

BERTRAND AND WALRAS EQUILIBRIUM

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Abstract: Extending to a general equilibrium context Bertrand's critique of Cournot, we present a game-theoretic model of a pure exchange, monetary economy, in which buyers as well as sellers announce both prices and quantities. Our main result is that those Nash equilibria such that there are at least two buyers and two sellers actively trading on every market yield allocations which are "competitive" allocations for the underlying exchange economy. Under this characterization of strategic behavior, then, "two is enough for competition."

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I. INTRODUCTION

Modern Walrasian analysis (see, for example, Debreu (6)) focuses on the correspondence between economies, defined by agents' characteristics and technological possibilities, and their competitive allocations. In recent years a considerable research effort has been directed towards both enriching and also constructing game-theoretic foundations for this analysis. We shall refer to this effort generically as the "Noncooperative Approach to Economic Equilibrium" (NAEE).

Contributions to the NAEE enrich the Walrasian model by imposing upon it some additional structure, specifically a "mechanism," to be viewed as a stylized representation of the economic institutions of trade. Unlike the Walrasian model, which leaves open the question of how competitive allocations come to be realized, the NAEE does in fact take a position on this issue, specifying an economic process through which trades are actually effected.

The other focus of the NAEE is foundational: it investigates the conditions on the primitives of economic analysis under which the Walrasian model may be considered to be an appropriate idealization of some less abstract economic structure. Since Walrasian theory is founded upon some rather strong assumptions, a natural prerequisite that the Walrasian model must satisfy is that these assumptions should be relatively "innocuous," in some well-defined sense. The particular assumption which has received most attention is that agents act as if they have no effect on price; that is, they maximize their utility (or profit) functions subject to the constraint that prices are exogenously given. The NAEE relaxes this

characterization, then, the size of individual agents relative to the market is immaterial; in short, "two is enough for competition."*

Most game-theoretic contributions to competitive theory invoke a notion of strategic behavior originating with Cournot's classic study of duopoly (Cournot (5)). We gather together the features which characterize this notion to form a rather loosely defined construct, which we call the Cournot paradigm. Models based on this paradigm are of the following type: agents announce decisions about the quantities that they wish to trade, while the rates at which commodities are ultimately exchanged in the economy are determined by the "impersonal forces of the market."** The essential difference between the Cournot paradigm and the Walrasian model is that at least some of the agents are assumed to recognize, and to exploit any relationship which may exist between the quantities which they announce and the prices which subsequently prevail.

An alternative characterization of strategic behavior, which we develop in this paper, can be viewed as a generalization of the classic critique of Cournot by Bertrand (3). Under this characterization, the prices at which agents trade are determined not by the "impersonal forces of the market" but by the agents themselves, who

*This result is also obtained by Dubey, in a recent paper (Dubey (8)). Grossman, also, has results which are very similar to mine in spirit, but which obtain for rather different reasons (see Grossman (11)). The relationships between Dubey's paper and mine is discussed in detail in Section IV and VI.

**The phrase is Scitovsky's (24). In an Historical Note (Simon (29)), we emphasize that while this specification is formally equivalent to Cournot's original model, the story which Cournot himself tells about his model differs in some important respects from this characterization, which is the contemporary interpretation, of his model.

simply announce these prices. While Bertrand discussed Cournot's model of a (proto-) production economy, in which proprietors announced prices and consumers acted as price-takers, we model a pure exchange economy in which both sellers and buyers specify the prices at which they will trade. In our model transactions are effected via a process of intermediation.

We construct a game-theoretic model of a pure exchange, monetary economy in which agents function simultaneously as both buyers and sellers. Strategies are quadruples: an offer to sell, specifying both quantities and prices, and a collection of alternative bids to buy, also specifying quantities and, for each quantity, a "price." More specifically, an agent, acting as a seller, announces a vector representing the quantities of each commodity which he is willing to supply, together with a vector of ask-prices, the rates at which he is prepared to sell these commodities in exchange for money. As a buyer, the agent announces an "acceptance set" of vectors, each element in the set representing a commodity bundle, together with a set of associated "bid-prices." A buyer thus indicates his willingness to buy any one of a number of alternatives. Each vector in the "acceptance set," paired with the associated "price," can be interpreted as a bid to purchase some quantity of a well-defined composite commodity or "package deal," as well as a "bid-price," specifying the per-unit rate at which he is prepared to pay, in money, for this composite commodity.

The formal specification of our model is completed by the addition of a strategic outcome function, a map from strategies to

assumption. Agents are modelled as acting strategically in the following sense: they acknowledge, and exploit to their best advantage, whatever capacity they have to affect the prices which constrain their trading opportunities. The following question can then be addressed: under what conditions on the parameters which define an economy will the allocations which result, when agents act strategically, be "close" to the competitive allocations for the underlying Walrasian economy?

This paper contributes to our understanding of the relationship between these two facets of the NAE. One aim of our study is to indicate how sensitive the foundational question is to the particular specification of the economic institutions of trade. Specifically, the main theorem which we present suggests that a fundamental theme of the NAE is less robust than is commonly supposed. The theme is that "the size of the market relative to the individual agent will be a key explanatory variable for the tendency of noncooperative behavior to approximate perfect competition." (Mas-Colell (16), p. 122). Our claim is that the robustness of this theme can be attributed to the ubiquity, in the Noncooperative Approach, of what we shall refer to as the "Cournot paradigm" of strategic behavior. We suggest an alternative, and, arguably, no less plausible, characterization of strategic behavior, together with a stylized notion of an "economic process." It will be shown that, if this alternative specification is adopted, then the Nash and Walrasian allocations will coincide if all markets are "thick," where a market is said to be thick if there are at least two buyers and two sellers active in that market. Under this

outcomes. To motivate this function we introduce the fiction of a competitive arbitrage sector. The outcome of our game would arise if there existed a profit-maximizing, competitive arbitrage industry consisting of "firms" which bought commodities from sellers, paying in money, at the rates specified by each seller, and sold back to buyers "package deals," in exchange for money, at the rates specified by the buyers. Specifically, the outcome is a random vector, the support of whose distribution is the volume-maximal subset of the arbitrage-profit-maximal subset of the set of individually and collectively feasible allocations.*

The remainder of this paper is organized as follows. Section II begins with a discussion of the foundational aspect of the NAE. We then place our model in its historical context, discussing Cournot, Bertrand and Edgeworth. The section concludes with a discussion of the differing implications of the Cournot paradigm and our model for the foundations of competitive analysis. Section III is a critical evaluation of the Cournot paradigm. In section IV we relate our work to Dubey's. Section V contains the formal presentation of the model and statement of the results. Section VI is devoted to a rather important aspect of our model; we model buyers as announcing "acceptance sets" and sets of associated prices rather than simply prices and quantities. Proofs are gathered together in the final section.

