ECONOMETRIC ANALYSIS BY CONTROL METHODS*

INTRODUCTION

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Econometric Analysis by Control Methods

PREFACE

by Gregory C. Chow

Since the early 1970's, economists have become increasingly interested in methods of stochastic control. From 1972 to 1978, a series of annual conferences on stochastic control in economics were held successively at Princeton University, the University of Chicago, the Board of Governors of the Federal Reserve System, the Massachusetts Institute of Technology, Stanford, Yale, and the University of Texas, under the sponsorship of the National Bureau of Economic Research and with financial support from the National Science Foundation. Selected papers from these conferences were published in various issues of the *Annals of Economic and Social Measurement*. In 1979, an international Society for Economic Dynamics and Control was formed with its first annual conference held at Cambridge University. Nineteen-seventy-nine also saw the publication of the new *Journal of Economic Dynamics and Control*.

The author's text, *Analysis and Control of Dynamic Economic Systems* (Wiley, 1975), was an attempt to summarize some of the important methods of stochastic dynamics and control together with economic applications for the students and professionals who wish to learn the subject. Since then, not only have new methods been developed, but many novel and important applications have appeared, as we will report in this book. This book is divided into four parts. Part I is concerned with control techniques. Methods for large nonlinear systems of simultaneous equations are discussed in Chapters 1 through 5. When the author's 1975 book was published, it was considered difficult or impractical to control such systems. Now, algorithms have appeared and computer programs are available. In addition, the method of Kalman filtering has been further developed and recognized to be useful for the estimation of econometric models, as we will point out in Chapter 6.

Part II of this book, including Chapters 7 through 14, is concerned with economic applications of control methods. Chapter 7 suggests an economic definition of the inflation-unemployment tradeoff relationship which is implicit in an econometric model and is derived by stochastic control techniques. Chapter 8 provides a method to evaluate the success or failure of historical macroeconomic policies. Chapter 9 attempts to answer the question, "Has government policy contributed to economic instability?", by studying analytically the dynamic properties of the Michigan Quarterly Econometric Model. Chapter 10 recommends the application of the control framework to estimate the amount of resources available for government investment in Taiwan. Chapter 11 shows how imperfect models can be used for the formulation of stabilization policies. Chapter 12 provides a practical guide for using econometric models in macroeconomic policy formulation. Chapter 13 deals with the applications of control methods to the econometric analysis of Soviet economic planning. Finally, Chapter 14 indicates how control techniques can be used to compare econometric models.

In the late 70's it was the view of some economists that the use of optimal control for the formulation of macroeconomic policies is inconsistent with the assumption of rational expectations. It has now become evident, however, that the use of control techniques for the evaluation of economic policies and for the derivation of optimal policies is not only possible but is essential under rational expectations. Moreover, control techniques are useful for the estimation of macroeconomic models under the assumption of rational expectations. This theme is developed in Chapters

15 through 17, which form Part III of this book. While previous chapters have been concerned with control techniques in discrete time and mainly with the study of aggregate economic activities, the last two Chapters 18 and 19 in Part IV provide an exposition of stochastic control techniques in continuous time and apply these techniques to the study of problems in microeconomics.

It is hoped that this volume will provide its readers with a fairly broad treatment of the techniques of stochastic control and its important economic applications. This book can be used in conjunction with the author's 1975 text. It can also be used independently because the material is self contained, its main prerequisite being some basic training in econometrics at the level of, for instance, R. J. Wonnacott and T. H. Wonnacott's Econometrics (Wiley, 1979).

In the preparation of this book, I have benefitted from the collaboration of Sharon Bernstein Megdal (Chapters 4 and 7), Ettie H. Butters (Chapter 5), Suzanne Heller (Chapter 9), and Donald W. Green (Chapter 13). Pia Ellen and Constance Dixon have provided excellent assistance in typing various drafts of the manuscript. Ms. Ellen, in addition, has helped in editing the manuscript and preparing the index. To them, my sincere thanks.

