AUTOCORRELATED ERRORS IN SIMPLE DISEQUILIBRIUM MODELS

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by

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1. Introduction

The simplest disequilibrium model is given by

$$\begin{aligned} D_t &= \alpha_1' x_{1t} + u_{1t} \\ S_t &= \alpha_2' x_{2t} + u_{2t} \\ Q_t &= \min\ (D_t, S_t) \end{aligned} \tag{1.1}$$

$t = 1, \ldots, T$, where D_t and S_t represent the demand and supply of some commodity and are unobserved, x_{1t}, and x_{2t} are vectors of exogenous variables, u_{1t} and u_{2t} are error terms with a joint pdf $f(u_{1t}, u_{2t})$, and where Q_t is the actual quantity transacted in the market. The model given by (1.1) corresponds to a situation of completely rigid prices (included on the right hand side among the x's) in which case, under voluntary exchange, the observed quantity Q_t is equal to the lesser of the quantities demanded and supplied.

Two subvarieties of the model given by (1.1) exist. (1) It may happen that, although the investigator does not observe D_t and S_t, it is known for each sample point whether $D_t \leqq S_t$ or $D_t > S_t$. In this case at least

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the sample partition is known. (2) Alternatively, it may happen that even this sample partition information is absent. It is well known that in Case (1) the likelihood functions is

$$L = \prod_{D_t \leq S_t} \int_{Q_t}^{\infty} g(Q_t, S_t)\,dS_t \prod_{D_t > S_t} \int_{Q_t}^{\infty} g(D_t, Q_t)\,dD_t \tag{1.2}$$

and for Case (2) is

$$L = \prod_{t=1}^{T} \left[\int_{Q_t}^{\infty} g(D_t, Q_t)\,dD_t + \int_{Q_t}^{\infty} g(Q_t, S_t)\,dS_t \right] \tag{1.3}$$

where $g(D_t, S_t)$ is the joint pdf of D_t, S_t derived from $f(u_{1t}, u_{2t})$ and (1.1). (Maddala and Nelson [8], Laffont and Monfort [6], Bowden [2]). Further, slightly more complicated, variants of the basic model include the case in which price is an endogenous variable which is assumed to move according to excess demand. The system (1.1) is typically augmented by an equation such as $p_t = p_{t-1} + \gamma(D_t - S_t)$ or $p_t = p_{t-1} + \gamma(D_t - S_t) + u_{3t}$ where u_{3t} is a random error and where p_t is understood to be included among the right hand variables x_{1t} and x_{2t}. We concentrate on the simple model without a price adjustment equation.

Most theoretical work and all empirical work with models of this type is based on the assumption that u_{1t} and u_{2t} are serially uncorrelated. A single exception to this is Laffont and Monfort [7] who derive the likelihood function for the case of autocorrelated errors (as well as for the case in which lagged values of D and S appear in the equations) on the assumption that the sample partition is known. In the present paper we consider the case in which the sample partition is not known.

2. The Likelihood Function

We now assume in addition to (1.1) that

$$\begin{aligned} u_{1t} &= \rho_1 u_{1t-1} + \varepsilon_{1t} \\ u_{2t} &= \rho_2 u_{2t-1} + \varepsilon_{2t} \end{aligned} \tag{2.1}$$

and that ε_{1t} and ε_{2t} are i.i.d. as $N(0,\sigma_1^2)$, $N(0,\sigma_2^2)$ respectively. Defining

$$\begin{aligned} k_{1t} &= - \alpha_1' x_{1t} + \rho_1 \alpha_1' x_{1t-1} \\ k_{2t} &= - \alpha_2' x_{2t} + \rho_2 \alpha_2' x_{2t-1} \end{aligned}$$

we can write the conditional joint pdf $g(D_t, S_t | D_{t-1}, S_{t-1}) = g_1(D_t|D_{t-1}) g_2(S_t|S_{t-1})$ where

$$\begin{aligned} g_1(D_t|D_{t-1}) &= \frac{1}{\sqrt{2\pi}\,\sigma_1} \exp\left\{-\frac{1}{2}\left(\frac{D_t - \rho_1 D_{t-1} + k_{1t}}{\sigma_1}\right)^2\right\} \\ g_2(S_t|S_{t-1}) &= \frac{1}{\sqrt{2\pi}\,\sigma_2} \exp\left\{-\frac{1}{2}\left(\frac{S_t - \rho_2 S_{t-1} + k_{2t}}{\sigma_2}\right)^2\right\} \end{aligned} \tag{2.2}$$