*A detailed explanation and economic motivation of the strategic outcome function is deferred until the formal presentation of the model in Section V.

II. THE PRICE-TAKING HYPOTHESIS

The hypothesis that agents take prices as given is viewed by the Noncooperative Approach to Economic Equilibrium as the most questionable of the assumptions underlying Walrasian analysis. The other assumptions, such as perfect information, perfect trust, and the absence of any kind of transactions costs, all relate to the nature of the economic environment. It can be argued that, for certain economic problems, to abstract from the complexities of the environment in which agents act is a sound research strategy. The price-taking hypothesis, on the other hand, is an assumption of a qualitatively different type, being a restriction on the behavior of the fundamental units of economic analysis, the agents themselves. Furthermore, it would appear that the requirement that agents behave naively with respect to price constitutes a violation of the classical dictum that Economic Man is a selfish, rational and sophisticated operator. (See Arrow (2) for an early discussion on this topic.)

The Noncooperative Approach to this issue is to construct models in which agents act strategically. A question of particular interest is: when do the Nash allocations for these models approximate the Walrasian allocations for the associated economies? Our paper demonstrates that the question at issue here is not simply: to what extent can agents affect the prices at which they trade? There is a second question which must also be addressed: even if agents can affect prices a great deal, under what circumstances does it make a difference that they have this capacity? Our model stands as an extreme illustration of the relevance of this second question. Since agents in our model actually determine the prices at which they trade, simply by announcing

those prices, the first question is resolved by construction. It turns out, however, that so long as all markets are "thick," a property of the Nash equilibrium is that all agents choose to announce precisely those prices which, as price-taking agents in the associated Walrasian economy, they would have taken as given in the Walrasian equilibrium. (This "thickness condition" is, of course, "endogenous," in the sense that it relates to the strategies announced by agents and cannot be derived from the primitives which define the model. We do, however, state conditions on these primitives under which the Nash equilibria will always exhibit a property which is essentially equivalent to the property mentioned above.)

In order to motivate our model and, incidentally, to place it in a classical context, we now proceed to trace the progression from Cournot, through Bertrand and Edgeworth, to the present paper, with the aid of a simple, classical example. The following economy is similar to the one considered by Edgeworth (10) in his extension of Bertrand's critique of Cournot. Two proprietors, s_1 and s_2 , can costlessly produce up to $1\frac{1}{2}$ units, and no more, of mineral water of identical quality. There are two identical consumers, b_1 and b_2 , who act as price-takers and are each parameterized by the demand schedule dd' , so that the aggregate demand schedule is DD' (see Figure II.1). The Walrasian price for this economy is zero.

We now associate with this economy a game which is equivalent to Cournot's original model of duopoly. Each proprietor, independently of the other,

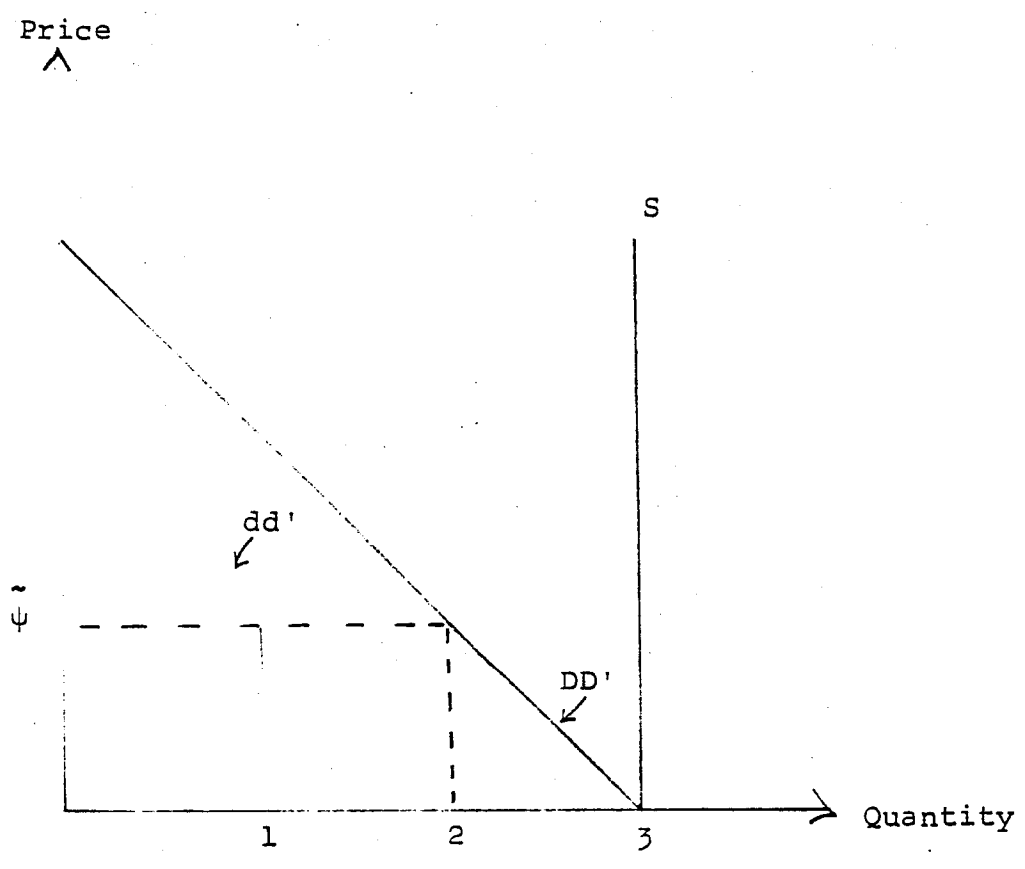


Figure II.1

decides on a level of production; the price which is generated is such that the collective demands of the two consumers exactly equals the aggregate production decision of the two proprietors. A Cournot-Nash equilibrium for this game is a pair of quantities, one for each proprietor, such that, given the action of the other, neither proprietor can increase his profits by changing his quantity action.

The equilibrium for this game is the prototypical illustration of the central theme of the Noncooperative Approach, discussed above. Both proprietors command nonnegligible market shares and, in accordance with the theme, the price associated with the equilibrium exceeds the competitive price of zero. Bertrand's criticism of this result (the text of which appears in our Historical Note (Simon (29))) was, in effect, that a nonzero price was sustainable only because Cournot imposed, axiomatically, the Law of One Price, that is, the requirement that identical commodities must trade at the same price.* Bertrand asserted that this presumption was unreasonable and pointed out the consequences of relaxing it. We paraphrase his argument in the context of our example. Suppose that the Cournot-Nash equilibrium pair is $(1,1)$ and that the equilibrium price is $\tilde{\psi} > 0$. Once the Law of One Price is relaxed, this cannot be an equilibrium, since, for example, s_1 could offer to sell $1\frac{1}{2}$ units at a price infinitesimally below $\tilde{\psi}$.

