THE OPTIMAL WEIGHTING OF

INDICATORS FOR A CRAWLING PEG

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August 1980

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I. Introduction

The problem of choosing indicators for exchange-rate adjustment will be relevant for some time to come. Most countries do not permit their exchange rates to float freely, and therefore must choose some rule, implicit or explicit, for adjusting the rate. In any program of exchange-rate surveillance, the IMF staff will have to take a view on the appropriate indicators. Indeed, in the early seventies, some work was done at the Fund along these lines, namely by Underwood (1973) and Williamson (1973), and the problem was discussed by the Committee of Twenty (1974).

To our knowledge, the definitive study of the problem to date is Kenen's (1975). He did an extensive simulation study of numerous alternative schemes and found that a reserve change or basic balance indicator would not stabilize reserves, whilst a reserve indicator resulted in large fluctuations in the current account. He also noted that this reflected the problem of simple proportional or integral stabilization rules analyzed by Phillips (1954), but did not pursue a Phillips' analysis. Even though the importance of Phillips' early contribution has been more widely noted in the macro literature, perhaps because the title of his paper refers to a "closed economy", he also notes that "The general principles of stabilization (...) could equally well be used, for example, in investigating the stability of adjustment in international trade or the problems involved in commodity price stabilization schemes."[1]

In this paper we generalize Kenen's results and relate them to an optimal control approach to the problem. Kenen studied arbitrarily specified adjustment mechanisms; we wish to derive an optimal specification explicitly. In section II we generalize Kenen's results analytically, and derive weights for the current account and reserve target that yield monotonically stable adjustment. We see that a current account (or, in general, flow) indicator is stable but randomizes reserves, while a reserve indicator yields a limit cycle. Conditions for a Kenen-Phillips formula weighting the two to give stable results are presented and illustrated.

In section III we derive an optimal control solution for adjustment to a given current account disturbance. Since this formula is a linear control rule, it can be derived as the solution of the minimization of a quadratic minimum energy loss function subject to a linear equation of motion. Thus optimal weights for the current account and reserve target are derived for various values of the derivative of the current account with respect to the exchange rate (B_e), and the weights of the output and control variables in the social loss function.

Finally in section IV we derive the adjustment equation in a stochastic framework with continuous current account and exchange rate multiplier shocks. The separation theorem of stochastic control, known in economics as the principle of certainty equivalence, implies that the linear control rule remains applicable,

given the expectation on the state variables conditional on the path of the output variables. If the two are uncorrelated, we therefore have an expression in the Kenen-Phillips form of section II.

The values of the optimal weights for various values of the parameters of this problem are compared with those obtained for the deterministic case. It is found that the weight of the current account in the optimal control rule is generally higher than the welfare weight, and also higher than the lower bound of Section II. When the variance of the effect of the exchange rate on the current account increases, however, the optimal weight approaches the lower bound and becomes smaller than the welfare weight, as was to be expected from the static analysis in Brainard (1967). The results are summarized and conclusions are drawn in Section V, which includes the summary Tables 5 and 6.

II Flow vs. Stock Indicators

The purpose of this section of the paper is to expose the analytical problem of the choice of indicators as clearly as possible, setting the stage for the optimizing approaches of the following sections. Therefore we begin with the simplest model that illustrates the problem. Assume that the monetary authority in a given small open economy has already decided not to permit its exchange rate to float freely. This is necessary for the question of choice of indicators to arise. Further, assume zero capital mobility so that the current account balance B in

foreign currency is the rate of accumulation of reserves,

(1) $\dot{R} = B$.

These two assumptions are consistent; with no stabilizing speculation on capital account, a foreign exchange market based on trade flows alone might well be unstable.[2]

The average current account balance over a period long enough to ignore J-curve effects of changes in relative prices will be an increasing function of the real exchange rate, defined as

$$p = eP*/P.$$

Here e   is measured in units of home currency per units of foreign exchange and $P*/P$ is an appropriate relative price index of domestic and foreign goods.

The current account balance can be written as

$$B(p) = X(p)/p - M(p),$$

where X denotes exports in domestic currency and M denotes imports in foreign currency. The effect of a real devaluation around equilibrium is given by

$$dB = \frac{dp}{p} \; M* \; (d_x + d_m - 1) \; ,$$

$$\text{where } d_i = \frac{\varepsilon_i(1+\eta_i)}{\eta_i+\varepsilon_i} \quad , \qquad i = x,m \; ,$$

is a combination of supply (η) and demand (ε) elasticities of exports and imports. By appropriate choice of units, equilibrium imports and the equilibrium exchange rate are set to one, so that $\frac{dB}{dp} = d_x + d_m - 1$, where d_x and d_m are the absolute values of the export and import

elasticities.

Furthermore, we neglect income effects on the current account, as well as the effects of monetary policy and income on domestic prices, by assuming that domestic absorption is manipulated by aggregate demand policy to keep internal balance. We therefore set $P = P^* = 1$ and focus on the dependence of the current account on changes in the nominal exchange rate.[3] We also define the policy horizon so that the "Marshall-Lerner condition" holds.

Under these conditions, we can express the current account balance as an increasing function of e :

(2) $B = B(e)$; $B_e > 0$.

Note that the model could be amended to include capital flows as a function of uncovered interest rate differentials. In that case, given interest rates, exchange rate expectations would have to depend on the current level of the exchange rate, and be such that $B_e > 0$, including capital flows in B. This is consistent with a variety of expectations formation mechanisms. Adding capital movements in this "old" way would reduce analytical clarity without adding anything to the results.

A position of external balance is defined by the attainment of a given target level of reserves R*, with a zero balance on current account.[4] The latter condition defines a target value for the exchange rate:

(3) $B(e^*) = 0$,

and $R = R^*$, $e = e^*$ defines external balance. The problem of choice of objective indicators is to choose a rule for adjusting e, following

observations on B or R, that converges to external balance.