$$g(D_t, S_t | D_{t-1}, S_{t-1}) = \frac{1}{2\pi\sigma_1\sigma_2} \exp\left\{-\frac{1}{2}\left[\frac{(D_t - \rho_1 D_{t-1} + k_{1t})^2}{\sigma_1^2} + \frac{(S_t - \rho_2 S_{t-1} + k_{2t})^2}{\sigma_2^2}\right]\right\} \tag{2.3}$$

It follows that

$$h(Q_t | D_{t-1}, S_{t-1}) = \int_{Q_t}^{\infty} g(Q_t, S_t | D_{t-1}, S_{t-1})\, dS_t + \int_{Q_t}^{\infty} g(D_t, Q_t | D_{t-1}, S_{t-1})\, dD_t$$

$$= \frac{\exp\left\{-\frac{1}{2}\left(\frac{Q_t - \rho_1 D_{t-1} + k_{1t}}{\sigma_1}\right)^2\right\}}{\sqrt{2\pi}\,\sigma_1} \int_{Q_t}^{\infty} \frac{\exp\left\{-\frac{1}{2}\left(\frac{S_t - \rho_2 S_{t-1} + k_{2t}}{\sigma_2}\right)^2\right\}}{\sqrt{2\pi}\,\sigma_2} dS_t \; +$$

$$\frac{\exp\left\{-\frac{1}{2}\left(\frac{Q_t - \rho_2 S_{t-1} + k_{2t}}{\sigma_2}\right)^2\right\}}{\sqrt{2\pi}\,\sigma_2} \int_{Q_t}^{\infty} \frac{\exp\left\{-\frac{1}{2}\left(\frac{D_t - \rho_1 D_{t-1} + k_{1t}}{\sigma_1}\right)^2\right\}}{\sqrt{2\pi}\,\sigma_1} dD_t \qquad (2.4)$$

Given known initial values D_0 , S_0 , and conditional pdf's $f_1(D_{t-1}|D_0)$, $f_2(S_{t-1}|S_0)$, the (marginal) pdf for Q_t is

$$h(Q_t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(Q_t|D_{t-1},S_{t-1}) f_1(D_{t-1}|D_0) f_2(S_{t-1}|S_0)\, dD_{t-1} dS_{t-1} \qquad (2.5)$$

We therefore need expressions for $f_1(D_{t-1}|D_0)$ and $f_2(S_{t-1}|S_0)$. Since

$f_1(D_{t-1}|D_0) = \int \cdots \int g_1(D_{t-1}|D_{t-2}) \cdots g_1(D_1|D_0)\, dD_{t-2} \ldots dD_1$ and similarly for $f_2(S_{t-1}|S_0)$, we have

$$f_1(D_{t-1}|D_0) = \frac{\exp\left\{-\frac{(D_{t-1} - \rho_1^{t-1} D_0 + k_{1t-1} + \rho_1 k_{1t-2} + \ldots + \rho_1^{t-2} k_{11})^2}{2\sigma_1^2\left(1 + \rho_1^2 + \rho_1^4 + \ldots + \rho_1^{2(t-2)}\right)}\right\}}{\sqrt{2\pi}\,\sigma_1\left(1 + \rho_1^2 + \rho_1^4 + \ldots + \rho_1^{2(t-2)}\right)^{1/2}} \qquad (2.6)$$

$$f_2(S_{t-1}|S_0) = \frac{\exp\left\{-\frac{\left((S_{t-1} - \rho_2^{t-1}S_0 + k_{2t-1} + \rho_2 k_{2t-2} + \ldots + \rho_2^{t-2}k_{21}\right)^2}{2\sigma_2^2\left(1 + \rho_2^2 + \rho_2^4 + \ldots + \rho_2^{2(t-2)}\right)}\right\}}{\sqrt{2\pi}\,\sigma_2\left(1 + \rho_2^2 + \rho_2^4 + \ldots + \rho_2^{2(t-2)}\right)^{1/2}} \tag{2.7}$$

