direct proof.

But to suppose an expenditure system is normal, that is uniformly responsive and consistent, is enough to get, in a behaviouristic fashion—that is, directly in terms of expenditures, rather than hypothetical underlying constructions—all the characteristics of the systems by which, either explicitly or implicitly, the consumer is usually pictured in economics; that is, with a numerically represented preference structure with smooth, strictly convex levels. This concept of normality seems to provide the simplest adequate analysis of this picture. A commitment to viewing the considered problem in terms of such systems is desirable. But the possibility of convexification for such systems is not yet demonstrated, though it is plausible.

A way around this difficulty, without altering this commitment, would be to show that it is possible to approximate uniformly the levels of a continuously differentiable function, with strictly convex levels in a compact domain, by the levels of a strictly convex function. Again, this possibility is plausible though it is not yet proved. Also, it would suffice for the general validity of the method which is going to be investigated.

Should convexification, and even approximate convexification, fail, normal expenditure systems must be defined more restrictively, to be differentiable. Then twice differentiable preference functions are obtained, and Fenchel's result may be used. But the validity of the method to be investigated must be considered correspondingly limited.

These questions will be left tentative for the moment. In the exposition, it is taken, as seeming a safe conjecture, that convexification is possible for preference functions of normal expenditure systems whose domains are compact. But approximate convexification would suffice for the validity of the final results. In the meanwhile, an independent proof will be given for the convexification of continuously twice-differentiable functions with strictly convex levels, which can also be derived as a special case of Fenchel's theorem. It may be noted that this will give another approach to the possibility of approximate convexification, if the following question could be answered affirmatively. Let the ϵ-average of a continuous function in a compact domain φ be the function obtained by replacing the value of φ at x by its average in the sphere centre x and radius ϵ. If φ is differentiable, the second ϵ-averages will be continuously twice differentiable approximations to φ. If the levels of φ are strictly convex, it is asked if the levels of these approximations are also strictly convex for sufficiently small ϵ. If this can be asserted, then the possibility of approximate convexification follows from the possibility of the convexification of these approximations.

Let $\mathcal{E}$ be an invertible and continuously differentiable expenditure system, so the matrix $x_{u'}$ of partial derivatives of the elements of x with respect to the elements of u' everywhere exists, and is regular and continuous. Assume the domain, and therefore also the range C_o of ξ to be compact.

Let

$$s = x_{u'}(1-ux') .$$

Then a necessary and sufficient condition for the consistency of ξ is that s be symmetric, and its quadratic form be negative in every direction not parallel to u :

$$s = s' , \quad du'sdu < 0 \quad (du \nparallel u) .$$

The existence, continuity and symmetry of s implies the existence of a function φ , with gradient φ_x , which is such that

$$\varphi_x = u \lambda ,$$

where $\lambda = x'\varphi_x$ since $u'x = 1$, and which is continuously twice-differentiable, determining the inverse $u_{x'}$ of the continuous matrix $x_{u'}$ from the relation

$$(1-ux')\varphi_{xx'} = (u_{x'} + uu')\lambda .$$

The negativity condition then implies P_ξ to be a scale, and φ to be a function with the property

$$\varphi(x) > \varphi(y) \Longleftrightarrow xP_\xi y ,$$

and which is strictly increasing, and has smooth strictly convex levels.

The strict level-convexity of φ provides that $dx'\varphi_{xx'}dx$ be negative definite under the constraint $\varphi_{x'}dx = 0$. The wanted convexity condition is that $\varphi_{xx'}$ be unrestrictedly negative definite; and it may not be satisfied. It is asked if φ can be transformed by a continuously twice-differentiable increasing function $\omega(t)$ to an equivalent function $\varphi^* = \omega(\varphi)$ for which this condition will be satisfied.

Now, for any such transform,

$$\varphi^*_x = \omega'(\varphi)\varphi_x$$

and

$$\varphi^*_{xx'} = \omega''(\varphi)\varphi_x\varphi_{x'} + \omega'(\varphi)\varphi_{xx'}$$

$$= \omega'(\varphi)(\varphi_{xx'} - \rho\varphi_x\varphi_{x'})$$

where

$$\rho = -\frac{\omega''(\varphi)}{\omega'(\varphi)}, \text{ and } \omega'(\varphi) > 0 .$$

Thus it is just asked that there exists a function ω such that $\omega'(\varphi) > 0$, and

$$\varphi_{xx'} - \rho\varphi_x\varphi_{x'}$$

be negative definite, it being given that $\varphi_{xx'}$ is negative definite under the constraint determined by φ_x. But, by a general theorem on constrained quadratic forms, with this given and not otherwise, all that is required is that

$$\rho > \sigma ,$$

where σ is defined as the maximum root of the equation

$$|\varphi_{xx'} - \sigma\varphi_x\varphi_{x'}| = 0 ,$$

and is therefore a continuous function of $x \in C_o$. At points where $\varphi_{xx'}$ is regular, it is easy to see that

$$\sigma = (\varphi_{x'}\varphi^{-1}_{xx'}\varphi_x)^{-1} .$$

If φ is already convex, so $\varphi_{xx'}$ is negative definite, then $\sigma < 0$. In this case $\rho > \sigma$ if ω is a convex function, since then $\rho > 0$; in which case φ^* will just be another convex function, equivalent to φ. Otherwise $\sigma \geq 0$, and it is required to find an ω such that $\rho > \sigma$ everywhere in C_o.

Since C_o is compact, and φ, σ are continuous, it is possible to define

$$\bar{\sigma}(t) = \max_{x \in C_o, \varphi(x)=t} \sigma(x) ,$$

and this will be continuous in C_o, since φ, σ are continuous, and

C_o compact. Without loss in generality, assume

$$\min_{x \in C_o} \varphi(x) = 0 .$$

With an arbitrary continuous function $\epsilon(t) > 0$, there can be determined a unique continuously twice differentiable function $\Omega(t)$ such that

$$-\frac{\Omega''(t)}{\Omega'(t)} = \bar{\sigma}(t) + \epsilon(t) ,$$

and

$$\Omega(0) = 0 , \; \Omega'(0) = 1 ;$$

and then Ω is an example of a function ω with the desired properties.

THEOREM. If φ is a continuously twice-differentiable function in a compact domain C_o, with strictly convex levels, then it can be transformed into a strictly convex function $\varphi^* = \omega(\varphi)$, by an increasing, continuously twice-differentiable function $\omega(t)$.

Let $x = \xi(u)$ $(u'x = 1)$ be an expenditure system satisfying the regularity condition

$$| \xi(\frac{u}{\rho}) - \xi(\frac{u}{\sigma}) | < M |\rho - \sigma| ,$$

so called by Uzawa in his investigation, and which was introduced by Houthakker in proving his fundamental theorem.