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Chapter 1

Introduction

The science of economics, as it is generally acknowledged, is a study of means for allocating scarce resources to satisfy competing and not totally attainable objectives. Mathematically speaking, an optimal control problem is concerned with the determination of the best ways to achieve a set of objectives as indexed by a criterion function when the performance is judged over many periods and when the dynamic behavior of the system is subject to a set of constraints. Therefore, dynamic economics and optimal control are conceptually very similar, if not formally identical. The differences, if they exist, lie mainly in the specification of special assumptions concerning the objective function or the dynamic system made in the optimal control literature for the sake of mathematical convenience. Such assumptions restrict the applicability of certain optimal control techniques to dynamic economics. Hopefully they will be relaxed or modified as the science progresses, but restrictive assumptions yield the benefits of analytical power and insights, and may indeed be relevant in many applications.

The scope of this volume is further limited in its treatment of both techniques and applications. The techniques are mainly, though not entirely, generalizations of the methods used for solving the elementary optimal control problem of minimizing a quadratic loss function subject to the constraint of a linear model. The applications are mainly, though again not entirely, econometric in nature, with due emphasis on statistical measurements.

The purpose of this book is to present further developments in methods and applications of stochastic control in economics which the author has pursued since the publication of his Analysis and Control of Dynamic Economic Systems (1975). In order to survey its contents, it will be convenient to set up the basic optimal control problem of minimizing the expectation of a quadratic loss function subject to the constraint of a linear model with additive stochastic disturbances. Much of this volume will be concerned with extensions, modifications and applications of this basic control problem.

1. Optimal Control of a Linear Model with Known Parameters

Let the environment facing the economic decision makers be represented by a linear system

$$y_t = A_t y_{t-1} + C_t x_t + b_t + u_t \tag{1}$$

where y_t is a vector of p state variables, x_t is a vector of q control variables, A_t , C_t and b_t are matrices of known constants, and u_t is a random vector which is serially independent and identically distributed. It is understood that state variables have been introduced to eliminate y_{t-2} , y_{t-3} ,..., x_{t-1} , x_{t-2} , etc. from the system, and to incorporate x_t as a subvector of y_t so that the loss function below need not have x_t as an argument. For example, if there are 50 endogenous variables in the original system and the model includes a variable $y_{6,t-2}$, an identity $y_{51,t} = y_{6,t-1}$ can be introduced. Given this identity, $y_{6,t-2}$ can be written as $y_{51,t-1}$ and the second-order lag disappears.

If $y_{6,t-3}$ is present, another identity $y_{52,t} = y_{51,t-1}$ can be used and $y_{6,t-3}$ can be written as $y_{52,t-1}$. Let there be 40 identities of this type to rid the system of second and higher order lagged endogenous variables. If $x_{1,t-1}$ and $x_{1,t-2}$ are present, we can define $y_{91,t} = x_{1,t-1}$ and write $x_{1,t-2}$ as $y_{91,t-1}$, etc. Let there be 10 identities of this type. We then define $y_{101,t} = x_{1t}$, $y_{102,t} = x_{2t}$ to incorporate x_t as a subvector of y_t. The loss function measuring the preference of the decision makers is quadratic

$$W = \sum_{t=1}^{T} (y_t-a_t)'K_t(y_t-a_t) \;. \tag{2}$$

The optimal control problem is to find strategies for x_t in order to minimize expected loss.

This elementary linear-quadratic optimal control problem can be solved by dynamic programming. First, we find the optimal policy for the last period T, given all the information up to (the end of) period $T-1$. Denote by V_T the expectation, conditioned on information up to $T-1$, of the loss for period T, which is a function of the policy x_T,

$$V_T = E_{T-1}(y_T-a_T)'K_T(y_T-a_T) = E_{T-1}(y_T'H_Ty_T-2y_T'h_T+c_T) \tag{3}$$

where we have set $K_T = H_T$, $K_Ta_T = h_T$ and $c_T = a_T'K_Ta_T$. Substituting $A_Ty_{T-1} + C_Tx_T + b_T + u_T$ for y_T in (3) and minimizing V_T with respect to x_T by differentiation, we find the optimal policy for the last peri

$$\hat{x}_T = G_T y_{T-1} + g_T \tag{4}$$

where

$$G_T = -\ (C_T' H_T C_T)^{-1} (C_T' H_T A_T) \tag{5}$$

$$g_T = -\ (C_T' H_T C_T)^{-1} C_T' (H_T b_T - h_T)\quad . \tag{6}$$

The minimum expected loss for the last period is obtained by substituting for x_T in V_T ,

$$\begin{aligned} \hat{V}_T = {} & y_{T-1}' (A_T + C_T G_T)' H_T (A_T + C_T G_T) y_{T-1} \\ & + 2 y_{T-1}' (A_T + C_T G_T)' (H_T b_T - h_T) \\ & + (b_T + C_T g_T)' H_T (b_T + C_T g_T) - 2 (b_T + C_T g_T)' h_T \\ & + c_T + E_{T-1} u_T' H_T u_T \quad . \end{aligned} \tag{7}$$