Let $\Phi(x)$ denote $\int_{-\infty}^{x} \frac{e^{-z^2/2}}{\sqrt{2\pi}}\,dz$, introduce (2.6) and (2.7) in (2.4) and integrate. This yields

$$h(Q_t) = \int_{-\infty}^{\infty} \frac{\exp\left\{-\frac{1}{2}\left(\frac{Q_t - \rho_1 D_{t-1} + k_{1t}}{\sigma_1}\right)^2\right\}}{\sqrt{2\pi}\,\sigma_1} f_1(D_{t-1}|D_0)\,dD_{t-1} \int_{-\infty}^{\infty}\left(1-\Phi\left(\frac{Q_t - \rho_2 S_{t-1} + k_{2t}}{\sigma_2}\right)\right) \times$$

$$f_2(S_{t-1}|S_0)\,dS_{t-1}$$

$$+ \int_{-\infty}^{\infty} \frac{\exp\left\{-\frac{1}{2}\left(\frac{Q_t - \rho_2 S_{t-1} + k_{2t}}{\sigma_2}\right)^2\right\}}{\sqrt{2\pi}\,\sigma_2} f_2(S_{t-1}|S_0)\,dS_{t-1} \int_{-\infty}^{\infty}\left(1-\Phi\left(\frac{Q_t - \rho_1 D_{t-1} + k_{1t}}{\sigma_1}\right)\right) \times$$

$$f_1(D_{t-1}|D_0)\,dD_{t-1}$$

$$= \frac{\exp\left\{-\frac{\left(Q_t - \rho_1^t D_0 + k_{1t} + \ldots + \rho_1^{t-1} k_{11}\right)^2}{2\sigma_1^2\left(1 + \rho_1^2 + \rho_1^4 + \ldots + \rho_1^{2(t-1)}\right)}\right\}}{\sqrt{2\pi}\,\sigma_1\left(1 + \rho_1^2 + \rho_1^4 + \cdots + \rho_1^{2(t-1)}\right)^{1/2}} \int_{-\infty}^{\infty} \left(1-\Phi\left(\frac{Q_t - \rho_2 S_{t-1} + k_{2t}}{\sigma_2}\right)\right) \times$$

$$f_2(S_{t-1}|S_0)\,dS_{t-1}$$

$$+ \frac{\exp\left\{-\frac{\left(Q_t - \rho_2^t S_0 + k_{2t} + \ldots + \rho_2^{t-1} k_{21}\right)^2}{2\sigma_2^2\left(1 + \rho_2^2 + \rho_2^4 + \ldots + \rho_2^{2(t-1)}\right)}\right\}}{\sqrt{2\pi}\,\sigma_2\left(1 + \rho_2^2 + \rho_2^4 + \ldots + \rho_2^{2(t-1)}\right)^{1/2}} \int_{-\infty}^{\infty} \left(1-\Phi\left(\frac{Q_t - \rho_1 D_{t-1} + k_{1t}}{\sigma_1}\right)\right) \times$$

$$f_1(D_{t-1}|D_0)\,dD_{t-1} \qquad (2.8)$$

The integrals remaining in (2.8) need to be evaluated by numerical quadrature methods. The likelihood function is

$$L = \prod_{t=1}^{T} h(Q_t) \qquad (2.9)$$

It is well-known that the likelihood functions of disequilibrium models in which sample partitioning information is absent are unbounded at certain boundary points of the parameter space (where either σ_1 or σ_2 equals zero) (Goldfeld and Quandt [4]). It is easily shown that the same property holds in the present case. In practice, however, an interior maximum can nearly always be found and corresponds to a consistent estimator (Amemiya and Sen [1], Hartley and Mallela [5]).

3. A Computational Example

The Model. We employ the aggregate labor market model of Rosen and Quandt [10] as reformulated by Romer [9]. The demand and supply functions for labor are

$$\ell nD_t = \alpha_0 + \alpha_1 \ell n w_t + \alpha_2 \ell n X_t + \alpha_3 t + u_{1t} \tag{3.1}$$

$$\ell nS_t = \beta_0 + \beta_1 \ell n w_{nt} + \beta_2 \ell n P_t + u_{2t} \tag{3.2}$$

where w_t is total wages and salaries in the U.S. private sector in 1958 dollars, divided by the number of private hours worked, $w_{nt} = w_t(1-\theta_t)$ where θ_t is the ratio of personal taxes to personal income, X_t is GNP in 1958 dollars and P_t is the potential number of hours worked and equals the civilian population between 16 and 64 multiplied by the average annual hours worked. All variables except the time trend were expressed in logarithms and the period covered was 1929 - 1973. The min condition is