Let $\mathcal{L}$ be any path between u_o and u_1, described by a continuously differentiable function $u(t)$ $(0 \leq t \leq 1)$ such that

$$u(o) = u_o , \; u(1) = u_1 .$$

Then the differential equation

$$\xi(\frac{u}{\rho})' \frac{du}{d\rho} = 1 , \text{ with } \rho(o) = 1 ,$$

has a unique solution $\rho(t)$, determining a number $\rho_{o1}(\mathcal{L}) = \rho(1)$, depending on the path $\mathcal{L}$ between u_o and u_1. The path $\mathcal{K}$ described by $\frac{u}{\rho}$ defines the correction of the path $\mathcal{L}$. No further modification of a path is obtained by repeated correction. The number $\rho_{o1}(\mathcal{L})$ defines the total correction between u_o and u_1, along the path $\mathcal{L}$.

The expenditure system is called integrable if the differential form $\xi(u)'du$ is integrable, having an integrating factor μ and an integral ψ such that

$$\mu\xi(u)'du = d\psi .$$

In this case $d\psi = 0$ along a corrected path. For, by virtue of the balance constraint

$$\xi(\frac{u}{\rho})'(\frac{u}{\rho}) = 1 ,$$

the differential equations

$$\xi(\frac{u}{\rho})'\frac{du}{d\rho} = 1 \quad \text{and} \quad \xi(\frac{u}{\rho})'\frac{d(\frac{u}{\rho})}{dt} = 0$$

are equivalent.

Integrability is equivalent to the condition that, for any path $\mathcal{L}$ which is a cycle, the corrected path $\mathcal{K}$ is also a cycle. It is also equivalent to the condition that the total correction $\rho_{o1}(\mathcal{L})$ along a path $\mathcal{L}$ is the same for all paths with the same extremities.

Now Houthakker's conditions of regularity and consistency, in view of his result by the method of ascending and descending sequences, imply integrability, and therefore that $\rho_{o1} = \rho_{o1}(\mathcal{L})$ is a number defined independently of the path $\mathcal{L}$ for its construction. This number is identified with the threshold ratio in which expenditure must change to compensate in preference for a change of balance between u_o and u_1. The continuous process of compensating adjustment, keeping the integral $\psi(\frac{u}{\rho})$ constant along any path of u, is defined by the differential equation, and then determines the total compensating adjustment from one extremity to the other. This adjustment ρ_{o1} is also defined irrespective of any consideration of a path between u_o, u_1. It yields a unique point

$x_1^{(o)} = \xi(\frac{u_1}{\rho_{o1}})$, on a balance parallel to u_1 , and equivalent in preference to the point $x_o = \xi(u_o)$ on u_o , which replaces the point $x_1 = \xi(u_1)$ when expenditure is changed in the ratio ρ_{o1} while money prices remain fixed. The number ρ_{o1} is also the unique number defined by the condition

$$\psi(\frac{u_1}{\rho_{o1}}) = \psi(u_o) ,$$

where ψ is the function, the same as the indirect preference function of Houthakker, whose levels can be scanned by the method of path-correction, and which can be constructed by a process of integration.

2. Admissible preference hypotheses

Let $\mathcal{F} = \{E_r\} = [U;X]$ be an expenditure configuration with figures $E_r = [u_r;x_r]$ forming balance and composition sets $U = \{u_r\}$, $X = \{x_r\}$, such that $u_r \supset 0$, $x_r \supset 0$ and $u_r'x_r = 1$ $(r = 1,\ldots,k)$. It is assumed that $x_r \neq x_s$, $u_r \neq u_s$ $(r \neq s)$.

Let C denote the composition space, this being the positive orthant of a Euclidean space, whose points are given by the vectors $x \supset 0$. Let B denote the balance space, this being a replica of C , whose points are the balance vectors $u \supset 0$.

The cross-structure of $\mathcal{F}$ is given by $D_{\mathcal{F}} = \{D_{rs}\}$, where $D_{rs} = u_r'x_s - 1$. The base-preference relation $Q_{\mathcal{F}}$ of $\mathcal{F}$ is defined by

$$x_r Q_{\mathcal{F}} x_s \equiv D_{rs} \leq 0 \quad (r \neq s) .$$

The <u>preference relation</u> $P_{\mathcal{F}}$ of $\mathcal{F}$ is the transitive closure of the base-preference relation

$$P_{\mathcal{F}} = \vec{Q}_{\mathcal{F}} .$$

Thus the relation $P_{\mathcal{F}}$ is transitive; and the configuration $\mathcal{F}$ is called <u>consistent</u> if this relation is also irreflexive, and therefore an order.

Let $\mathcal{S}$ denote the normal preference scales on C. For any such scale $S \in \mathcal{S}$ there exists a continuously differentiable function φ, called a <u>gauge</u> for S, which represents S by the property

$$\varphi(x) > \varphi(y) \iff xSy .$$

It is to be granted that φ, in any case increasing, and with convex levels, can be chosen convex on any given compact domain C_o in C.

Any normal scale containing $P_{\mathcal{F}}$ defines an <u>admissible preference hypothesis</u> for $\mathcal{F}$. The <u>preference spread</u> of $\mathcal{F}$ is defined by the totality $\mathcal{S}_{\mathcal{F}}$ of admissible preference hypotheses:

$$S \in \mathcal{S}_{\mathcal{F}} \equiv S \in \mathcal{S} \wedge P_{\mathcal{F}} \subset S .$$

For every $S \in \mathcal{S}_{\mathcal{F}}$ there exists a non-empty class Γ_S^o of gauges of S which are convex on a compact domain C_o of C chosen to contain $\hat{x}$. The collection of these, for all scales admissible for $\mathcal{F}$, is

$$\Gamma_{\mathcal{F}}^o = \bigcup_{S \in \mathcal{S}_{\mathcal{F}}} \Gamma_S^o .$$

<u>The scales in</u> $\mathcal{S}_{\mathcal{F}}$, <u>restricted to the domain</u> C_o, <u>determine and are determined by the functions in</u> $\Gamma_{\mathcal{F}}^o$, <u>also restricted to</u> C_o.

The configuration $\mathcal{F}$ is called <u>normal</u> if $\mathcal{S}_{\mathcal{F}} \neq 0$, that is, if there exists a normal scale S which is an admissible preference hypothesis for $\mathcal{F}$.