One candidate suggested by Cooper (1970) would be to key adjustment of e to reserve changes, which, given (1), amount to the current account balance:

(4)　$\dot{e} = -\lambda B(e)$.

This is a proportional stabilizer, in Phillips' terms. As Kenen says, "It matches a flow control to a flow target."[5] Given our assumption that $B_e > 0$, it is stable around e*. Linearizing, we have

$$\dot{e} = -\lambda B_e (e - e^*),$$

and $d\dot{e}/de = -\lambda B_e < 0$. But there is no mechanism to move R to R* with this rule. The time path of R will resemble a random walk. A current account disturbance moving e* will be eliminated gradually as the adjustment rule (4) moves e to the new e*. During the adjustment period R will change. When e reaches e* and B is again zero, there will be no further change in R.

Another candidate, proposed in 1972 by the U.S. Secretary of the Treasury, would be to key adjustment of e to deviations of reserves from the target:[6]

(5)　$\dot{e} = -\lambda(R - R^*)$.

This is an integral stabilizer in Phillips' terms. As he says, "A country which attempts to regulate its current balance of payments, whether by means of internal credit policy or quantitative import restrictions, and in doing so responds mainly to the size of its foreign reserves (i.e., to the time integral of its current balance

of payments), is applying an integral correction policy which is likely to cause cyclical fluctuations similar to those illustrated [in his paper]. The short cycles that have occurred in the balance of payments of a number of countries since the war may be in part the result of such action". In Kenen's words, "the rule marries a _flow_ control to a _stock_ target, a union that is always apt to be unstable."[7]

Indeed, it leads to a limit cycle in e(t). To see this, differentiate the rule in (5) with respect to time, and linearize around e*.

$$\dot{e}^{\cdot} = \lambda \dot{R} = - \lambda B(e) = - \lambda B_e (e - e*).$$

The roots of this second-order differential equation are purely imaginary and equal to $\pm i \sqrt{\lambda B_e}$. If the system were to begin at R = R*, e = e*, a current account disturbance would set off an infinite cycle in e, B, and R.

Phillips' prescription was to combine the two rules in (4) and (5). The essential idea is to add a bit of the integral stabilizer as in (5) to the proportional rule of (4) in order to keep a stable adjustment system moving toward the reserve target. We can express this by weighting the two rules:

(6) $\dot{e} = - \theta[\gamma B(e) + (1-\gamma)(R-R*)]; \ 0 \leq \gamma \leq 1.$

Here θ gives the sensitivity or speed of adjustment of e with respect to the weighted average of off-target values of B and R. By appropriate

choice of units, we can scale θ to unity.

We can find a range of values for which will yield monotonically stable adjustment of e as follows. Differentiate (6) with respect to time and linearize B around e* to obtain the second-order differential equation.

$$\ddot{e} + \gamma \, B_e \dot{e} + (1-\gamma) \, B_e e = 0.$$

The roots are given by

$$r_1, \ r_2 = \frac{1}{2}(- \gamma B_e \pm \sqrt{\gamma^2 B_e^2 - 4 (1-\gamma) \, B_e})$$

For monotonic stability, γ should be chosen such that both roots are real and negative, which requires that the square root term be positive, or that

$$\frac{\gamma^2}{4} > \frac{1-\gamma}{B_e}$$

This, in turn, gives us a quadratic equation in γ with roots given by

$$r_1, \ r_2 = \frac{2}{B_e}(-1 \pm \sqrt{1+B_e})$$

Since γ is positive, we discard the negative root and obtain the expression for the permissible range of γ, depending on B_e.

$$(7) \quad 1 > \gamma > \frac{2}{B_e} \ (- 1 \pm \sqrt{1 + B_e} \ ) > 0.$$

To obtain an intuitive understanding of the result, recall that $B_e = d_x + d_m - 1$. If both demand elasticities are unity (in absolute value) so that $B_e = 1$, we have $\gamma > 0.83$. As B_e goes to zero, the bound for γ approaches unity; as B_e goes to infinity, it approaches zero. Some values for B_e and γ are given in Table 1.

Table 1: Lower Bound for γ, Depending on B_e

B_e	.1	1.0	10.0	100.0
γ	.98	.83	.46	.18

As B_e increases, less weight must be given to the current account balance to get a given adjustment of the current account. With B_e around ten, a "reasonable" weighting of the two targets would be 0.5 for the current account and 0.5 for the reserve deviation. A lower weight on the current account would cause instability.

III. Optimal Adjustment in a Deterministic Framework

Equation (7) and Table 1 give the permissible range of weights for the stock and flow targets that yields monotonically stable adjustment to a current account disturbance in the Kenen-Phillips framework. Still considering adjustment to a one-shot current account disturbance, we now turn to optimal control analysis from the viewpoint of external balance. In this context, the problem involves minimizing the square of the difference between actual and desired current account and level of reserves, with minimum exchange rate changes. The desired level of the current account and the deviation of reserves from some given level R* is taken to be zero. The quadratic minimum-energy loss function is then

$$L = \frac{1}{2} (1-\alpha)B(e)^2 + \alpha(R-R^*)^2 + c(\dot{e})^2].$$

Here $1-\alpha$ is the welfare weight for current account imbalance, and α weights the distance from the reserve target, both being measured relative to the cost of exchange rate variability, c.

As in section II the model of the economy is given by equation (1):

$$\dot{R} = B(e) = B_e(e-e^*),$$

where $B(e^*) = 0$ defines e*. In this simple case the output vector is just

$$w = [\begin{smallmatrix} R - R^* \\ B \end{smallmatrix}] \ ,$$

and we write

(8) $$w = Cz,$$

where $C = \begin{bmatrix} 1 & 0 \\ 0 & B_e \end{bmatrix}$,

and $z = [\begin{smallmatrix} R - R^* \\ e - e^* \end{smallmatrix}]$ is the state vector.