To obtain the optimal policies for the last two periods, we observe that $\hat{x}_T$ is already found which would yield the minimum loss $\hat{V}_T$, and that, by the principle of optimality of dynamic programming, we need only to find x_{T-1} to minimize

$$\begin{aligned} V_{T-1} & = E_{T-2} [(y_{T-1} - a_{T-1})' K_{T-1} (y_{T-1} - a_{T-1}) + \hat{V}_T] \\ & = E_{T-2} (y_{T-1} H_{T-1} y_{T-1}' - 2 y_{T-1}' h_{T-1} + c_{T-1}) \end{aligned} \tag{8}$$

where, using the expression (7) for $\hat{V}_T$, we have defined

$$H_{T-1} = K_{T-1} + (A_T + C_T G_T)' H_T (A_T + C_T G_T)\ , \tag{9}$$

$$h_{T-1} = K_{T-1} a_{T-1} - (A_T + C_T G_T)' (H_T b_T - h_T)\ , \tag{10}$$

$$c_{T-1} = a'_{T-1}K_{T-1}a_{T-1} + (b_T+C_Tg_T)'H_T(b_T+C_Tg_T) \tag{11}$$

$$- 2(b_T+C_Tg_T)'h_T + c_T + E_{T-1}u'_TH_Tu_T \quad .$$

Since the second line of (8) is identical with the last expression of (3) with T replaced by $T-1$, the solution for $\hat{x}_{T-1}$ is identical with (4) with T replaced by $T-1$, where G_{T-1} and g_{T-1} are defined by (5) and (6) respectively with a similar change in time subscripts. Accordingly, $\hat{V}_{T-1}$ will be given by (7) with the subscripts T replaced by $T-1$.

When we attempt to solve the problem for the last three periods, we observe that $\hat{x}_T$ and $\hat{x}_{T-1}$ have been found which would yield the minimum expected loss $\hat{V}_{T-1}$ for the last two periods and that, by the principle of optimality, we need only minimize

$$V_{T-2} = E_{T-3}[(y_{T-2}-a_{T-2})'K_{T-2}(y_{T-2}-a_{T-2}) + \hat{V}_{T-1}]$$

with respect to x_{T-2} , and so forth. At the end of this process, we find $\hat{x}_1 = G_1y_o + g_1$ as the optimal policy for the first period, and the associated minimum expected loss $\hat{V}_1$ for <u>all</u> periods (or from period 1 onward). Computationally, we solve equations (5) and (9) with t replacing T for G_t and H_t backward in time, for $t = T$, $T-1$,..., 1 . We then solve equations (6) and (10) with t replacing T for g_t and h_t backward in time, for $t=T$, $T-1$,..., 1 . Finally, solution of (11) with t replacing T backward in time yields c_1 which is used to evaluate $\hat{V}_1$ given by (7), with 1 replacing T .

Note that the expression $\hat{V}_t$ given by (7) can be used to obtain the values (shadow prices) of the initial resources y_{t-1} . The vector of the shadow prices of these resources is simply the derivative of $-\hat{V}_t$

(negative loss or benefits) with respect to y_{t-1}, namely,

$$-\frac{\partial \hat{V}_t}{\partial y_{t-1}} = -\,2(A_t+C_tG_t)'[H_t(A_t+C_tG_t)y_{t-1} + H_tb_t - h_t] \quad . \qquad (12)$$