*The reader will have noted that this is not the usual characterization of the issue between Cournot and Bertrand. We claim, however, that ours is the more historically accurate representation of this issue and document our claim in Simon (29).

Assuming now that s_2 's offer to sell at \tilde{p} still stands, both consumers would choose to purchase from the cheaper supplier, so that s_1 could increase his profits virtually by 50%. Bertrand thus contended that, once price-cutting competition of this kind ceased to be proscribed by the imposition of the Law of One Price, no price above the competitive price could be sustained as a noncooperative equilibrium.

Bertrand's criticism shows that the divergence between the outcomes resulting from strategic and from price-taking behavior can be attributed not to some intrinsic property of strategic behavior, but to strategic behavior in a particular, and an apparently rather arbitrarily imposed, institutional context.

Edgeworth's contribution to this exchange was to draw attention to a problem associated with Bertrand's model under an alternative specification of the production technology. Bertrand, following Cournot, had assumed that unlimited quantities of mineral water could be costlessly produced. Edgeworth considered the case, illustrated by our example, in which production was subject to a capacity constraint.* He demonstrated that under this specification no price, including the competitive price, could be sustained as a noncooperative equilibrium. Following Edgeworth, we consider the following pair of offers: s_1 and s_2 each offer to sell $1\frac{1}{2}$ units of mineral water at a price of zero. For concreteness we will assume

*The reader will observe that we have been faithful in our representation of Cournot and Bertrand, even though our specification of the technology is Edgeworth's, since up to this point we have not had any occasion to refer to the capacity constraint.

that consumer b_1 purchases from proprietor s_1 and that b_2 purchases from s_2 . This cannot be an equilibrium since, for example, s_1 can attain a higher level of profits by offering to sell one unit at price $\tilde{\psi}$. Consumer b_1 would, of course, prefer to purchase from s_2 at a price of zero, but b_2 is already purchasing all that s_2 has to offer. Since b_2 acts as a price-taker he will in fact purchase the quantity which s_2 offers at the price specified. In short, when Edgeworth's modification of Bertrand's model is formalized into a game, no Nash equilibrium in pure strategies exists.

We now demonstrate that one way to resolve this non-existence problem is to model the two sides of the market more symmetrically. We work in the context of a pure exchange economy and model both sellers and buyers as strategic actors, specifying an appropriate mechanism through which trades are effected.* For the game outlined in Section I there exists a Nash equilibrium such that the agents trade at the Walrasian price and receive their Walrasian allocation. The equilibrium list of strategies is: sellers s_1 and s_2 each offer to sell, and buyers b_1 and b_2 each offer to buy, $1\frac{1}{2}$ units of water at a price of zero.** To verify that this is indeed

* We are able to make the transition from a production to a pure exchange context and still maintain the same example for the purposes of exposition, because production in our example is costless. In particular, a proprietor who produces some commodity at zero cost but is subject to a capacity constraint is formally equivalent to an agent in a pure exchange economy who is endowed with a given quantity of that commodity.

** Recall that agents acting as buyers, in fact announce "acceptance sets" and sets of "prices." In this instance they would announce singleton sets.

an equilibrium, we need only consider once again the alternative strategy which established nonexistence in the Edgeworth model: s_1 offers to sell one unit at a price of $\tilde{\psi}$. It is no longer the case that s_1 can obtain higher "profits" by announcing this alternative offer. If he does so, then he will have a zero probability of realizing any sales, since, if water were to be purchased from s_1 (by some firm in the fictional competitive arbitrage sector) and sold back to either of the two buyers, then arbitrage losses at a rate of $\tilde{\psi}$ dollars per unit would be incurred. Hence, no allocation such that s_1 realizes any sales can be arbitrage-profit-maximal. To summarize the essential difference between our model and Edgeworth's, in Edgeworth's model a proprietor was able to take advantage of the naivety of consumers and increase his profits by announcing a price exceeding zero; in our model, on the other hand, the strategic outcome function is constructed in such a way that buyers, by acting strategically, are able to ensure that sellers are "locked into" the competitive price.

The remainder of this section will be devoted to an heuristic explanation of the divergence between our results and those associated with the Cournot paradigm of strategic behavior.* The skeleton of the argument developed below is as follows. We define a construct which we call the "perfect elasticity property" (PEP); we argue that a sufficient condition for the coincidence of the Nash allocations of a market game with the Walrasian allocations of the associated economy is that the Nash equilibrium has the

* This section was motivated by, and is rather closely related to, Ostroy's discussion of two constructs, which are introduced in Ostroy (19): the no surplus condition and the perfectly determinate price equilibrium.

PEP; we then show that, in models based on the Cournot paradigm this property obtains generically if and only if agents are negligible in size, but that, in our model, the Nash equilibria have the PEP under conditions which have nothing to do with the size of agents relative to the market.

The "perfect elasticity property" is a natural extension to a game-theoretic context of the familiar partial equilibrium notion of perfect elasticity. For concreteness, we will explain it in the context of our example and we will ignore the fact that the outcome of our game is a random vector rather than a point. Consider a seller s_i , who, given a Nash equilibrium list of strategies for an arbitrary market game, realizes a sale of a quantity q^* at a price of p^* , and for whom the following two conditions obtain: firstly, there exists no alternative strategy which s_i can announce such that, given the actions of the other players, s_i would realize positive sales at a price exceeding p^* ; secondly, there exists q' strictly greater than q^* such that, by announcing an appropriate strategy, s_i could realize sales of q' at a price less than, but arbitrarily close to p^* . If, given a Nash equilibrium for some market game, both these conditions are satisfied for all sellers and the analogous conditions (defined by reversing the inequalities relating to prices) are satisfied for all buyers, then we say that this equilibrium has the "perfect elasticity property." It will be immediately evident that, at a Nash equilibrium exhibiting the PEP, the predicament of, for example, a seller in this game is analogous to that of a monopolist who is faced with a perfectly elastic demand curve. The first condition is analogous to "perfect elasticity to the left" (dq negative), while the second is analogous to "perfect elasticity to the right" (dq positive).