The state vector is also expressed as a difference of actual and given desired values, so that we can treat the problem as a simple time⁻invariant output regulator problem. Strictly speaking, when the desired values are not zero we cannot assume that the minimum loss of the time⁻invariant problem is finite, or that a control exists. By writing the state variables in deviation form, however, we are able to ignore the forcing function and therefore work with infinite horizon. [8/] We write our objective function as

$$J = \frac{1}{2} \int_0^\infty (w'Dw + cu'u)dt \ ,$$

where $u = [\begin{smallmatrix} 0 \\ e \end{smallmatrix}]$ is the control vector,

and $D = \begin{bmatrix} \alpha & 0 \\ 0 & 1-\alpha \end{bmatrix}$.

Given (8), we have a convenient state space representation of our problem. [9/]

(9) $$\text{Min } \frac{1}{2} \int_0^\infty (z'Qz + cu'u) \ dt$$

subject to $\dot{z} = Az + Bu$ and $z(0) = z_0$,

where $Q = C'DC = \begin{bmatrix} \alpha & 0 \\ 0 & (1-\alpha) \ B_e{}^2 \end{bmatrix}$;

$A = \begin{bmatrix} 0 & B_e \\ 0 & 0 \end{bmatrix}$;

$B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Defining the vector of costate variables $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ we have the Hamiltonian

$$H = \frac{1}{2}(z'Qz + cu'u) + y'(Az + Bu),$$

which is minimized at each instant of time. The marginal conditions then are

$$(10) \qquad \frac{\partial H}{\partial u} = cu' + y'B = 0,$$

since $\dfrac{\partial^2 H}{\partial u^2} = cI$ is positive definite. By the minimum principle, we have

$$\frac{\partial H}{\partial y'} = \dot{z} = Ax + Bu;$$

$$-\frac{\partial H}{\partial z'} = \dot{y} = -Qz - A'y.$$

Using (10) transposed we write the canonical equations

$$\dot{z} = Az - \frac{1}{c}B'y;$$

$$\dot{y} = -Qz - A'y.$$

For time invariant A, B, Q and an infinite horizon problem, the costate variable is given by

$$(11) \qquad y = Kz,$$

where $K = \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} = \lim_{t \to \infty} K(t)$

is a positive matrix given by the Riccati equation:

$$(12) \qquad -KA - A'K + \frac{1}{c}K\,BB'K - Q = 0.$$

By substitution we derive from (12) equations for the elements of K:

$$\frac{1}{c} k_{12}^2 = \alpha \;;$$

(13)
$$\frac{1}{c} k_{12}k_{22} = k_{11}B_e \;;$$

$$\frac{1}{c} k_{22}^2 = 2B_e k_{12} + (1-\alpha) B_e^2.$$

Given the positive definiteness of K, $k_{11}, k_{12}, k_{22} > 0$, so that the solution is

$$k_{12} = \sqrt{c\alpha} \;;$$

$$k_{22} = B_e \sqrt{2c \sqrt{\alpha c}/B_e + (1-\alpha)c} \;;$$

$$k_{11} = \sqrt{\alpha(2\sqrt{\alpha c}/B_e + (1-\alpha))} \;.$$

The optimal control $\tilde{u}$ is linear in the state vector z, and from (10) and (11), satisfies the equation[10/]

(14)
$$\tilde{u} = -\frac{1}{c} B'Kz,$$

so that it can be written as a function of k_{12} and k_{22}:

$$\tilde{u}_t = \sqrt{\alpha/c}(R_t - R^*) - \sqrt{2\sqrt{\alpha/c}/B_e + \frac{1}{c}(1-\alpha)} \; B_e(e_t - e^*) \;.$$

Defining $\theta = \sqrt{\alpha/c} + \sqrt{2\sqrt{\alpha/c}/B_e + (1-\alpha)/c}$,

and $1 - \gamma = \frac{1}{\theta}\sqrt{\alpha/c}$,

this yields a familiar expression for the optimal rate of crawl

$$\dot{e} = -\theta[\gamma B(e) + (1-\gamma)(R - R^*)].$$

Some values of θ and γ for values of B_e, α and c are shown in Table 2.

Table 2

Alternate values of Θ and γ

Values of $1-\alpha$

c = 1	1		.5		0	
Value of B_e	Θ	γ	Θ	γ	Θ	γ
0.1	1	1	4.54	.84	5.47	.82
1.0	1	1	2.09	.66	2.41	.59
10.0	1	1	1.51	.53	1.45	.31
100.0	1	1	1.43	.50	1.14	.12
c = .5	1		.5		0	
Value of B_e	Θ	γ	Θ	γ	Θ	γ
0.1	1.14	1	5.58	.82	6.73	.79
1.0	1.14	1	2.73	.63	3.10	.54
10.0	1.14	1	2.10	.52	1.95	.27
100.0	1.14	1	2.01	.50	1.58	.11
c = .1	1		.5		0	
Value of B_e	Θ	γ	Θ	γ	Θ	γ
0.1	3.16	1	9.29	.76	11.11	.72
1.0	3.16	1	5.31	.58	5.67	.44
10.0	3.16	1	4.57	.51	3.96	.20
100.0	3.16	1	4.48	.50	3.41	.07

As long as there is a reserve target, so that $\alpha \neq 0$, the table is consistent with the result of the previous section that the weight on the current account increases as B_e decreases. In the case of $B_e = 1$, weighting equally the two targets and the control implies that the optimal current account weight is .63 rather than .83 from Table 1. Note, however, that θ is 2.73, compared with the implicit value of unity in Table 1. Thus the implicit lower found for this value of B_e is 0.30 (=0.83/2.73). This confirms the need to supplement the instability analysis of section II by an optimizing approach.