2. Techniques of Stochastic Control

The above brief summary of the theory of optimal control for a linear stochastic model and a quadratic objective function provides a convenient setting to introduce the remaining chapters of this book, which is divided into four parts. Part I deals with basic techniques of stochastic control. Chapter 2 applies the method of dynamic programming to find a nearly optimal policy for controlling a nonlinear simultaneous-equation model in econometrics and illustrates the method with an application to the Klein-Goldberger Model of the U.S. economy. The method of section 1 above is thus generalized to deal with a system of nonlinear simultaneous equations. A number of interesting characteristics of the Klein-Goldberger Model are uncovered when it is used to perform the optimal control exercise. In particular, it has been found that according to this Model, there is no need to trade a higher price level for an increase in real GNP or in total employment. That is, a Phillips curve does not exist for this model. By manipulating the policy variables, the government can achieve any combination of the general price index and real GNP (or total employment) as it wishes. This characteristic of the model is to be contrasted with the natural rate hypothesis or a vertical Phillips curve; it implies that the Phillips curve can be shifted at will by government policy. It is also to be contrasted with the negatively sloping Phillips curves uncovered in

Chapter 7 for the St. Louis Model and the Michigan Quarterly Econometric Model.

While Chapter 2 has a brief section on the control of nonlinear systems of simultaneous equations with unknown parameters, it mainly deals with the case of known parameters. The problems associated with unknown parameters are treated more thoroughly in Chapter 3. Chapter 4 is concerned with problems associated with large-scale nonlinear econometric models, discusses the computational problems in some detail and illustrates the method of approximately optimal feedback control by the use of the Michigan Quarterly Econometric Model. Chapter 5 is a guide to a prospective user of a computer program for performing the approximately optimal control calculations using a nonlinear econometric model based on the methods described in Chapters 2, 3 and 4. The computer program itself (written in Fortran) is available in the Econometric Research Program of Princeton University upon request.

Today, the ability of the method presented in Chapter 2 using an algorithm described in Chapter 5 to solve optimal control problems involving over two hundred nonlinear simultaneous equations is taken for granted. The author has used it to control,the model of Fair (1976) and a 1977 version of the MIT-Penn-SSRC model which contains about 250 simultaneous equations. However, as late as 1978, it was exciting to discover that the algorithm actually works for such large systems. Furthermore, the algorithm produces as a by-product a linearized version of the model for each time period. From these linear models one can derive analytically the important dynamic properties of the system. The dynamic properties include various multipliers measuring the effects of changes in current and/or past exogenous variables (taking place in one period or in several consecutive periods) on the current endogenous variables. They also include the autocovariance matrix and the spectral density matrix which measure, in the time domain and in the frequency domain respectively, the cyclical properties of the individual time series and the cyclical relationships between time series generated by the model. These concepts are discussed in Chow (1975). Until recently, it was common practice for econometricians to compute multipliers from a nonlinear model by computer simulation. The simulation process amounts to changing the values of the exogenous variable of interest in appropriate time periods, computing the resulting solution paths for the endogenous variables by iterative methods, and measuring the rate of change in the latter with respect to the former. Using the linearized model generated by our algorithm, the multipliers can be easily identified from the coefficients of the model. Similarly, the autocovariance and spectral density matrices can be derived analytically using the linearized model without resorting to expensive simulations. Chapter 6 presents the method of Kalman filtering and its applications to the estimation of econometric models.

3. Economic Applications of Stochastic Control

Part II of this book deals with economic applications of stochastic control techniques. It begins, in Chapter 7, with an econometric definition of the inflation-unemployment tradeoff relation implicit in an econometric model. It is argued that such a relation can be ascertained only by optimal control techniques. Economists have tried to vary the time paths of certain exogenous variables in an econometric model and observe the resulting paths of inflation and unemployment generated by simulations of the model. High inflation and low unemployment have often been found to associate with expansionary fiscal and monetary policies according to the model used. It is hoped that, by simulations, the tradeoff relationship between inflation and unemployment can be ascertained. However, it is quite likely that the policies used in the simulation experiment can be dominated, in the sense that an outcome corresponding to both lower inflation and lower unemployment rates could be obtained. By the use of stochastic control techniques, one can trace out the lowest inflation rate compatible with a given unemployment rate according to the specification of an econometric model and given a set of initial conditions, thus yielding an econometric definition of the inflation-unemployment trade-off. This has been done for the St. Louis Model and the Michigan Model. The method suggested can also be used to compare the characteristics of econometric models, as will be discussed in Chapter 13.