$$\ell nQ_t = \min(\ell nD_t, \ell nS_t) \tag{3.3}$$

where Q_t is total private hours worked. The model in (3.1) to (3.3) differs from these considered by Rosen and Quandt in two respects: (a) as a result of Romer's [9] modification nonlabor income was excluded from (3.2); (b) the wage adjustment equation, which stipulated that the (real) wage responds positively to excess demand, was eliminated. Although this assumption implies an extreme autonomy in wage movements and may lessen the economic content of the model, it has the virtue of simplifying the computations. It may also help in avoiding anomalous results due to the fact that the real wage rose in all but three years during the period.

Computational Considerations. The likelihood function of the above model with and without the assumption of serially correlated errors was maximized numerically, using first the Davidon-Fletcher-Powell (DFP) algorithm followed by the Quadratic Hill Climbing (GRADX) algorithm (Goldfeld and Quandt [3]). Derivatives were evaluated numerically. Equ. (2.8) requires knowledge of D_0 and S_0. It was assumed here that the initial observation was an equilibrium observation so that $D_0 = S_0 = Q_0$. (Alternatively one might have assumed priors for D_0 and S_0 and integrated them out.) In evaluating the likelihood function (2.9), 2T numerical quadratures are needed for every function evaluation. Each of these was obtained by adding the results of a sequence of Gaussian quadratures over a sequence of subintervals of width $\sigma_i(1 + \rho_i^2 + \ldots + \rho_i^{2(t-2)})^{1/2}$, starting from the means of the densities f_i and moving successively further to the right and left until the contribution to the integral of the marginal interval became negligible. Asymptotic standard errors were estimated from the negative inverse Hessian of the loglikelihood function. Computations were substantially more costly for the model involving serial correlation. A single function evaluation for the model without autocorrelation took about .004 seconds on an IBM 3033, whereas in the model with autocorrelation a function evaluation took about 1/2 second.

Results. Table 1 displays the estimates and asymptotic standard errors (in parentheses) of Romer's results and the two versions estimated here. The two simple disequilibrium models computed here have coefficients that resemble each other strongly as well as those obtained in Romer's endogenously wage-adjusting version. Although the likelihood ratio test statistic $-2\log\lambda = 27.02$ is greater than $\chi^2_{.01}(2)$, indicating rejection of the null hypothesis that $\rho_1 = \rho_2 = 0$, neither estimated autocorrelation coefficient is individually significant and certainly neither is of consequential magnitude. With either the autocorrelated or nonautocorrelated versions periods of excess demand for labor are predicted for 1943-1945, 1951-53, 1955 and 1965-1968. Excess supply is predicted for all other years.

4. Conclusions

The likelihood function for a simple disequilibrium model with autocorrelated error terms was derived and estimated with aggregate U.S. data. Although the estimation process is time consuming and complicated by the fact that every function evaluation requires 2T numerical quadratures, maximization of the likelihood function was feasible. In the example examined, the autocorrelation of the error terms was only of moderate magnitude.

Extension of this modification to more complicated models such as those involving a price adjustment equation can be based on the same principles but will involve substantially greater numerical and computational complexity.

Table 1. Estimation Results

Coefficients	Romer's Results	Simple Disequilibrium With No Autocorrelation	Simple Disequilibrium With Autocorrelation
α_0	-.455 (.066)	-.173 (.070)	-.299 (.066)
α_1	-.477 (.070)	-.424 (.068)	-0.429 (.069)
α_2	.948 (.010)	.897 (.011)	.922 (.010)
α_3	-.011 (.002)	-.010 (.002)	-.011 (.002)
β_0	-1.608 (.033)	-1.815 (.028)	-1.892 (.031)
β_1	-.189 (.021)	-.210 (.017)	-.212 (.019)
β_2	1.175 (.006)	1.213 (.005)	1.227 (.006)
σ_1^2	NA.	.001 (.0003)	.0006 (.0002)
σ_2^2	NA.	.0003 (.0002)	.0002 (.0001)
ρ_1	-	-	-.185 (.673)
ρ_2	-	-	.0000 (1.495)

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