If $S \in \mathcal{S}$, and $\varphi \in \Gamma_S^o$ then a necessary and sufficient condition for $S \in \mathcal{S}_{\mathcal{F}}$ is given by the <u>equilibrium conditions</u> of φ with

$$(1 - u_r x_r')g(x_r) = 0 \quad (r = 1,\ldots,k) ,$$

where g is the gradient of φ.

Reversely, if φ is a differentiable function, increasing and convex in C_o, it determines a normal scale S for which it is a gauge, convex on C_o : $\varphi \in \Gamma_S^o$. And if φ satisfies the equilibrium conditions with $\mathcal{F}$, then, and only then, $S \in \mathcal{S}_{\mathcal{F}}$.

It follows that *the functions in* $\Gamma_{\mathcal{F}}^o$ *are the differentiable functions which are increasing and convex in* C_o *and satisfy the equilibrium conditions with* .

Thus the following appears:

THEOREM. *The differentiable functions which are increasing and convex in any compact domain* C_o *containing* $\hat{X}$, *and satisfy the equilibrium conditions with* $\mathcal{F}$, *are gauges for scales which represent in* C_o *all the normal preference scales which are admissible hypotheses for* $\mathcal{F}$.

Thus, investigation of the scales which are admissible preference hypotheses for $\mathcal{F}$ is reduced to investigation of all the differentiable functions which satisfy the equilibrium conditions with $\mathcal{F}$, and are increasing and convex in an arbitrary compact domain C_o containing $\hat{X}$. Many of these functions, each of which must represent one of these scales, will represent the same scale. But each of these scales will be represented by at least one, and in fact an infinity, of these functions.

3. Gradient directors

A *gradient director* specifies the direction of the gradient of a differentiable function at each of a set of points.

The equilibrium conditions, for a differentiable increasing function φ with an expenditure configuration $\mathfrak{F}$, are just that $\mathfrak{F}$, considered as a gradient director, should admit φ . For the conditions are

$$g(x_r) = u_r\lambda(x_r) , \quad (r = 1,\ldots,k)$$

where $\lambda = x'g$ and g is the gradient of φ . Now $x \supset 0$; and, since φ is increasing, $g \supset 0$; so that $\lambda > 0$. Given a differentiable function φ , the function λ will be called the conjugate multiplier function; and it is always positive, for an increasing function in the positive orthant.

Thus the functions being investigated are all those which are increasing and convex, and which $\mathfrak{F}$ admits as a gradient director.

But a convex function which is increasing at a set of points must also be increasing everywhere in the convex closure of those points. Therefore if $\mathfrak{F}$, as a gradient director, admits a convex function φ , then $g(x_r) = u_r\lambda(x_r)$ where $\lambda(x_r) > 0$, so that $g(x_r) \supset 0$, since also $u_r \supset 0$. Hence φ is increasing at the points of X ; and since convex, must be increasing throughout the convex closure $\hat{X}$, and therefore has a continuation which is differentiable, convex and increasing in the compact domain C_o containing X .

THEOREM I. The differentiable convex functions which are increasing in C_o and satisfy the equilibrium conditions with an expenditure configuration $\mathfrak{F}$ are continuations of those which are just defined on $\hat{X}$ and are admitted by the gradient director provided by $\mathfrak{F}$.

A convex gradient director is defined to be a gradient director which admits a convex function. If the gradient director provided by an expenditure configuration $\mathfrak{F}$ is convex, it follows that there exist normal preference scales which are admissible hypotheses for $\mathfrak{F}$, or that $\mathfrak{F}$ is normal; and conversely.

THEOREM II. The conditions for the normality of an expenditure configuration and its convexity as a gradient director, are equivalent.

4. Gradient configurations

A gradient configuration determines the gradient at each of a set of points of any differentiable function which possesses it.

Any set of gradients $G = \{g_r\}$ in the directions given by $U = \{u_r\}$ are of the form

$$g_r = u_r \lambda_r ,$$

corresponding to any set $\Lambda = \{\lambda_r\} \supset 0$, of multipliers $\lambda_r > 0$.

The gradient configurations admitted by the gradient director provided by the expenditure configuration $\mathcal{J}$ are those of the form (X, G), for any $\Lambda \supset 0$, possessed by any differentiable function with gradient g_r at x_r. The functions admitted by the gradient director are precisely those which possess some gradient configuration admitted by the director. To every such function φ, there corresponds a multiplier set $\Lambda = \{\lambda_r\} \supset 0$, determined, by the relation $\lambda_r = \lambda(x_r)$, from the conjugate multiplier function λ.

The existence of a convex function admitted by a gradient director is thus equivalent to the existence of a multiplier set $\Lambda \supset 0$ such that there exists a convex function possessing the corresponding gradient configuration admitted by the director.

5. Skeletons

A functional skeleton specifies the value and the gradient at each of a set of points of any differentiable function which is on it.

Thus, a set of points $X = \{x_r\}$, together with a set of levels

$\Phi = \{\varphi_r\}$ and a set of vectors $G = \{g_r\}$ defines the skeleton

$$\Sigma = (X,\Phi,G) = \{x_r,\varphi_r,g_r\}$$

with <u>base</u>, <u>level</u> and <u>gradient sets</u> X, Φ and G. The triads (x_r,φ_r,g_r) $(r = 1,\ldots,k)$ are the skeleton components. The condition for a function φ with gradient g to be on Σ is that

$$\varphi(x_r) = \varphi_r \ , \ g(x_r) = g_r \quad (r = 1,\ldots,k) \ .$$

Also, with φ and X given, Σ can be defined as <u>the skeleton of</u> φ <u>on</u> X.

A skeleton Σ is also determined when a gradient configuration (X,G) is taken together with a level set Φ. However, a gradient configuration is determined when a gradient director $\mathcal{F}$ is taken with a multiplier set Λ. Therefore, to a gradient director $\mathcal{F}$ together with multiplier and level sets Λ,Φ there corresponds a skeleton, which can be denoted by $\Sigma_{\mathcal{F}}(\Lambda,\Phi)$, and called <u>the skeleton on the director</u> $\mathcal{F}$ <u>for the multiplier and level sets</u> Λ, Φ.

Reversely, given a skeleton $\Sigma = (X,\Phi,G)$ with components (x_r,φ_r,g_r) such that $\lambda_r = x_r'g_r \neq 0$, it determines a unique gradient director $\mathcal{F}$ with director figures $[u_r;x_r]$ such that $u_r'x_r = 1$, where $u_r = \frac{g_r}{\lambda_r}$. And then $\Sigma = \Sigma_{\mathcal{F}}(\Lambda,\Phi)$, where $\Lambda = \{\lambda_r\}$.