Chapter 8 provides a theoretical framework to evaluate historical macroeconomic policies formulated in an environment of uncertainty. Uncertainty in the consequences of the policies adopted because of random elements in the economy makes the evaluation of historical policies difficult. Policies should not be judged merely by the ensuing outcomes which were partly due to chance or luck, but by the expected loss or gain calculated at the time when the decisions were made. Following this principle, we provide in Chapter 8 a method to evaluate historical policies using stochastic control techniques.

Chapter 9 attempts to answer the question, has government policy been destabilizing? It provides an answer by studying analytically the dynamic properties of the Michigan Quarterly Econometric Model under three different policy regimes. The first specifies that the policy variables follow their reaction functions as estimated by historical data. These reaction functions are used to represent the policy behavior of the government. The second specifies that the policy variables follow constant rates of change, representing passive behavior of the government. The third consists of optimal control policies obtained by stipulating certain quadratic loss function to measure economic instability. Undoubtedly the third regime is superior to the first two by construction, but it provides a limit as to what could possibly be achieved under idealized, and perhaps unrealistic, conditions. It is interesting to note, however, that the historical policy reaction functions, when combined with the equations of the Michigan Model, have produced an economy with a higher measure of stability than a set of passive policies under the second regime.

While the applications of Chapter 7, 8 and 9 are oriented toward the economist who wishes to study or analyze macroeconomic policies from the perspective of an observer, the following three chapters are concerned with possible applications of optimal control techniques to the actual formulation of economic policies. Chapter 10 briefly indicates how total government investment expenditures should be determined using the method of optimal control. The problem was presented to the author when he served as consultant to the Economic Planning Council in Taiwan in 1978 and the solution was later implemented by the Council. Unfortunately, the Chapter contains only the method without an accompanying empirical study of the Taiwan economy.

Chapter 11 points out the possible use of imperfect econometric models for the formulation of stabilization policies. It has been suggested that because econometric models are imperfect and different models may yield different economic projections given the same hypothetical time paths for the policy variables, they cannot serve as a guid to produce actual policy recommendations. Chapter 11 argues that this conclusion is invalid, and that imperfect models can be fruitfully employed as an aid to the formulation and analysis of macroeconomic policies. Based partly on the concepts of 11, Chapter 12 goes further by outlining twelve concrete steps in applying control techniques to improve the formulation of macroeconomic policies.

Chapter 13 suggests three applications of optimal control techniques to the econometric analysis of Soviet economic planning. They involve checking the validity of the constraints specified by the model with respect to production relationships, tracing out the production frontiers including the tradeoff relation between consumption and defense, and, thirdly, the estimation of the preference function of the Soviet government for the purpose of forecasting its future behavior and thus the future course of the Soviet economy. Only the first of the three topics is treated in detail in the empirical sections of the Chapter by analyzing an econometric model of the Soviet economy.

Finally, in Chapter 14, we turn away from the subject of analysis and formulation of macroeconomic policies and to a comparison of econometric models by the use of stochastic control techniques. A number of tools are suggested, including the inflation-unemployment tradeoff relationship implicit in an econometric model discussed in Chapter 7 and the control tools to evaluate historical policies presented in Chapter 8. Furthermore, the stability characteristics of an econometric model subject to different policy regimes as explored in Chapter 9 could also be used to compare different econometric models.

4. Stochastic Control under Rational Expectations

Recently, there has been much interest among economists in exploring the implications of the hypothesis of rational expectations for the evaluation and formulation of macroeconomic policies. Part III of this book contains three chapters on the applications of stochastic control techniques to the evaluation and formulation of economic policies, and to the estimation and control of econometric models, under the assumption of rational expectations. It has sometimes been alleged that economic policy evaluation and optimization are impossible if economic agents form rational expectations. Chapter 15 is an attempt to dispel this mistaken belief by showing how the consequences of given economic policies can be evaluated, and how optimal policies based on economic models can be found by optimal control techniques, under the assumption of rational expectations. Here rational expectations simply means that all expectations attributed to the economic agents which are used in an econometric model are identical with the conditional expectations of the corresponding endogenous variables generated by the econometric model itself, and not formed by some process such as a distributed lag scheme other than the econometric model itself. This use of the term rational expectations requires only that the econometrician and the economic agents use the same model to form their expectations, without necessarily implying that the latter's behavior is based on maximizing some objective function.