Clearly, a sufficient condition for the coincidence of the Nash and Walrasian allocations is that the Nash equilibria of the given game have the PEP, since in this case no player can induce a perceptible change in the prices at which he trades by deviating locally from his equilibrium strategy. Thus if the PEP obtains it will be impossible to distinguish between strategic behavior, which takes into consideration the (nonexistent) impact that the agent's actions have on his trading constraints, and "price-taking" behavior, which is premised on the assumption that these constraints are parametric.

We now turn to consider the conditions under which the Nash equilibria of games based on the Cournot paradigm will exhibit the PEP. Games of this kind satisfy what has been called the "aggregation axiom" (the term was coined in Dubey, Mas Colell and Shubik (9)). Paraphrasing, this axiom is satisfied if the means of each of the possible lists of strategies which agents can play constitute the domain of the strategic outcome function. Specifically, strategic behavior in the context of the Cournot paradigm amounts to choosing quantities with a view to manipulating aggregate quantities. Thus in games based on the Cournot paradigm, unless an agent's market share is negligible any finite deviation by that agent from his equilibrium strategy would induce a finite percentage change in some quantity aggregate, which would result, in the generic case when the price mapping is locally injective, in a finite percentage change in market-clearing prices. Clearly, then, a necessary (and in fact sufficient) condition for the Nash equilibria of such games to have, generically, the PEP, is that the market share of each player is negligible.

In our model, on the other hand, the notion of strategic behavior is quite different and so, accordingly, are the conditions under which the Nash equilibria have the PEP. These conditions are that all markets are thick and that all trade on every market is at the same price. Consider the situation of an active seller under these conditions. If the seller were to raise his ask-price by an arbitrarily small amount his sales would fall to zero, since any transaction in which that seller participated would generate negative arbitrage profits. If, on the other hand, he were to lower his ask-price by an arbitrarily small amount he would be able to capture the market shares of all his rival sellers, at least one of whom is guaranteed, by the thickness condition, to exist. While the market share available for him to capture might be small, it must be bounded away from zero. Thus both facets defining the PEP would be satisfied. In short, "two is enough for competition."

III. THE COURNOT PARADIGM

One of the aims of the Noncooperative Approach to Economic Equilibrium is to enrich the Walrasian model by superimposing upon it a stylized representation of the economic institutions of trade. The formal counterparts of these institutions, in game-theoretic models, are the properties of the strategic outcome function and the restrictions defining agents' strategy sets. Two strands within the literature may be distinguished: one is prescriptive in spirit, the other descriptive. The prescriptive strand takes as its starting point certain properties of allocations, typically "desirable" properties such as Pareto efficiency, and investigates the conditions which models must satisfy if their equilibrium allocations are to exhibit these properties. (For an extensive discussion and bibliography of this strand of the literature see Postlewaite-Schmeidler (21) and Schmeidler (25).)

Our concern is not with this, but with the descriptive strand of the literature. Descriptive models endeavor to characterize, in a highly stylized way, some facet of actual economic activity. A valid criterion by which to evaluate a descriptive (but not, of course, a prescriptive) study is the extent to which it succeeds in describing, explaining or predicting some aspect of economic reality. This section is a selective critique of the Cournot paradigm, based on this criterion alone. We have chosen to focus on those particular areas in which our model

has the distinct comparative advantage; we do not claim to be comprehensive.

The first issue we address is the most immediate. The essence of the Cournot paradigm is that agents announce quantities, while prices are determined by "the impersonal forces of the market." While it is possible to find examples of markets which function essentially in this way (agricultural examples are frequently cited), most economists' intuitive impression is that, like quantities, the prices at which trade takes place are set, not by impersonal forces, but by some of the economic agents who actually participate in the market in question. Clearly, our model accords more closely with this intuition than does the Cournot paradigm.

The second issue to be addressed relates to a dissonance which exists between the role which prices play in the Cournot paradigm and our intuitive conception of the actual function of prices. In models based on the Cournot paradigm, the function of prices is to clear markets. In Cournot's original model, for example, price is defined by the condition that demand equals the aggregate quantity supplied. The role of prices, then, is to render consistent the decentralized actions of each of the economic agents in the model. In short, prices can be thought of as fulfilling a "system function."

This conception of prices is inconsistent with our intuitive understanding of the role which prices play in actual economic systems. The natural view of prices is that they are the

vehicles by which individual agents compete with each other for market-shares. This is certainly the view which predominates in the less abstract literatures of economic theory, such as Industrial Organization, and which is formalized in the model which we present below. These two conceptions of the function of prices cannot easily be reconciled; it is difficult to conceive of a model in which prices both serve the "system," by clearing markets, and serve the individual, as the vehicle of competition for market-share.

The remainder of this section is devoted to a discussion of two features which are embedded into the institutional structure of the Cournot paradigm: the "Law of One Price" and the "law" that supply must equal demand. In our model, by contrast, while neither is imposed a priori, both features are exhibited by the equilibria of our game. Rather than assuming them from the start, we derive both as results of the analysis, as consequences of more fundamental, behavioral assumptions, which are imposed on the primitives of the analysis.

The Law of One Price stipulates that commodities which are identical must always trade at the same price. In game-theoretic models based on the Cournot paradigm, this Law has the status of an axiom: it is a property, not just of the outcomes which are generated by equilibrium strategies, but of every conceivable outcome. In these models, the Law of One Price has "a life of its own," as it were, independent of the behavioral assumptions which define the strategic actors.

Our quarrel with this aspect of the Cournot paradigm is based on the following reasoning. There are compelling economic arguments which suggest that the Law of One Price should be a property of equilibrium allocations. It is difficult to conceive, however, of any justification, based either on empirical grounds or on the demands of logical consistency, for the a priori imposition of this law as a part of the institutional environment. Furthermore, by insisting that identical goods must always trade at the same prices, the Cournot paradigm suppresses, by fiat, a potent "competitive" force, which is virtually ubiquitous in actual economic situations: the capacity of agents to "compete by price." (Clearly, this and our earlier discussion about whether prices serve the system or the individual are closely related.) In our model, on the other hand, this competitive force is unleashed and the Law of One Price is derived as a result rather than imposed from the start.

Expressed more abstractly, our criticism is a rather basic game-theoretic point. Formally, the difference between the Cournot and Bertrand "games" is that, while Cournot restricts the set of possible strategy profiles to those such that all trade takes place at the same price, Bertrand does not. Cournot, it would appear, excludes all other strategy profiles (i.e., yielding trade at more than one price), on the grounds that no conceivable configurations of agents' preferences exist for which such profiles could be equilibria. Our observation is, simply, that this is not a valid criterion upon which to reduce

a game: profiles are not inessential to the structure of a game merely because they can never be equilibria. A strategy profile (s_1, s_2, \dots, s_n) will not be an equilibrium if, say, the first agent can obtain a higher payoff by announcing s_1' rather than s_1 . The alternative profile (s_1', s_2, \dots, s_n) need not itself be an equilibrium. Thus profiles which can never be equilibria may nevertheless be essential to the structure of a game, because they constitute the alternatives which establish that other profiles are not equilibria.