With a fixed gradient director $\mathcal{F}$ given, there is a one-to-one correspondence between the skeletons Σ which are on it, and the multiplier and level sets (Λ,Φ).

A <u>convex skeleton</u> is defined to be a skeleton which has some convex function on it. Accordingly, a gradient director is convex if it can be taken together with some level set to form a convex skeleton; and a gradient director $\mathcal{F}$ is convex if it can be taken together with some multiplier and level sets Λ,Φ to form a skeleton $\Sigma(\Lambda,\Phi)$ which is convex.

6. Consistency, convexity and normality

Consider a given expenditure configuration $\mathfrak{F}$. It is consistent if its preference relation $P_{\mathfrak{F}}$, in any case transitive, is irreflexive, and therefore an order.

Also $\mathfrak{F}$ is normal if its normal preference spread $\mathcal{S}_{\mathfrak{F}}$ is non-empty: $\mathcal{S}_{\mathfrak{F}} \neq 0$. In this case there exists a scale $S \in \mathcal{S}_{\mathfrak{F}}$ such that

$$P_{\mathfrak{F}} \subset S .$$

Since a scale S is irreflexive, it follows that $P_{\mathfrak{F}}$ is irreflexive; that is, $\mathfrak{F}$ is consistent. Therefore normality for $\mathfrak{F}$ implies its consistency.

Now the convexity of $\mathfrak{F}$, as a gradient director, has been seen to be equivalent to its normality. It is to appear that convexity is equivalent to consistency. It will follow that the three conditions of consistency, convexity and normality for an expenditure configuration are equivalent.

It only has to be shown that consistency implies convexity.

Consistency is directly equivalent to the condition:

(D) $D_{rs} \leq 0$, $D_{st} \leq 0$, $\ldots$, $D_{qr} \leq 0$ <u>is impossible for all distinct elements</u> $r,s,t,\ldots,q$ <u>from</u> $1,\ldots,k$.

But this condition (D) has been shown (Res. Mem. No. 21) equivalent to the condition:

(D') <u>There exists a multiplier set</u> Λ <u>such that</u>

$$\lambda_r > 0, \ \lambda_r D_{rs} + \lambda_s D_{st} + \ldots + \lambda_q D_{qr} > 0$$

<u>for all sets of distinct elements</u> $r,s,t,\ldots,q$ from $1,\ldots,k$.

But, further, this condition (Λ) has been shown (Res. Mem. Nos. 18,21) equivalent to the condition

(D") <u>There exist multiplier and level sets</u> Λ, Φ <u>such that</u>

$$\lambda_r > 0, \lambda_r D_{rs} > \varphi_s - \varphi_r$$

for all distinct elements r,s from $1,\dots,k$.

Thus the three conditions D, D' and D" on the cross-structure $D = \{D_{rs}\}$ of an expenditure configuration $\mathfrak{F}$ are equivalent.

Moreover, it is easy to see that D' is satisfied for the Λ in every Λ,Φ which satisfy D" . Also it can be shown (Res. Mem. No. 18) that for every Λ which satisfies D' , there exists a Φ such that Λ,Φ satisfy D" .

Now consider any skeleton Σ , and the conditions on this skeleton given by

$$\mathcal{N}(\Sigma) : g_r \supset 0 , (x_s - x_r)' g_r > \varphi_s - \varphi_r .$$

Also consider the condition

$$\mathcal{N}_{\mathfrak{F}}(\Lambda,\Phi) : \lambda_r > 0, \lambda_r D_{rs} > \varphi_s - \varphi_r$$

on multiplier and level sets Λ,Φ in respect to an expenditure configuration $\mathfrak{F}$.

With $\Sigma = \Sigma_{\mathfrak{F}}(\Lambda,\Phi)$, these conditions are equivalent:

$$\mathcal{N}(\Sigma_{\mathfrak{F}}(\Lambda,\Phi)) \Longleftrightarrow \mathcal{N}_{\mathfrak{F}}(\Lambda,\Phi) .$$

When $\mathcal{N}(\Sigma)$ is seen equivalent to the condition that Σ admit a function which is convex and increasing, and therefore a gauge for a normal preference scale, then the existence of Λ,Φ such that $\mathcal{N}_{\mathfrak{F}}(\Lambda,\Phi)$, already observed to be equivalent to the consistency of $\mathfrak{F}$, will be seen equivalent to the existence of a normal preference scale which $\mathfrak{F}$ admits on a gradient director. This latter condition is equivalent to the existence of a normal preference scale which is an admissible hypothesis for $\mathfrak{F}$, in other words, to the condition $\mathcal{S}_{\mathfrak{F}} \neq 0$, for the normality of $\mathfrak{F}$. It will follow thus that the condition for the consistency and normality of $\mathfrak{F}$ are equivalent.

THEOREM I. <u>A necessary and sufficient condition for a functional skeleton with components</u> (x_r, φ_r, g_r) <u>to be convex is that</u>

$$(x_r - x_s)' g_s > \varphi_r - \varphi_s .$$

A necessary and sufficient condition for a differentiable function φ, with gradient g, to be convex is that

$$(y-x)' g(x) > \varphi(y) - \varphi(x) \quad (y \neq x).$$

Therefore, if the function is convex, and is admitted by the skeleton, then, with $x = x_s$, $y = x_r$ and $g(x_s) = g_s$, it follows that the skeleton must satisfy the condition in the theorem.

Now suppose this condition holds for the skeleton. Consider the piece-wise linear function

$$\varphi(x) = \min_{r=1,\ldots,k} \{\varphi_r + (x - x_r)' g_r\} .$$

It is an almost everywhere differentiable convex function, differentiable in the neighborhood of every point x_r, with gradient g_r and value φ_r at x_r. It thus fails to be admitted by the skeleton only in that it fails in being everywhere differentiable.

It fails to be differentiable only at the boundaries of a finite polyhedral dissection of the x-space. In a neighborhood of these boundaries which excludes the points x_r, it can be smoothed, in an infinity of possible ways, into a differentiable function, preserving the convexity and without disturbing the value and the gradient in the neighborhood of these points. Then an everywhere differentiable convex function will be obtained which is admitted by the skeleton. It is possible, moreover, to construct convex functions on the skeleton which possess any number of derivatives.