If, in addition, we assume that the economic agent's behavior results from maximization, a more powerful theory or model results. The tools of optimal control are extremely useful for the specification of such optimization models under rational expectations. If the economic agents are assumed to maximize the expectation of an objective function over time in a stochastic environment, the tools of optimal control can be used to determine their optimal decision rules. For example, equation (1) of this chapter may describe the stochastic environment facing the economic decision makers and equation (2) may represent their preference function; equation (4) with t replacing T then represents their optimal decision functions. In most of our previous applications, the decision makers have been identified with the government authorities. In fact, they may represent economic agents of the private sector, such as consumers and producers. Equation (4) then represents the optimal consumption function, investment function, or demand function for inputs, as the case may be. Chapter 16 is concerned with the statistical estimation of such models using optimal control techniques. It is important to note that the parameters to be estimated include not only the parameters of (1), with time-invariant coefficients A, C and b, but the parameters of the preference function (2), with $K_t = \beta K$ (β being a discount factor) and $a_t = a$. Methods of maximum likelihood and two-stage least squares are proposed to estimate these parameters. Having estimated the parameters of (1) and (2), which correspond to the "structural" parameters of simultaneous-equation models, one can ascertain how the behavioral equations (4), with time-invariant coefficients G and g (which correspond to reduced-form parameters), will change when government policy changes.

To make explicit the effect of government policy on the environment facing the private economic agents, consider two sets of decision makers, one representing the private sector which controls x_{1t} and the second representing the government which controls x_{2t}. The econometric model (1) with time-invariant coefficients can be written as

$$y_t = Ay_{t-1} + C_1x_{1t} + C_2x_{2t} + b + u_t \quad . \tag{13}$$

If the government follows a rule $x_{2t} = G_2y_{t-1} + g_2$, the environment facing the private agents will be

$$\begin{aligned} y_t &= (A+C_2G_2)y_{t-1} + C_1x_{1t} + (b+C_2g_2) + u_t \qquad (14) \\ &= A_1y_{t-1} + C_1x_{1t} + b_1 + u_t \end{aligned}$$

which shows how government policy affects the environment facing the stochastic environment of the private agents. Since the latter obtains its optimal policy $x_{1t} = G_1 y_{t-1} + g_1$ by maximizing the expectation of its objective function subject to the constraint (14), one can apply the theory of optimal control to deduce the behavior of the private agents when the government's policy changes. Chapter 17 is concerned with a dynamic game model involving two players. It provides a method to find the optimal policy for the second player (the government) who anticipates that its policy $x_{2t} = G_2 y_{t-1} + g_2$ will affect the behavior of the private sector. This is the solution to a dynamic game with the government as the dominant player. A solution is also provided when the game is assumed to be in a Nash (a Cournot) equilibrium. Furthermore, methods to estimate the parameters of (13) and of the preference functions of the two players will also be provided, under the assumption of a dominant player or a Nash equilibrium.

5. Optimal Control Methods for Stochastic Models in Continuous Time

Part IV of this book contains two chapters. Chapter 18 provides an elementary exposition of the model of stochastic differential equations where the random disturbance is driven by a Wiener process or Brownian motion, and of the method of optimal control applicable to this model. The exposition is built upon concepts of dynamic systems and of optimal control in discrete time by showing how the model and the method of dynamic programming would become when the time interval between successive observations is made very small. Applications to microeconomics will also be provided, including the determination of consumption and portfolio selection over time, the theory of mutual funds and capital asset pricing, and the pricing of stock options. Chapter 19 studies the optimum consumption and exploration of a limited natural resource. Although the subject matter of Part IV is different from the remainder of this book, the methods employed are quite similar. In fact, the methods of stochastic control in continuous time are also applicable to the study of macroeconomic problems when the macromodel is formulated in continuous time. See Gertler (1980) for an example. Like the previous chapters, Part IV present a set of powerful tools for dynamic analysis and control, and addresses important economic problems using these tools. They are the main purposes of the studies contained in this volume.

Reference

Gertler, Mark, "Uncertain Lags and Optimal Monetary Rules," presented before the meetings of the Society for Economic Dynamics and Control at Princeton University on June 4, 1980.