We raise a parallel objection to the second institutional feature of the Cournot paradigm: the axiomatic equality of supply with demand, that is, markets clear, whether or not outcomes are generated by equilibrium strategies. Such phenomena as queues, rationing and inventory accumulation are precluded, again by fiat. These phenomena are not merely of considerable empirical significance as manifestations of disequilibrium. It could be argued that it is precisely these manifestations that constitute the signals which induce agents to modify their behavior, so that the Cournot paradigm in fact suppresses the very phenomena which are central to the operation of actual, decentralized equilibrating processes. In our model, rationing is associated with "disequilibrium" in a very natural, if stylized, way: strategies which generate outcomes such that rationing occurs cannot be equilibrium strategies, because an agent on the long side of the market will, in general, have an incentive to deviate from his original action, in order to secure for himself, at the expense of his rivals, a larger ration.

IV. RELATED WORK

For an extensive survey of the related contemporary literature, the reader is referred to Simon (27). That survey is divided into two parts. We first mention several examples of models based on the Cournot paradigm: Dubey (7), Dubey-Mas-Colell and Shubik (9), Hart (12), Novshek (17), Novshek-Sonnenschein (18), Ostroy (19), Roberts-Postlewaite (23), Postlewaite (20), Postlewaite-Schmeidler (22) and Shapley-Shubik (26). The survey concludes with a discussion of five papers whose results bear some resemblance to ours: Schmeidler (25), Hurwicz (14), Grossman (11), Dubey (8) and Wilson (31).

We report here only on the study most closely related to ours, by Dubey (8). Like us, Dubey presents a model in which both buyers and sellers act strategically, announcing both prices and quantities. While his main result is equivalent to ours, there are two important respects in which our models are substantially different. The first relates to the designs of our respective strategic outcome functions. If quantity announcements are such that a market fails to clear, then, in Dubey's model, agents are rationed proportionately. A rationed agent can always secure for himself a larger share of the commodities in short supply, simply by appropriately magnifying his quantity announcements. He has no need to

"compete-by-price." Our game, on the other hand, is stochastic: in particular, when there are several agents on the long side of some market, all announcing the "marginal" price, the market share of each of them is a random variable. An agent who announces more than he wishes to trade risks actually realizing his announced, rather than his desired trade. To guarantee himself a given increase in market share, a seller (buyer) must shave (increase) his ask-price (bid-price). We feel that, in this respect our model can be reconciled more readily with intuitive conceptions of the way in which economic agents actually do "compete-by-price." Certainly our approach is more faithful to the spirit of Bertrand. The second distinction relates to our respective treatments of buyers: Dubey's buyers announce quantity- and price-vectors, while our buyers announce "acceptance sets" of quantities and sets of associated bid-prices. We argue elsewhere¹¹ that the former approach, while more natural at first sight, fails to extend adequately the spirit of "price competition a la Bertrand" to a general equilibrium context, because it precludes the possibility of "price competition across markets." We mention only two of the remaining distinctions between our models. In Dubey's, trades are made at the prices specified by the buyers, so that the "surplus" arising from the difference between bid- and ask-prices is received by the sellers; in our model, this surplus is earned by the fictional arbitrageurs. Dubey's traders have access to unlimited funds, at zero interest rates, from which to finance trades; in our model, trades must be financed out of initial money holdings.

V. THE FORMAL PRESENTATION

This section is organized into six parts. The first three subsections develop the model and correspond to the three main components of a Strategic Market Game: a pure exchange economy, which is the primitive of our analysis; the strategy sets; and the strategic outcome function. In the fourth subsection we discuss equilibrium notions. The main result is presented in the fifth subsection. The last is devoted to a corollary of the main result. Before presenting the model, however, we draw attention to a difference between the pure exchange economy of our Market Game and the conventional Walrasian model of pure exchange.

Our objective is to compare the outcomes which result from strategic behavior (the Nash allocations of our Market Game) to those which result from "price-taking," or "competitive behavior" (the equilibrium allocations for the underlying pure exchange economy). The specification of this exchange economy, however, involves a modification of the usual notion of "price-taking behavior." Specifically, agents in this economy are subject to a "liquidity constraint," in addition to the conventional Walrasian constraints. We add this additional constraint because a corresponding liquidity constraint is built into the structure of the Game itself; if we are to compare outcomes, then the constraints which agents face in the Game must be matched by constraints which they face as "price-takers."

The liquidity constraint which binds agents in the Game is that they can offer no more than their initial endowments of money in exchange for purchases. We invoke this restriction in order to sidestep the following difficulty: in our model, an agent could be rationed as a seller yet realize his original purchase plans; had he planned, in the Walrasian spirit, to finance his purchases with the revenue from his sales, he could find himself required to purchase a commodity bundle for which, ex post, he could not afford to pay. The matching constraint for the exchange economy is known in monetary theory as the "Clower expenditure constraint" (see Clower (4)): given a price vector, the value of an agent's purchases cannot exceed the value of his endowment of money. We define a corresponding equilibrium notion: a constrained Walrasian equilibrium is a price-allocation pair such that all agents maximize utility over their expenditure constrained budget sets and net trades sum to zero.

V.1 The Primitive: A Pure Exchange Economy.

An exchange economy, E , is an ordered pair, $((u_\alpha)_{\alpha \in A}, \langle w_{\alpha 0}, w_\alpha \rangle_{\alpha \in A})$. A is a finite, nonempty set of agents. There is a finite set, $L = \{1, \dots, \ell\}$, of non-monetary commodities. The 0-th good is commodity money, which is universally desired, and which is both a medium of exchange and unit of account, its price being fixed at unity. Agents are characterized by utility functions, representing preferences over lotteries. We first assume that these are "von Neumann-Morgenstern utility functions": the

utility of a lottery is just the expected utility of its prizes (see Varian (30), pp. 105-7). Having made this assumption, we can, in the usual way, completely characterize agents' preferences over lotteries by characterizing their utilities over the prizes themselves: in our model, these prizes are net-trade vectors. Formally, for each agent $\alpha \in A$, $u_\alpha: R \times R^l \rightarrow R$ satisfies the following assumptions:*

$$A1: u_\alpha \text{ is continuous} \quad (5.1)$$

$$A2: u_\alpha \text{ is quasi-concave}^{**} \quad (5.2)$$

$$A3: u_\alpha \text{ is monotonic.***} \quad (5.3)$$

[We emphasize, again, that all agents derive utility from the commodity money.] The vector $\langle w_{\alpha 0}, w_\alpha \rangle \in R_+ \times R_+^l$ is the endowment of the α -th agent. We assume that each agent is endowed with a positive quantity of money and that the aggregate endowment of each nonmonetary commodity is positive, that is:

$$A4: w_{\alpha 0} > 0, \quad \text{for each } \alpha \in A \quad (5.4)$$

$$A5: \sum_{\alpha} w_\alpha \gg 0 \quad (5.5)$$

* R , R_+ and R_- denote, respectively, the real line, the non-negative real line and the non-positive real lines. R^n , R_+^n , R_-^n are defined analogously, for n-dimensional Euclidean space.