The table shows also that the effect of the weight attached to exchange rate variability is to increase γ, particularly for low values of B_e. When $B_e = 0.1$ and $\alpha = .5$, the weight on the current account increases from .76 to .84 as c rises from 0.1 to 1. However, when $B_e = 100$, γ is unchanged at .5. This effect is weaker when there is a pure reserve target, so that $\alpha = 1$. Note also that values of c larger than one would tend to lower θ so that we could obtain an implied lower bound for θ larger than unity.

In the case of a pure current account target, $\alpha = 0$, it is clear from Table 2 that the weight on the current account does not change with B_e, but that the implied lower bound declines with the decline in c from the values in Table 1 where c = 1. Since the effect of changes in c is quite clear , we will set it

at unity for the remainder of the paper, so that we will in fact

be measuring α in terms of a unit cost of exchange rate variability.

Let us now consider the case in which future utility is dis-

counted at a rate ρ, so that the minimand in (9) becomes

(15) $\quad \frac{1}{2} \int_0^\infty (z'Qz + u'u)e^{-\rho t} \, dt.$

Then the Riccati equation in (12) becomes

(16) $\quad -KA - A'K + \rho K + KBB'K - Q = 0,$

and equations (13) for its elements become

$$k_{12}^2 - \quad + \rho \, k_{11} = 0 \ ;$$

$$k_{12}k_{22} - k_{11}B_e + \rho k_{12} = 0 \ ;$$

$$k_{22}^2 - 2k_{12}B_e - (1 - \alpha) \, B_e^2 + \rho k_{22} = 0.$$

Solving out for k_{11}, which does not enter the optimal control

solution in (14), we obtain two second-order equations which can be

represented in k_{12}, k_{22} space. This is done in Figure 1. It is

clear that the intersection of the two curves at E_ρ is to

the soutwest of R, where the rate of discount is zero and the co-

efficient on reserves is independent of the coefficient on the current

account. Note that-given c- larger values of B_e bring E_ρ closer to R,

as shown in the θ column of Table 3, which is comparable to the θ

column of Table 2 above. The weights γ are however closer to those of

Table 2 for lower values of B_e. As expected, discounting reduces the

sensitivity of the rate of crawl to the indicators but this reduction is

more than offset by a large value of the "Marshall-Lerner condition" so

that when $B_e = 100$ (and $\alpha = .5$), $\gamma = .47$ rather than .50 as in Table 2.

Figure 1

The coefficients of the Riccati matrix
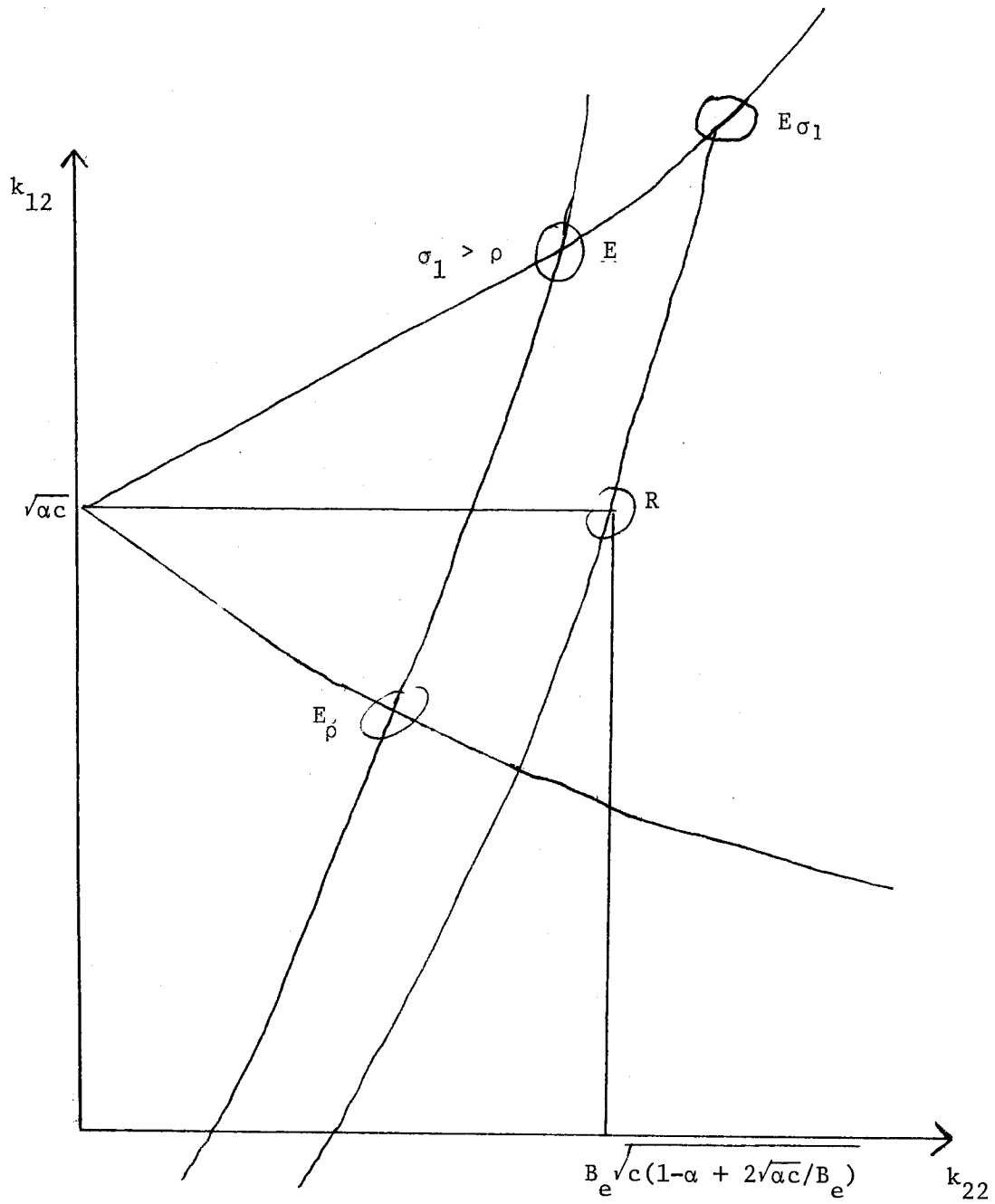
Table 3