A smoothing process which achieves this result is in accordance with an averaging method due to H. E. Bray. For information of this general method, I am indebted to J. M. Danskin. Let the ϵ-average of the continuous function φ, for $\epsilon > 0$, be defined as the function $\bar{\varphi}_\epsilon$ whose value at x is the average of the function φ in the sphere centre x at radius ϵ. Then φ is convex and differentiable. With ϵ less than the Euclidean distance of every point x_r from the nearest singularity of φ, the functions φ and $\bar{\varphi}_\epsilon$ coincide in the neighbourhood of each point x_r. Hence $\bar{\varphi}_\epsilon$ is a convex differentiable function admitted by the skeleton. By taking the ϵ-average repeatedly, with any sufficiently small ϵ, convex functions with any number of derivatives can be obtained, which coincide with φ in the neighbourhood of each point x_r, and are therefore admitted by the skeleton. It is noted that the derivative of the ϵ-average $\bar{\varphi}_\epsilon$ of φ is the everywhere defined ϵ-average $\bar{g}_\epsilon$ of the almost everywhere defined and continuous derivative g of φ.

COROLLARY. A necessary and sufficient condition for the existence of a differentiable function φ with gradient g such that

$$\varphi(x_r) = \varphi_r , \quad g(x_r) = g_r ,$$

and which is convex and increasing in a convex neighborhood containing the points x_r, is that

$$g_r \supset 0 \quad \text{and} \quad (x_r - x_s)' g_s > \varphi_r - \varphi_s .$$

For, by the theorem, there exists such a function which is convex. Now the corollary follows from Theorem 2.I.

Let $K_{\mathfrak{F}}$ denote all those skeletons $\Sigma_{\mathfrak{F}}(\Lambda, \Phi)$ such that $\mathcal{N}_{\mathfrak{F}}(\Lambda, \Phi)$. These will be called the normal skeletons for $\mathfrak{F}$. Let $\hat{\Gamma}_{\mathfrak{F}}$ denote the convex functions defined on the convex closure $\hat{X}$ of X which are admitted by a $\Sigma \in K_{\mathfrak{F}}$. Let $\Gamma_{\mathfrak{F}}$ denote the convex increasing functions defined on C which are admitted by a $\Sigma \in K_{\mathfrak{F}}$. Then the functions in $\Gamma_{\mathcal{F}}$ are the convex increasing continuations over C of the functions in $\hat{\Gamma}_{\mathcal{F}}$. These functions $\Gamma_{\mathcal{F}}$, considered restricted to C_o, provide the functions $\Gamma^o_{\mathcal{F}}$, already described, and which are the object of the analysis.

THEOREM II. *The consistency of an expenditure configuration* $\mathcal{F}$ *is equivalent to its normality, and to the existence of multiplier and level sets* Λ, Φ *satisfying the condition* $\mathcal{N}_{\mathcal{F}}(\Lambda,\Phi)$, *and therefore also to the existence of a non-empty class* $K_{\mathcal{F}}$ *of skeletons* $\Sigma_{\mathcal{F}}(\Lambda,\Phi)$ *such that* $\mathcal{N}_{\mathcal{F}}(\Lambda,\Phi)$. *These skeletons are such as to admit a non-empty class* $\Gamma^o_{\mathcal{F}}$ *of convex increasing functions defined on any compact domain* C_o *containing* $\hat{X}$. *The functions in* $\Gamma^o_{\mathcal{F}}$ *are the convex gauges existing in* C_o *for all normal preference scales* $\mathcal{S}_{\mathcal{F}}$ *which are admissible hypotheses for* .

Thus, given $\mathcal{F}$ and a compact domain C_o containing $\hat{X}$, for every $S \in \mathcal{S}_{\mathcal{F}}$, such as must exist provided $\mathcal{F}$ is consistent, there will exist a Λ, Φ such that $\mathcal{N}_{\mathcal{F}}(\Lambda,\Phi)$, and a convex increasing function φ on $\Sigma_{\mathcal{F}}(\Lambda,\Phi)$ which is a gauge for S in C_o.

Also, given any Λ, Φ such that $\mathcal{N}_{\mathcal{F}}(\Lambda,\Phi)$, such as exists provided $\mathcal{F}$ is consistent, and for every convex increasing function φ on $\Sigma_{\mathcal{F}}(\Lambda,\Phi)$, such as exists provided the condition $\mathcal{N}_{\mathcal{F}}(\Lambda,\Phi)$ holds, there corresponds a scale $S \in \mathcal{S}_{\mathcal{F}}$, for which it is a gauge.

Thus the problem of investigating the scales $S \in \mathcal{S}_{\mathcal{F}}$ in the region C_o is reduced to an investigation of all the convex increasing functions on the skeletons $\Sigma_{\mathcal{F}}(\Lambda,\Phi)$ such that $\mathcal{N}_{\mathcal{F}}(\Lambda,\Phi)$. To this end, there will first be considered the class of such functions on any one such skeleton.

7. Convex skeleton envelopes

Let $\Sigma = \Sigma_{\mathcal{F}}(\Lambda,\Phi)$, where $\mathcal{N}_{\mathcal{F}}(\Lambda,\Phi)$. Then

$$\Sigma = (X,\Phi,G) = \{x_r, \varphi_r, g_r\}$$

is a skeleton such that

$$g_r \supset 0 , \quad (x_r - x_s)' g_s > \varphi_r - \varphi_s .$$

In (x,φ)-space, consider the half-spaces Π_r defined by

$$\Pi_r : (x - x_r)' g_r \geq \varphi - \varphi_r ,$$

at the points V_r defined by

$$V_r = (x_r, \varphi_r) .$$

Every V_r lies on the frontier of Π_r , and in the interior of every Π_s $(s \neq r)$.

Also consider all the half-spaces Π defined by

$$\Pi : \omega + g'x \geq \varphi$$

where

$$g \supseteq 0 , \quad \omega \text{ finite} ,$$

and

$$\omega + g'x_r \geq \varphi_r \quad (r = 1,\ldots,k) .$$

Every Π_r is a Π ; and every Π contains every V_r .

Let W_i, W_o be the intersections of the Π_r , and all the Π , respectively, and let Φ_i, Φ_o be their boundaries.

Then Φ_i is a convex polyhedron with the Π_r for its faces, representing a convex increasing function φ_i ; and Φ_o is a convex polyhedron with vertices V_r , representing a convex, non-decreasing function φ_o .