** A function $f: X \rightarrow R$ is quasi-concave if, for every $r \in R$, $\{x \in X: f(x) \geq r\}$ is a convex set.

*** A function $f: X \rightarrow R$ is monotonic if, for all $x, x' \in X$, $x \geq x'$, $x \neq x' \Rightarrow f(x) > f(x')$.

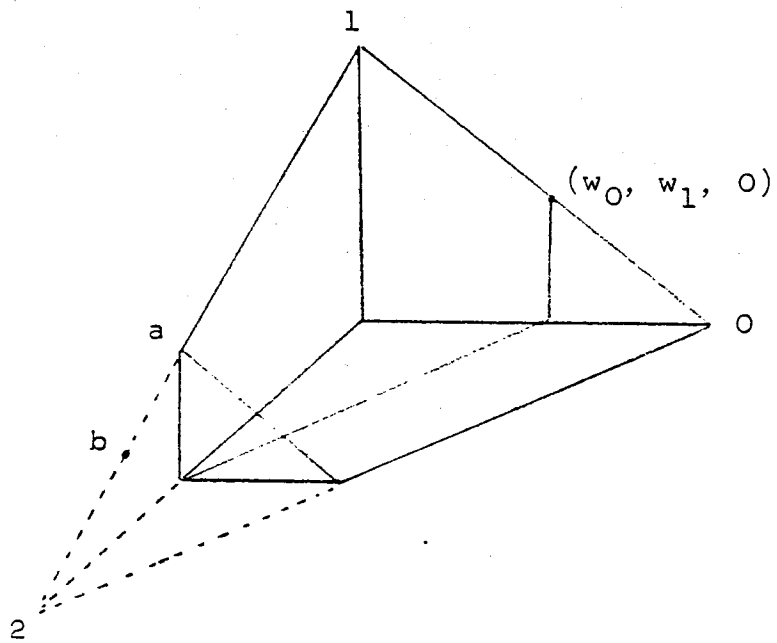


Figure V.1

[We maintain the following convention for vector orderings: for any vectors $x, y \in R^n$, we define the symbols \geq , $>$, \gg by: $x \geq y \Leftrightarrow x_k \geq y_k$, for $k = 1, \dots, n$; $x > y \Leftrightarrow x \geq y$ and $x \neq y$; $x \gg y \Leftrightarrow x_k > y_k$, for $k = 1, \dots, n$.]

The difference between the expenditure constrained and the Walrasian budget sets is illustrated in Figure V.1. The solid lines delineate the former, indicating that the effect of the expenditure constraint is to truncate the Walrasian budget set.

The agent whose budget set is illustrated in the figure is endowed with positive quantities of money and the first commodity. The point "a" represents this agent's expenditure of his entire money endowment on the second commodity. Net trades such as "b," involving the financing of additional purchases of the second commodity by sales of the first, are attainable for a Walrasian agent, but not for an agent in our "constrained economy."

Formally, we define, for each α , the constrained budget correspondence, $\beta_\alpha: R_+^l \rightarrow R^l$, by

$$\beta_\alpha(p) = \{x \in R^l: p \cdot x^+ \leq w_{\alpha 0}; w_\alpha + x \geq 0\} \quad (5.6)$$

[Recall that the price of the zero-th good is fixed at unity. The symbols x^+ and x^- refer, respectively, to the nonnegative and the nonpositive components of the vector x : given $x \in R^n$, $x^+ = (x_1^+, \dots, x_n^+)$, where $x_k^+ = \max(x_k, 0)$, $k = 1, \dots, n$; similarly $x^- = (x_1^-, \dots, x_n^-)$, where $x_k^- = \min(x_k, 0)$, $k = 1, \dots, n$; "." denotes the inner product, $x \cdot y = \sum_{k=1}^n x_k y_k$.] We define constrained demand for α by

$$\xi_{\alpha}(p) \cdot \{x \in \beta_{\alpha}(p) : x' \in \beta_{\alpha}(p) \Rightarrow$$

$$u_{\alpha}(\langle -p \cdot x, x \rangle) \geq u_{\alpha}(\langle p \cdot x', x' \rangle)\} \quad (5.7)$$

Finally, a price-allocation pair (\tilde{p}, \tilde{x}) , where $\tilde{p} \in \mathbb{R}_+^l$ and $\tilde{x} = (\tilde{x}_{\alpha})_{\alpha \in A}$, $\tilde{x}_{\alpha} \in \mathbb{R}^l$, is said to be a constrained Walrasian pair for E if

$$\tilde{p} \gg 0 \quad (5.8a)$$

$$\tilde{x}_{\alpha} \in \xi_{\alpha}(\tilde{p}) \quad \text{for all } \alpha \in A \quad (5.8b)$$

$$\sum_{\alpha} \tilde{x}_{\alpha} = 0 \quad (5.8c)$$

Proposition: Given A1-A5, a constrained Walrasian pair for E exists.

V.2 Strategy Sets

Agents in our Game act simultaneously as sellers and as buyers. As a seller, an agent announces a strictly positive

vector of ask-prices,* $p \in \overset{\circ}{R}_+^{\ell}$,** and a vector of sale-offers of nonmonetary commodities, $z \in R_-^{\ell}$. As a buyer, an agent specifies a set of alternative purchase offers for nonmonetary "commodity bundles," together with a set of "per-unit prices," one for each of the alternative commodity bundles. A commodity bundle is a non-negative, ℓ -dimensional vector. Commodity bundles which lie in the $(\ell-1)$ dimensional unit simplex, $\Delta^{\ell-1}$, are called package deals (where, for $n \in N$,

$$\Delta^n = \{x \in R_+^{n+1} : \sum_{k=1}^{n+1} x_k = 1\} \quad (5.9)$$

Any commodity bundle can be uniquely represented as a scalar multiple of some package deal. Thus the package deals may serve as the units of measurement for commodity bundles. Each buyer specifies two functions, ζ and ρ , both mapping $\Delta^{\ell-1}$ to R_+ . ζ defines an "acceptance set" of alternative commodity bundles, that is, a set of alternative scalars, one