Alternate values of

θ and γ

for $\rho = .2$

1-α B e	1		.5		0	
	θ	γ	θ	γ	θ	γ
.1	.41	1	2.51	.84	3.34	.81
1	.91	1	1.78	.67	2.07	.58
10	.99	1	1.35	.47	1.38	.31
100	.99	1	1.35	.47	1.13	.12

IV Optimal Adjustment in a Stochastic Framework

To analyze the problem within a stochastic framework, we will modify the model given by equations (1) and (2) above. While we still assume that the exchange rate can be controlled exactly, we introduce an additive current account disturbance ω_2. We also introudce uncertainty regarding the effect of the exchange rate on the current account. This "state-dependent noise" is modelled as an additive disturbance ω_1 to B_e, possibly correlated with ω_2. If we assume further that ω_1 and ω_2 are Brownian motion with σ_1^2 dt and σ_2^2 dt as variances of their respective increments, the change in reserves can be approximated around equilibrium by a linear stochastic differential equation of the form

$$dR = (B_e dt + d\omega_1)(e - e*) + d\omega_2.$$

The state representation of our system becomes

(18) $dz = (Az + Bu)dt + S_1 z dt + S_2 1\, dt,$

where $S_i = \begin{vmatrix} \overline{\sigma_i} & \overline{0} \\ \underline{0} & \underline{0} \end{vmatrix}$; $i = 1, 2$;

$\underset{\sim}{1}' = [1 \quad 1].$

We now wish to minimize

$$E_0 \int_0^T \frac{1}{2}(z'Qz + u'u)dt,$$

where the expectation E_0 is taken conditional on the steady state of the system, $z = 0$, subject to (18) and to $z(0) = z_0$.

Define $\underline{11/}$

$$J(z,t) = \min_u E_t \int_t^T \frac{1}{2}(z'Qz+u'u) \; ds,$$

and

(19) $$\Phi(u,z,t) = \frac{1}{2}(z' \; Qz+u'u) + \mathcal{L}(J),$$

where $\mathcal{L}$ is the Dynkin operator.

By Ito's Lemma we find

$$dJ = J_t + \sum_{i=1}^{2} J_{z_i} \; dz_i + \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} J{z_i z_j} \; dz_i \; dz_j \; .$$

By definition

$$\mathcal{L}(J) = \frac{1}{dt} E_t \; (dJ),$$

so that

(20) $$\mathcal{L}(J) = J_t + J_z'(Az+Bu) + \frac{1}{2}(z'S_1' + \underset{\sim}{1}'S_2') \; J \; (S_1 z + S_2 \underset{\sim}{1}) \; .$$

By Bellman's theorem we know that there exists a control rule $\tilde{u}$

such that, from (19),

(21) $$\Phi(\tilde{u},\tilde{z},t) = 0.$$

Given (21), we minimize (19) with respect to u and obtain the control

rule

(22) $$\tilde{u} = - B'J_z.$$

Substituting (22) into (20) we can write the optimal value of Φ as

(23) $$\frac{1}{2}z'Qz - \frac{1}{2} J_z' \; BB'J_z + J_z'Az + \frac{1}{2} z'S_1'J_{zz}S_1 z + \underset{\sim}{1}'S_2'J_{zz}S_1 z$$

$$+ \frac{1}{2}\underset{\sim}{1}'S_2'J_{zz}S_2\underset{\sim}{1} + J_t = 0 \; .$$

To evaluate the partial derivatives of J consider terminal loss

from time zero to time zero $+ \Delta$.[12/] This can be written as

$$J_T(z_o, t_\theta) = J_z(z_o, t_o + \Delta) + E_u^{min} \int_{T-\Delta}^{T} \frac{1}{2}(z'0z + u'u)dt .$$

Make T very large so that the expected value of the integral

approaches the steady state value L. Then, dropping T subscripts,

we obtain

$$J(z,t) = J(z, t + \Delta) + \Delta L ,$$

so that

$$J_t = -L .$$

Consider now J as a polynominal in z such as

$$J = z'Kz + kz + c,$$

where

$$k = (k_1 \quad k_2) ,$$

so that

$$J_z = Kz + k', \text{ and}$$

(24) $J_{zz} = K$.

Substituting into (23) and collecting terms we have

$$\frac{1}{2} z' [KA + A'K - KBB'K + S_1'KS_1 + Q] z$$

$$+ [k'A - k'BB'k + \underset{\sim}{1}'S_2'KS_1] z$$

$$-L - \frac{1}{2} kBB'k + \frac{1}{2} \underset{\sim}{1}'S_2'KS_2\underset{\sim}{1} = 0 .$$

The terms in brackets are equations for $\frac{dK}{dt}$, $\frac{dk}{dt}$ and $\frac{dc}{dt}$, which for

sufficiently large T have approximate solutions

(25) $KA + A'K - KBB'K + S_1'KS_1 + Q = 0$;

(26) $k'A - k'BB'k + \underset{\sim}{1}'S_2'S_1 = 0$;

(27) $\frac{1}{2} \underset{\sim}{1}'S_2'KS_2\underset{\sim}{1} - \frac{1}{2} kBB'k' = L.$