These functions determine a pair of polyhedral dissections Δ_i, Δ_o of the x-plane, by projection of these surfaces parallel to the φ-axis. Each point x_r lies in the interior of one and only one cell of the dissection Δ_i , which may be denoted by N_r , throughout which the gradient of φ_i is constant, and equal to g_r . Any vertex of Φ_i is the intersection of certain hyperplanes $\Pi_r, \Pi_s, \ldots$ and may be denoted by $V_{r,s,\ldots}$. It projects into a point in the x-space which may be denoted by $x_{r,s,\ldots}$, which is a vertex of each of the cells $N_r, N_s, \ldots$ in the dissection Δ_o . It lies in the interior of one and only one cell of the dissection Δ_i which is determined as the polyhedron with vertices $x_r, x_s, \ldots$ and may be denoted by $N_{r,s,\ldots}$. Throughout each cell $N_{r,s,\ldots}$ of Δ_o the gradient of φ_o is constant, and given by some vector, which may be denoted by $g_{r,s,\ldots}$ whose direction is bounded in the convex closure of the directions of $g_r, g_s, \ldots$ so that $g_{r,s,\ldots} \supset 0$, since $g_r, g_s, \ldots \supset 0$. If $\Pi_{r,s,\ldots}$ is the hyperplane joining the points $V_r, V_s, \ldots$, then it is given by

$$\Pi_{r,s,\ldots} : (x-x_{r,s,\ldots})' g_{r,s,\ldots} = \varphi - \varphi_{r,s,\ldots}$$

for some $\varphi_{r,s,\ldots}$ determined together with $g_{r,s,\ldots}$ by the condition that $\Pi_{r,s,\ldots}$ passes through $V_r, V_s, \ldots$.

Let φ^* be any convex increasing function, with gradient g^* , which is on the skeleton Σ , so that

$$\varphi^*(x_r) = \varphi_r \ , \ g^*(x_r) = g_r \ .$$

Since it is increasing,

$$g^* \supset 0 \ ,$$

and since it is convex

$$(x-x_o)' g^*(x_o) > \varphi^*(x) - \varphi^*(x_o) \qquad (x \neq x_o) \ .$$

The points

$$V^*_{x_o} = (x_o, \varphi^*(x_o))$$

describe a convex surface Φ^* in (x,φ)-space representing the function

φ^*. The half-spaces Π^* of the form

$$\Pi^*_{x_o} : (x-x_o)'g^*(x_o) \geq \varphi(x) - \varphi(x_o)$$

intersect in a convex region W^*, whose boundary is Φ^*, which has the Π^* for its supports.

Since φ^* is on Σ, it follows that every Π_r is a Π^* and every Π^* is a Π. Hence, for the regions of intersection of these half-spaces,

$$K_o \subset K^* \subset K_i ;$$

and then, for the functions represented by the boundaries of these regions,

$$\varphi_o(x) \leq \varphi^*(x) \leq \varphi_i(x) ,$$

with the equalities attained just for $x = x_r$ $(r = 1,\ldots,k)$.

The polyhedral functions φ_o, φ_i are differentiable almost everywhere, failing in differentiability only at the boundaries of the dissections Δ_o, Δ_i with vertices $x_{r,s,\ldots}$ and x_r, respectively. Though they are not themselves admitted as being on the skeleton Σ, since they are not everywhere differentiable, they are limits of convex increasing function on Σ, since there exist such functions φ^* arbitrarily close to them.

Evidently

$$\varphi_i(x) = \min_{r=1,\ldots,k} \{\varphi_r + (x-x_r)'g_r\} .$$

A set of numbers $\alpha = \{\alpha_r\}$ such that

$$\Sigma\alpha_r = 1$$

will be called a distribution, and a positive distribution if $\alpha \supseteq 0$, that is $\alpha_r \geq 0$. Now for every positive distribution α, let

$$x_\alpha = \Sigma x_r \alpha_r ; \; \varphi_\alpha = \Sigma \varphi_r \alpha_r .$$

Then x_α describes the convex closure $\hat{X}$ of $X = \{x_r\}$ as α ranges through all possible distributions. For every $x \in \hat{X}$ there will be a variety of distributions α such that $x_\alpha = x$. Evidently

$$\varphi_o(x) = \max_{x=x_\alpha} \varphi_\alpha \quad (x \in \hat{X}) .$$

Let C_x denote the positive orthant space C translated to a point $x \in C$, so that

$$y \in C_x \equiv y \supseteq x .$$

Let C_X be the union of the C_x for all x in the convex closure $\hat{X}$ of X , so that

$$y \in C_X \equiv \bigvee_z z \in \hat{X} \wedge y \supseteq z .$$

Then

$$\varphi_o(x) = \begin{cases} \max\limits_{x \supseteq y \in C_x} \varphi_o(y) & (x \in C_X) \\ -\infty & (x \in \bar{C}_X) \end{cases}$$

where $\bar{C}_X$ is the complement in C of C_X .

The functions φ_o, φ_i thus defined are to be called the <u>outer</u> and <u>inner envelopes</u> of the convex increasing differentiable functions on the skeleton Σ .

THEOREM. <u>Let</u> φ^* <u>be any convex increasing differentiable function, with gradient</u> g^* , <u>such that</u>

$$\varphi^*(x_r) = \varphi_r , \quad g^*(x_r) = g_r \quad (r = 1,\ldots,k) .$$

<u>Let</u>

$$\varphi_i(x) = \min_{r=1,\ldots,k} \{\varphi_r + (x-x_r)' g_r\}$$

<u>and</u>

$$\varphi_o(x) \doteq \max_{x \supseteq x_\alpha} \{\varphi_\alpha , -\infty\}$$

<u>where</u>

$$x_\alpha = \Sigma x_r \alpha_r , \quad \varphi_\alpha = \Sigma \varphi_r \alpha_r$$

and

$$\alpha_r \geq 0\ ,\ \Sigma\alpha_r = 1\ .$$

Then

$$\varphi_o(x) \leq \varphi^*(x) \leq \varphi_i(x)$$

with the equalities attained just if $x = x_r$, for some r . For every $\epsilon > 0$ there exist such functions φ^* with either of the properties

$$\varphi^*(x) < \varphi_o(x) + \epsilon\ ,\quad \varphi^*(x) > \varphi_i(x) - \epsilon$$

so the functions φ_o and φ_i , though not differentiable, are limits of such differentiable functions φ^* .