* This assumption is purely for convenience, obviating the need to deal with messy, but uninteresting, special cases. Since we have assumed monotonicity, constrained Walrasian equilibrium prices are always strictly positive, and we could easily show that Nash equilibrium prices must, also, always be strictly positive.

$$\overset{\circ}{R}_+^{\ell} = \text{interior of } R_+^{\ell} .$$

for each possible package deal. ρ specifies a set of alternative per-unit bid prices, one for each possible package deal. A strategy for α , $s_\alpha = (p_\alpha, z_\alpha, \rho_\alpha, \zeta_\alpha)$ has the following interpretation: α offers to sell the vector, $-z_\alpha$, at prices, p_α , in exchange for money. He simultaneously bids to buy any one of the commodity bundles in the set $\{\zeta_\alpha(\eta)\eta : \eta \in \Delta^{\ell-1}\}$, paying money in exchange for the package deal, η , at the per unit rate of $\rho_\alpha(\eta)$ (that is, paying a total of $\rho_\alpha \zeta_\alpha(\eta) = \rho_\alpha(\eta)\zeta_\alpha(\eta)$ to receive the commodity bundle $\zeta_\alpha(\eta)\eta$). Under certain restrictions on ρ and ζ , each strategy $s = (p, z, \rho, \zeta)$ defines (a subset of) a "nice" "indifference surface" in $R \times R^\ell$,* that is

$$\{ \langle -(pz + \rho\zeta(\eta)), z + \zeta(\eta)\eta \rangle : \eta \in \Delta^{\ell-1} \} \quad (5.10)$$

will be a subset of the boundary of some convex, monotone set in $R \times R^\ell$. The restrictions we impose are that ζ be a convex function and that the product function, $\rho\zeta$, be a constant function. Denote by S the set of all possible strategies:

$$S = \{(p, z, \rho, \zeta) : p \in R_+^\ell ; z \in R_-^\ell \quad (5.11a)$$

$$\zeta : \Delta^{\ell-1} \rightarrow R_+ ; \rho : \Delta^{\ell-1} \rightarrow R_+ \quad (5.11b)$$

$$\zeta \text{ is convex} \quad (5.11c)$$

$$\rho\zeta(\eta) = \rho\zeta(\eta'), \text{ for all } \eta, \eta' \in \Delta^{\ell-1} \quad (5.11d)$$

A strategy is said to be allowable for a given agent if it is an element of S and satisfies certain additional

*I.e., a level set of a member of the class of utility functions satisfying (A1)-(A3).

requirements: no offer to sell a commodity can exceed the agent's endowment of that commodity (this restriction applies also to the monetary commodity in a sense made precise below (5.13b)); also an agent cannot offer to sell and bid to buy on the same market [this requirement greatly simplifies the exposition but is not essential to the argument]. The strategy set for α is the set of allowable strategies for α :

$$S_\alpha = \{(p, z, \rho, \zeta) \in \mathcal{S} :$$

$$z + w_\alpha \geq 0 \quad (5.13a)$$

$$\rho \zeta(\eta) \leq w_{\alpha 0}, \text{ for all } \eta \in \Delta^{\ell-1} \quad (5.13b)$$

$$\eta \cdot z < 0 \Rightarrow \zeta(\eta) = 0 \} . \quad (5.13c)$$

(5.13b) is the condition that a buyer cannot offer to pay more than his total endowment of money for any commodity bundle.

(5.13c) formalizes the proscription on buying and selling in the same market. A selection is a #A-dimensional list of allowable strategies. Let S denote the set of all selections, that is

$$S = \prod_{\alpha \in A} S_\alpha \quad (5.14)$$

An example will help to clarify the preceding exposition. Suppose that $\ell = 2$, the commodities being "gin" and "tonic." The package deal $(\frac{1}{2}, \frac{1}{2})$ is a "gin-and-tonic," mixed by adding a $\frac{1}{2}$ -unit of gin to a $\frac{1}{2}$ -unit of tonic. Suppose that a buyer α 's action is $(\rho_\alpha, \zeta_\alpha)$, defined by, for $\eta \in [0, 1]$, $\zeta_\alpha(\eta, 1-\eta) = \sqrt{4/\eta(1-\eta)}$ and $\rho_\alpha(\eta, 1-\eta) = 8/\zeta_\alpha(\eta, 1-\eta)$ (see Figure V.2). [The reciprocal of zero is taken to equal infinity.] We interpret α 's action to include, among other alternatives, a bid to buy

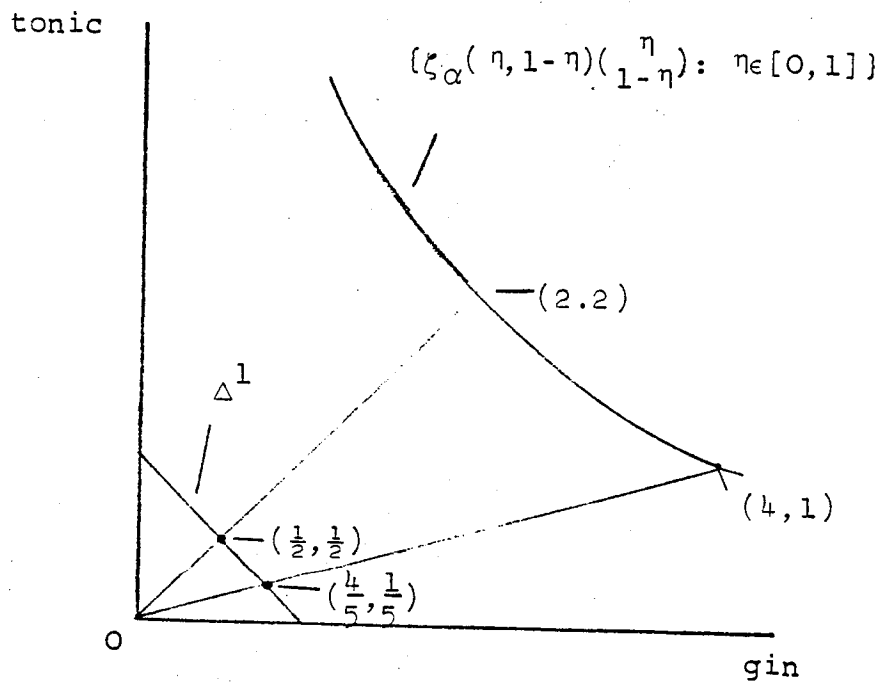


Figure V.2

either 4 units of $(\frac{1}{2}, \frac{1}{2})$ at a per-unit rate of \$2, or 5 units of $(\frac{4}{5}, \frac{1}{5})$ at a rate of \$1.6 per unit. We could construct utility functions for α (defined over net trades), for example $u_\alpha = (10 - x_0)x_1x_2$, such that the action $(\rho_\alpha, \zeta_\alpha)$ would accurately reflect α 's preferences over different mixes of gin and tonic.