From (25) we find the equations for the elements of the Riccati matrix as

$$k_{12}^2 - k_{11}\sigma_1^2 = \alpha ;$$

$$k_{12} \, k_{22} = k_{11} \, B_e ;$$

$$k_{22}^2 - 2 \, k_{12}B_e = (1-\alpha)B_e^2 .$$

Eliminating k_{11}, we find that the first and third equations define a hyperbola and parabola respectively in k_{12}, k_{22} space, just as in equations (17) above.[13/] Now the parabola is the same as the third equation in (13) above whereas the hyperbola is upward sloping; their intersection E_σ is to the north east of point R, as shown in Figure 1 above. The larger σ_1, the further away will E_σ be from R, in the same way that a larger ρ brought E_ρ closer to the origin and away from R.[14/]

Given the elements of K, we solve for k in (26) to find

$$k_1 = \frac{k_{22}^2}{B_e^2} \, \sigma_{12} ;$$

(28)

$$k_2 = \frac{k_{22}}{B_e} \, \sigma_{12} .$$

Even though we minimized loss conditional on a = 0, the variance terms make it non-zero in the steady-state, as can be seen by solving for the value of the loss function in (27):

$$(29) \quad L = \frac{1}{2} \sigma_2^2 \frac{k_{12}k_{22}}{B_e} - \frac{1}{2} \frac{k_{22}^2}{B_e^2} \sigma_{12}^2 \;.$$

Note that if the two disturbances are uncorrelated the optimal control cannot reduce the loss and that the zero loss optimal weights are independent of σ_2.

Now using (22) and the first equation in (24) we find the optimal control rule to be

$$(30) \quad \tilde{u} = - B'(Kz + k') \;.$$

Thus, as expected from the separation theorem, the rule is the same as in the deterministic case when the two disturbances are uncorrelated, so that the forcing term k' is zero. Using (28) we can express the rule in (30) in terms of the k_{12} and k_{22} coefficients, or in the θ, γ notation of (8) as:

$$\dot{e} = -\theta \; [\gamma \; (B(e) \; + \sigma_{12}) + (1-\gamma) \; (R-R^*] \; ,$$

where $\quad \theta = k_{12} + k_{22}/B_e \; ; \; 1-\gamma = k_{12}/\theta \; .$

Some values of θ and γ for the usual values of B_e and α and three values of σ_1, are shown in Table 4. The optimal rate of crawl depends, in addition, on the covariance term σ_{12}. If the additive and state dependent disturbances are negatively correlated, the optimal rate of crawl is less than if they are uncorrelated. However, in (29), loss

Table 4: Alternate Values of θ and γ for

Different Values of σ_1.

B_e \ $1-\alpha$	1		.5		0	
	θ	γ	θ	γ	θ	γ
$\sigma_1^2 = .1$						
$B_e = .1$	2.66	.91	5.35	.82	6.30	.79
1	1.22	.91	2.23	.65	2.50	.58
10	1.11	.91	.56	.52	1.51	.31
100	1.10	.91	1.45	.50	1.20	.12
$\sigma_1^2 = 1$						
$B_e = .1$	40.10	.50	40.55	.51	40.60	.51
1	4.82	.50	4.80	.48	5.02	.46
10	2.20	.50	2.14	.41	1.75	.29
100	2.02	.50	1.87	.39	1.20	.12
$\sigma_1^2 = 4$						
$B_e = .1$	400.06	.20	400.00	.20	400.00	.20
1	40.61	.20	40.35	.20	40.07	.20
10	7.39	.20	6.23	.20	4.56	.19
100	5.20	.20	4.00	.19	1.38	.11

only depends on σ_1^2 and the square of the correlation coefficient, so that the sign of the covariance has no effect on loss.

Aside from the effect of the covariance, which is not included in Table 4, the interpretation is similar to Tables 2 and 3 above. In fact, we notice that the offsetting of the overall sensitivity θ by the size of B_e, which was pointed out in connection with Table 3, holds for the stochastic case. The strong effect of the variance term of the state noise is also evident from the table. Indeed, when the standard deviation of the state-dependent noise is 2, the optimum weight on the current account is 0.2 for virtually all of the values of α and B_e that have been used. The exception is the combination of a pure reserve target and "infinite" elasticities ($\alpha= 1$ and $B_e = 100$). In that case the table shows a drop in the weight on the current account from 0.2 to 0.11. In the case of a pure flow target ($\alpha= 0$) we find that, just as in Tables 2 and 3, the optimum current account weight does not depend on B_e. However, while in the deterministic case the current account weight was always unity, now it ranges from 0.2 when variance is 4 to 0.91 when variance is 0.1. Also, in that case, the sensitivity parameter θ declines with increases in B_e whereas it increased with B_e in the discount case of Table 3.

In the equal weight case ($\alpha= .5$), low variance yields optimal weights that are very close to the ones obtained in the deterministic analysis. For large B_e, in fact, these weights are close to the no discount case of Table 2. For example, when $B_e = 10$, $\gamma = .53$ in Table 2 and $\gamma = .47$ in Table 3. When $B_e = 100$, γ remains unchanged in the

discount case of Table 3 and it is equal to 0.5 in the cases of Tables 2 and 4.

When variance is unity, however, the current account weight drops substantially and the range is reduced from .82 – .50 to .51 – .39. As mentioned, the uniform value for a variance of 4 is 0.2. As γ varies less, the range of the sensitivity parameter θ increases substantially with the variance. Thus, when $\sigma_1^2 = .1$, θ has a range of the same order of magnitude as in Table 2 (5.35 to 1.45 vs. 4.54 to 1.43), whilst in the high variance case the range is 400 to 4. It should be pointed out that if the variance of state-dependent noise is too large, an optimal control will not exist. The same would be true, a fortiori when control-dependent noise is incorporated in the analysis.