8. Polyhedral preference maps

Cutting the convex surfaces Φ_o, Φ_i by the hyperplane $\varphi = \varphi_r$ in (x,φ)-space, and projecting into the x-space parallel to the φ-axis, there is obtained a pair of convex polyhedra through x_r , which may be called the enveloping preference levels of x_r , associated with multiplier and level sets Λ, Φ for which $\mathcal{N}_{\mathcal{F}}(\Lambda,\Phi)$. They may be denoted by

$$\mathcal{P}_r^o = \mathcal{P}_r^o(\Lambda,\Phi)\ ,\ \mathcal{P}_r^i = \mathcal{P}_r^i(\Lambda,\Phi)\ .$$

Thus, $\mathcal{P}_r^i$ is the boundary of the region determined by the inequalities

$$(x - x_s)' g_s \geq \varphi_r - \varphi_s \qquad (s = 1,\ldots,k)\ .$$

And $\mathcal{P}_r^o$ is the boundary of the region composed of the points

$$x \supseteq x_\alpha \quad (\varphi_\alpha \geq \varphi_r)\ .$$

Now, for any scale $S \in \mathcal{S}_{\mathcal{F}}$, restricted to some compact region C_o containing $\widehat{X}$, there exists a convex gauge φ , which has, with respect to X , a skeleton Σ_φ which $\mathcal{F}$ admits as a gradient director, and is therefore of the form

$$\Sigma_\varphi = \Sigma\ (\Lambda,\Phi)\ ,\ \text{where}\ \ \mathcal{N}_{\mathcal{F}}(\Lambda,\Phi)\ ,$$

in which case S will be classified as a (Λ,Φ)-scale in $\mathcal{S}_{\mathcal{F}}$.

There is a multiple correspondence between the scales $S \in \mathcal{S}_{\mathcal{J}}$, and the multiplier and level sets Λ, Φ such that $\mathcal{N}_{\mathcal{J}}(\Lambda,\Phi)$, since every scale S has a variety of gauges, φ, each on a different skeleton $\Sigma_{\mathcal{J}}(\Lambda,\Phi)$, while on every such skeleton there are gauges for a variety of scales.

But if S is a (Λ,Φ)-scale, then it is known that the locus $\tilde{S}_r$ of points x equivalent to x_r in S is the strictly convex surface determined as the level surface $\varphi(x) = \varphi_r$ of the gauge φ, and therefore is bounded between the enveloping preference levels, $\mathcal{P}_r^o$, $\mathcal{P}_r^i$ obtained from the skeleton $\Sigma_{\mathcal{J}}(\Lambda,\Phi)$, the three surfaces having x_r as their only common point. This is true for every Λ,Φ such that S is a (Λ,Φ)-scale. Also, for every Λ,Φ such that $\mathcal{N}_{\mathcal{J}}(\Lambda,\Phi)$, there exists a scale $S \in \mathcal{S}_{\mathcal{J}}$ which is a (Λ,Φ)-scale; moreover, S can be chosen so that $\tilde{S}_r$, in any case between $\mathcal{P}_r^o$ and $\mathcal{P}_r^i$, is arbitrarily close to either of these surfaces.

All the admissible preference levels $\tilde{S}_r$ through x_r, determined relative to the scales $S \in \mathcal{S}_{\mathcal{J}}$, are thus characterized as the strictly convex surfaces confined between pairs of surfaces $\mathcal{P}_r^o(\Lambda,\Phi)$, $\mathcal{P}_r^i(\Lambda,\Phi)$ such that $\mathcal{N}_{\mathcal{J}}(\Lambda,\Phi)$.

THEOREM. For an expenditure configuration $\mathcal{J} = [U;X]$, where $U = \{u_r\}$, $X = \{x_r\}$, the preference levels $\tilde{S}_r$ through x_r determined, in any compact region C_o containing $\hat{X}$, relative to the scales $S \in \mathcal{S}_{\mathcal{J}}$, are the smooth, strictly convex surfaces through x_r between, and touching only at x_r, the pairs of surfaces $\mathcal{P}_r^i(\Lambda,\Phi)$, $\mathcal{P}_r^o(\Lambda,\Phi)$ which are the boundaries of the regions

$$(x-x_s)'g_s \geq \varphi_r - \varphi_s \quad (s = 1,\ldots,k)$$

and

$$x \supseteq x_\alpha \quad (\varphi_\alpha \geq \varphi_r)$$

where

$$g_r = u_r \lambda_r ,$$

and

$$x_\alpha = \Sigma x_r \alpha_r , \quad \varphi_\alpha = \Sigma \varphi_r \alpha_r$$

for

$$\alpha_r \geq 0 , \ \Sigma \alpha_r = 1 ,$$

and where

$$\Lambda = \{\lambda_r\} , \ \Phi = \{\varphi_r\}$$

satisfy the condition $\mathcal{N}_{\mathcal{F}}(\Lambda, \Phi)$ given by

$$\lambda_r > 0 , \ \lambda_r D_{rs} > \varphi_s - \varphi_r$$

with

$$D_{rs} = u_r' x_s - 1 .$$

Moreover, there exist such preference levels lying arbitrarily close to any of these boundaries.

9. Ranging the cost-of-living

To every scale $S \in \mathcal{S}_{\mathcal{F}}$ of the cost-of-living index, there corresponds a point-determination $\rho_{rs} = \rho_{rs}(S)$, this being the minimum relative cost of maintaining, at prices belonging to x_r, a standard not inferior to x_s, in the scale S:

$$\rho_{rs}(S) = \min_{x \bar{S} x_s} u_r' x .$$

and equivalently,

$$\rho_{rs}(S) = \min_{x \in \tilde{S}_s} u_r' x .$$

Now as S varies in $\mathcal{S}_{\mathcal{F}}$, $\rho_{rs}(S)$ has a corresponding variation throughout a certain range which is now to be determined.

Define

$$\rho^{i}_{rs}(\Lambda,\Phi) = \min \{u_r{}'x;\ (x-x_t)'g_t \geq \varphi_s - \varphi_t\ ,\ (t = 1,\ldots,k)\}$$

and

$$\rho^{o}_{rs}(\Lambda,\Phi) = \min \{u_r{}'x;\ x \supseteq x_\alpha\ ,\ \varphi_\alpha \geq \varphi_s\}$$

where

$$\Sigma = \Sigma_{\mathfrak{F}}(\Lambda,\Phi) \text{ and } \mathcal{N}_{\mathfrak{F}}(\Lambda,\Phi)\ .$$

Since $u_r \supset 0$, it follows that

$$\rho^{o}_{rs}(\Lambda,\Phi) = \min \{u_r{}'x_\alpha;\ \varphi_\alpha \geq \varphi_s\}\ .$$

Then it is known that, if S is a (Λ,Φ)-scale,

$$\rho^{i}_{rs}(\Lambda,\Phi) < \rho_{rs}(S) < \rho^{o}_{rs}(\Lambda,\Phi)\ .$$

Moreover, a (Λ,Φ)-scale S can be chosen so that $\rho_{rs}(S)$ is arbitrarily close to either of these limits. Since every scale $S \in \mathcal{S}_{\mathfrak{F}}$ is a (Λ,Φ)-scale for some Λ,Φ such that $\mathcal{N}_{\mathfrak{F}}(\Lambda,\Phi)$, it follows that if

$$\rho^{i}_{rs}(\mathfrak{F}) = \min \{\rho^{i}_{rs}(\Lambda,\Phi)\ ;\mathcal{N}_{\mathfrak{F}}(\Lambda,\Phi)\}$$

and

$$\rho^{o}_{rs}(\mathfrak{F}) = \max \{\rho^{o}_{rs}(\Lambda,\Phi)\ ;\ \mathcal{N}_{\mathfrak{F}}(\Lambda,\Phi)\}\ ,$$

then $\rho_{rs}(S)$ describes the open interval between the limits $\rho^{i}_{rs}(\mathfrak{F})$ and $\rho^{o}_{rs}(\mathfrak{F})$ as S ranges through $\mathcal{S}_{\mathfrak{F}}$.