V.   Summary and Conclusions

The numerical findings from section II – IV are summarized in Tables 5 and 6 where the values of the current account weight γ, the sensitivity coefficient θ, and the coefficient of the current account in the optimal rule (=$\gamma\theta$) are listed for $\alpha = .5$ and for $B_e = 1$ and $B_e = 100$ respectively. The implied lower bound is obtained by dividing .83 by θ and permits comparison of the weights for a given change in the exchange rate; this is subject to the proviso that, in the stochastic case, the change in the exchange rate would be larger or smaller depending on whether the state dependent noise is positively or negatively correlated with the additive disturbance. If the variances are equal, in fact,

this might mean a difference as high as $\pm$ 4 multiplying the co-efficient on B in the tables.

The tables bring out the results that have been emphasized earlier. They can be summarized as four points.

First, the low variance (σ_1^2 = .1 in the tables) and deterministic discount cases bracket rather tightly the deterministic no-discount case. For B_e = 1 the range of γ is 0.65 to 0.67 (no discount γ = .66) and the range of the implied lower bound is 0.37 to 0.47. For B_e = 100 the range of γ is 0.65 to 0.67 (no discount γ = .50) and the range of the implied lower bound is 0.124 to 0.133. Second, the effect of discounting in reducing the sensitivity of the rate of crawl to the indicators is more than offset by large values of B_e. For example, in Table 5, with B_e = 1, γ in the deterministic discount case is higher than both γ with no discount and γ in the stochastic low-variance case. In Table 6 where B_e = 100 γ with discount is lower than the other two. Third, the sensitivity parameter θ is reduced as B_e increases. This can be seen by comparing Tables 5 and 6. The effect is more pronounced in the stochastic low-variance case and less pronounced in the deterministic discount case. Fourth, and perhaps most important, both tables show again that the effect of a large variance - σ_1^2 = 4 or larger - are quite strong. When σ_1^2 = 4, the lower bound goes to .02 in Table 5 and to .05 in Table 6, while θ increases to 40 and 4 respectively. This is not surprising since, for larger values of the variance of the state-dependent noise, an optimal control rule will not exist.

Table 5

Values of Adjustment Parameters for $\alpha = .5$; $B_e = 1$

Section	Description	γ	θ	coefficient of B	implied lower bound
II	Lower bound	.83	1		
III	Deterministic no discount	.66	2.09	1.38	.40
III	Deterministic discount $\rho = .2$	.67	1.78	1.19	.47
IV	Stochastic $\sigma_1^2 = .1$	.65	2.23	1.44	.37
	$\sigma_1^2 = 1$	.48	4.80	2.30	.17
	$\sigma_1^2 = 4$	.20	40.35	8.07	.02

Table 6

Values of Adjustment Parameters

for $\alpha = .5$; $B_e = 100$.

Section	Description	γ	θ	coefficient of B	implied lower bound
II	Lower bound	.18	1		
III	Deterministic no discount	.50	1.43	.715	.126
III	Deterministic discount $\rho = .2$	.47	1.35	.635	.133
IV	Stochastic $\sigma_1^2 = .1$	.50	1.45	.725	.124
	$\sigma_1^2 = 1$	.39	1.87	.729	.096
	$\sigma_1^2 = 4$	.19	4.00	.760	.045

This paper has shown that the Kenen-Phillips formula for the optimal weighting of indicators in a crawling peg is obtained in the various situations of sections II - IV, given the simplest model of the economy and the definition of the loss function in terms of external balance only.

In all cases the optimal formula is a weighted combination of targets with an additional parameter for the desired speed of adjustment. With a low variance of the state dependent noise, equal weights given to the current account and reserve targets and a unit cost of exchange rate variability imply equal weights of the current account and reserve indicators when import and export elasticity are high. When elasticities are low, however, the optimal current account weight increases to about 2/3.

The optimal speed of adjustment θ is very sensitive to the estimated value of B_e and σ_1. However, if we normalize the speed of adjustment to unity, Tables 5 and 6 show the criterion of instability derived in section II substantially understates the current account weight and that the degree of understatement increases as elasticities increase.

In sum, while we have shown that the optimal indicator is in general a weighted combination of the flow and stock targets, the numerical results suggest that the quantitative choice of a formula will require careful econometric estimation in each particular case.

Appendix

The eigenvalues of the system (9), with $c = 1$ for simplicity, are given by

$$\text{(A1)} \qquad r_1, r_2 = -\frac{B_e}{2}\left[\sqrt{1-\alpha + \frac{2\sqrt{\alpha}}{B_e}} \pm \sqrt{1-\alpha - \frac{2\sqrt{\alpha}}{B_e}}\right].$$

The system can be seen to be stable. Using the same values of α and B_e as in Table 2, we compute alternative values of r_1 and r_2 in Table Al. As shown there, when $\alpha = .5$ the system is monotonically stable for high values of B_e and oscillatory for low values of B_e. Also, with no reserve target the system is always monotonically stable, but with no current account target it is always oscillatory.