THEOREM. <u>For an expenditure configuration</u> $\mathfrak{F} = [U;X]$, <u>where</u> $U = \{u_r\}$, $X = \{x_r\}$, <u>there exists a scale</u> $S \in \mathcal{S}_{\mathfrak{F}}$ <u>such that</u> $\rho_{rs}(S) = \rho_{rs}$ <u>if and only if</u>

$$\rho^{i}_{rs}(\mathfrak{F}) < \rho_{rs} < \rho^{o}_{rs}(\mathfrak{F})$$

<u>where</u>

$$\rho^{i}_{rs}(\mathfrak{F}) = \min_{\Lambda,\Phi}\ \min_{x}\ \{u_r{}'x;\ \mathcal{L}_s(\Lambda,\Phi,x)\}$$

$$\rho^{o}_{rs}(\mathfrak{F}) = \max_{\Lambda,\Phi}\ \min_{\alpha}\ \{u_r{}'x_\alpha;\ \varphi_\alpha \geq \varphi_s, \mathcal{N}_{\mathfrak{F}}(\Phi)\}$$

<u>with</u>

$$\Lambda = \{\lambda_r\}\ ,\ \Phi = \{\varphi_r\} \ \underline{\text{and}}\ D_{rs} = u_r{}'x_s - 1\ ,$$

and

$$x_\alpha = \Sigma x_r \alpha_r \; , \; \varphi_\alpha = \Sigma \varphi_r \alpha_r$$

for

$$\alpha_r \geq 0 \; , \; \Sigma \alpha_r = 1 \; ,$$

and

$$\mathcal{N}_{\mathcal{F}}(\Lambda, \Phi) \equiv \lambda_r > 0 \; , \; \lambda_r D_{rs} > \varphi_s - \varphi_r \quad (r \neq s; \; r,s = 1,\ldots,k)$$

and

$$\mathcal{N}_{\mathcal{F}}(\Phi) \equiv \mathcal{N}_{\mathcal{F}}(\Lambda, \Phi) \quad \text{for some} \quad \Lambda$$

and

$$\mathcal{L}_s(\Lambda, \Phi, x) \equiv \mathcal{N}_{\mathcal{F}}(\Lambda, \Phi) \; , \; (x - x_t)' u_t \lambda_t \geq \varphi_s - \varphi_t \quad (t = 1,\ldots,k) \; .$$

The interval $(\rho^i_{rs}, \rho^o_{rs})$ may be called the absolute interval for the cost-of-living index ρ_{rs}, on the data provided by the expenditure configuration $\mathcal{F}$. It is defined just if $\mathcal{F}$ is consistent, this being the condition for the existence of Λ, Φ such that $\mathcal{N}_{\mathcal{F}}(\Lambda, \Phi)$.

PREVIOUS RESEARCH MEMORANDA

"Beneficiation of United States Manganese Reserves."
Research Memorandum No. 1 (January 1957).

"Programming the Supply of Strategic Materials."
Research Memorandum No. 2 (September 1957).

"Econometric Analysis of the United States Manganese Problem."
Final Report, Part I, Research Memorandum No. 3 (October 1958).

"On the Magnification of Derived Demand." R. G. Davis and W. G. Mellon.
Research Memorandum No. 4 (November 1958).

"A General Theory of Measurement-Applications to Utility." J. Pfanzagl.
Research Memorandum No. 5 (December 1958).

"Estimates of National and Regional Steel Production in the United States for 1970." R. E. Kuenne. Research Memorandum No. 6 (February 1959).

"Value and Expenditure." S. N. Afriat.
Research Memorandum No. 7 (February 1959).

"Structures in Logistics Operations." H. D. Mills.
Research Memorandum No. 8 (February 1959).

"Stochastic Properties of Elementary Logistic Components." H. D. Mills.
Research Memorandum No. 9 (February 1959).

"Uncertainty and the Expansion of Bank Credit." Daniel Orr and Walter G. Mellon. Research Memorandum No. 10 (November 1959).

"The Concepts of Equivalence, Order and Scale in the Analysis of Preferences." S. N. Afriat. Research Memorandum No. 11 (January 1960).

"First Report of the Time Series Project." Clive W. J. Granger.
Research Memorandum No. 12 (February 1960).

"The Axiomatic Concept of a Scale and the Logic of Preferences in the Theory of Consumers' Expenditures." S. N. Afriat.
Research Memorandum No. 13 (March 1960).

"Econometric Analysis of the United States Manganese Problem." Final Report, Part II, Research Memorandum No. 14 (March 1960).

"Production Stability and Inventory Variation." Daniel Orr.
Research Memorandum No. 15 (May 1960).

"Statistical Projection." S. N. Afriat.
Research Memorandum No. 16 (June 1960).

"Preference Structures and the Measurement of Military Worth." Richard G. Davis. Research Memorandum No. 17 (September 1960).

"The System of Inequalities $a_{rs} > X_r - X_s$." S. N. Afriat. Research Memorandum No. 18 (October 1960).

"A New Look at Economic Time Series Analysis." Oskar Morgenstern. Research Memorandum No. 19 (January 1961).

"Gradient Configurations and Quadratic Functions." S. N. Afriat. Research Memorandum No. 20 (January 1961).

"Expenditure Configurations." S. N. Afriat. Research Memorandum No. 21 (February 1961).

"Subjective Programming." Robert J. Aumann. Research Memorandum No. 22 (February 1961).

"Almost Strictly Competitive Games." Robert J. Aumann. Research Memorandum No. 23 (February 1961).

"The Cost-of-Living Index: Algebraical Theory." S. N. Afriat, Research Memorandum No. 24 (March 1961).

"The n-Person Bargaining Game." Koichi Miyasawa. Research Memorandum No. 25 (March 1961).

"Utility Theory Without the Completeness Axiom." Robert J. Aumann. Research Memorandum No. 26 (April 1961).