Table Al

Values of B_e	Values of α					
	0		.5		1.0	
	r_1	r_2	r_1	r_2	r_1	r_2
.1	$-.1$	0	$-.192(1+i)$	$-.192(1-i)$	$-.223(1+i)$	$-.223(1-i)$
1.0	-1.0	0	$-.69(1+i)$	$-.69(1-i)$	$-.707(1+i)$	$-.707(1-i)$
10	-10	0	-7	-1	$-2.23(1+i)$	$-2.23(1-i)$
100	-100	0	-70.7	-1	$-7.07(1+i)$	$-7.07(1+i)$

The solution of (9) is of the form

$$z_t = \exp(Gt)z_o,$$

where $G = A - BB'K$, and,

$$\exp(Gt) = \frac{B_e}{r_2 - r_1} \begin{bmatrix} 1 & 1 \\ r_1/B_e & r_2/B_e \end{bmatrix} \begin{bmatrix} \exp(r_1 t) & 0 \\ 0 & \exp(r_2 t) \end{bmatrix} \begin{bmatrix} r_2/B_e & -1 \\ -r_1/B_e & 1 \end{bmatrix} .$$

Consider the case where reserves are initially at their target level, so that $R_o = R*$. Then the motion of the system is given by

(A2)
$$R_t - R* = \frac{1}{r_2 - r_1} [\exp(r_1 t) + \exp(r_2 t)] B(e_o) ;$$

$$e_t - e* = \frac{1}{r_2 - r_1} [-r_1 \exp(r_1 t) + r_2 \exp(r_2 t)](e_o - e*).$$

The minimum value of the loss function associated with the feedback rule in equation (14) can be written as:

$$\tilde{L}_t = \frac{1}{2} z'_t K z_t = \frac{1}{2} k_{11}(R_t - R*)^2 + \frac{1}{2} k_{22}(e_t - e*)^2 + k_{12}(R_t - R*)(e_t - e*).$$

Substituting the eigenvalues and the elements of the Riccati matrix for the deviations of the state variables from their given equilibrium levels, we can express minimum loss as a function of the two parameters α and B_e. Note that in this deterministic framework, loss is zero in steady-state where $z = 0$.

Substituting from (A1) into the exchange rate equation in (A2), we express the exchange rate path as a function of the two parameters α and B_e:

$$e_t - e* = \frac{1}{2}(e_o - e*)[(a_2 - a_1)/\psi - (a_2 + a_1)],$$

where
$$= \sqrt{\frac{B_e(1-\alpha) - 2\sqrt{\alpha}}{B_e(1-\alpha) + 2\sqrt{\alpha,}}}$$

and $a_j = \exp(r_j t)$; $j = 1, 2$.

In the case of a pure reserve target, $\alpha = 1$ and $\psi = i$ so that the exchange rate path oscillates. It is given by:

$$e_t = \frac{1}{2}(e_o - e^*) \ [(a_2-a_1) - (a_2+a_1)i] + e^*i.$$

In the case of a pure current account target, $\psi = 1$ and the path of the exchange rate is particularly simple. It is given by

(A3) $e_t = e_o \ a_1 + e^* \ (1-a_1).$

By integrating (A3) over a given interval, say one or two years, we can rationalize the neglect of J-curve effects mentioned in the text and express the average value of the exchange rate in period ℓ as:

$$e_\ell = e_o \theta_\ell \ + e^* \ (1- \theta_\ell),$$
$$\text{where} \ \theta_\ell = - \frac{1}{r_1} \exp \ [r_1(\ell-1)][1-\exp(r_1)].$$

As in the analysis of Kouri (1978), the average exchange rate for the ℓ'th period is a weighted average of the initial and long run levels, where the weight of the initial value declines over time. With $B_e = 1$, Table A1 above gives $r_1 = -1$ so that the weight on the initial value becomes 7.8×10^{-5} in the 10th year. Values for the first five years are given in Table A2 below.

Table A2

ℓ	1	2	3	4	5
θ_ℓ	.632	.233	.086	.031	.012

Footnotes

1 See Phillips (1954), p. 305 footnote 1 . The import-
   ance of this work is emphasized in the first paragraph of Turnovsky
   (1973) and in the preface of Aoki (1976), for example.

2 See Branson-Katseli (1978) for the fully developed argument.

3 While it would be easy to set up a simple model allowing for the
   usual macroeconomic features, the complete optimization problem
   is beyond the scope of this paper.

4 The extensive literature on optimal reserves was surveyed in
   Williamson (1973a). See also Bilson-Frenkel (1979).

5 See Kenen (1975) p. 128. Cooper's proposal is analyzed on p. 118
   and"given good marks" in the conclusion on p. 147.

6 The outline of the proposal presented at the Annual Fund Meeting
   is in IMF (1972), p. 34-44, esp. p. 39-40. The full document
   is in CEA (1973), p. 160-174. Alternative proposals are repro-
   duced in C-20 (1974) and discussed in Williamson (1977). See
   Underwood (1973) for a listing of similar proposals.

7 See Phillips (1954), p.298 footnote 1 and Kenen (1975) p. 128. The
   conditions for the instability of the reserve target indicator, as
   well as the changes in reserves indicator when there are capital
   flows, were derived in a complete model of the "small open inflation-
   ary economy" by Martirena-Mantel (1976), who concludes that her

results "seem to agree" with Kenen's. Recently,still in

connection with the "Southern Cone problem", this instability

was obtained in a variety of portfolio balance models.

8 The homogeneous system we work with has of course the same

eigenvalues as the inhomogeneous one. The consequences of

discounting are analyzed below.

9 Note that the system (8),(9) is both controllable and observable,

since the rank of $(B \vdots AB)$ and $(C' \vdots A'C)$ is two.

10 The minimum value of the loss function and the explicit solution

of the system are found in the Appendix.

11 See a similar derivation in Macedo (1979), Appendix 1. Also Chow

(1979).

12 See Chang-Sketler (1976) for a similar derivation.

13 If we were discounting and $\sigma_1^2 > \rho$ the intersection would be at E,

on the hyperbola to the left of R and on the parabola cutting the

k_{22} axis at $B_e(\sqrt{1-\alpha + \tilde{\rho}^2} - \tilde{\rho}$.

14 The similarity between state dependent noise and a "negative

discount" has been pointed out by Turnovsky (1973). Note how-

ever the difference in this model between (16) and (25) and the

difference in the magnitudes of the parameters discussed below

in the text.

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