The set

$$\{(-\rho_\alpha \zeta_\alpha(\eta), \zeta(\eta, 1-\eta)\eta \zeta(\eta, 1-\eta)(1-\eta)) : \eta \in [0, 1]\}$$

is a subset of the level set of u_α : $\{(x_0, x_1, x_2) \in \mathbb{R} \times \mathbb{R}^2 : u_\alpha(x_0, x_1, x_2) = 8\}$.

We will argue below, in Section VI, that there is a compelling reason for modelling buyers' actions in this way. The following discussion is intended merely to motivate our approach. In a monetary, as distinct from a barter economy, a significant asymmetry arises between the activities of buying and of selling. In general, agents can offer to sell only those commodities which they themselves own in the first place. Buyers, on the other hand, can bid to buy any commodity. The following examples illustrate this difference. Ignoring short sales, an investor on the stock market can sell only those stocks which are included in his initial portfolio; he can, however, buy any stock which is listed on the exchange. Similarly, a participant in the housing (used car) market can sell only the one house (car) he owns; he can, however, choose between any of the houses (cars) which are on the market. At least in these situations, the opportunities available to a buyer are much richer than those available to a seller. This difference is well captured by the asymmetry we introduce into our strategy sets.

V.3 The Strategic Outcome Function

We begin this subsection with the formal presentation and then offer an interpretation. A strategic outcome function (SOF) maps selections into outcomes. Before defining this mapping, we need some preliminary definitions. [We will be defining some auxiliary mappings, whose domains, for technical reasons, will be larger than the domain of our SOF.] An allocation $\theta = (\theta_\alpha)_{\alpha \in A}$, $\theta_\alpha \in \mathbb{R}^L$ is a $\#A$ -dimensional list of nonmonetary net-trade vectors. Given a list of strategies, $s \in \mathcal{S}^{\#A}$, we define an allocation to be s-allowable if (a) no agent sells more a commodity than he offers to sell, and (b) the total purchases of any commodity do not exceed the total sales of that commodity.

Formally, we define the allocation correspondence, $\Theta : \mathcal{S}^{\#A} \rightarrow (\mathbb{R}^L)^{\#A}$, which associates to each $\#A$ -dimensional list of strategies, s , the set of s -allowable allocations. Thus, for each $s \in \mathcal{S}^{\#A}$:

$$\Theta(s) = \{ \theta \in (\mathbb{R}^L)^{\#A} : \theta_\alpha \geq z_\alpha \} \quad (5.15a)$$

$$\left. \sum_\alpha \theta_\alpha \leq 0 \right\} \quad (5.15b)$$

It will have been observed that an allocation may be s -allowable even though agents receive more of some package deal than they bid to buy. We include such allocations for technical convenience (it is convenient to work with a convex-valued allocation correspondence) but, as will become clear below, we specify that agents actually pay for no more of a package deal

than they bid to buy. We are, in this way, equipping our SOF with a special kind of "free disposal capability": the system can "throw away" resources by giving them away, gratis, to agents. Since agents' utilities are monotonic (A3), this assumption is quite innocuous.

Given $s \in \mathcal{S}^{\#A}$, we associate to each s -allowable allocation, $\theta \in \Theta(s)$, a scalar $\pi(\theta, s)$, which we call the arbitrage profit generated from θ by s . $\pi(\theta, s)$ can be interpreted as the aggregate value of the commodity bundles which agents purchase (evaluated at the per unit rates which the agents specify) less the aggregate cost of the resources which make up these commodity bundles (evaluated at the prices which the sellers of these resources specify).

The rule by which the value of a commodity bundle is calculated is somewhat involved. The first step is to define a "normalization function," which assigns to each commodity bundle the package deal of which it is a scalar multiple. Formally, define $\eta: R_+^{\ell} \rightarrow \Delta^{\ell-1}$ by, for $x \in R_+^{\ell}$,

$$\eta(x) = \begin{cases} (1/\ell, \dots, 1/\ell) & x=0 \\ x/\|x\|_1 & x>0 \end{cases} \quad (5.16)$$

$\|\cdot\|_1$ is the ℓ_1 -norm: given $y \in R^n$, $\|y\|_1 = \sum_{k=1}^n |y_k|$.

Agent α 's purchase of the commodity bundle, θ_{α}^+ can be interpreted as a purchase of $\|\theta_{\alpha}^+\|_1$ units of the package deal $\eta(\theta_{\alpha}^+)$.

If the quantity, $\|\theta_\alpha^+\|_1$, of $\eta(\theta_\alpha^+)$ which α is allocated does not exceed the quantity, $\zeta_\alpha(\eta(\theta_\alpha^+))$, of $\eta(\theta_\alpha^+)$ which α bid to buy, then the value of θ_α^+ is just the quantity multiplied by the per-unit rate, i.e., $\rho_\alpha(\eta(\theta_\alpha^+))\|\theta_\alpha^+\|_1$. If $\|\theta_\alpha^+\|_1$ exceeds $\zeta_\alpha(\eta(\theta_\alpha^+))$, then the value of θ_α^+ equals $\rho_\alpha\zeta_\alpha(\eta(\theta_\alpha^+))$, that is, α receives the excess of $\eta(\theta_\alpha^+)$ over $\zeta_\alpha(\eta(\theta_\alpha^+))$ at no charge.

It is convenient to condense this rule into a "valuation function," $v(\cdot, \cdot)$, which, given a strategy, assigns a value to each commodity bundle. Thus for $s_\alpha \in \mathcal{S}$ and $\theta \in \theta(s)$,

$$v(\theta_\alpha^+, s_\alpha) = \rho_\alpha(\eta(\theta_\alpha^+)) \min \left[\|\theta_\alpha^+\|_1, \zeta_\alpha(\eta(\theta_\alpha^+)) \right]. \quad (5.17)$$

To illustrate, we refer again to the gin-tonic example above.

The value of the commodity bundle $(1\frac{1}{2}, 1\frac{1}{2}) = \rho_\alpha(\frac{1}{2}, \frac{1}{2}) \|(1\frac{1}{2}, 1\frac{1}{2})\|_1 = 2 \times 3 = \6 ; the value of $(3, 3)$, on the other hand, is $\$8$, since $\rho_\alpha(\frac{1}{2}, \frac{1}{2}) \|(3, 3)\|_1 = 12 > \rho_\alpha(\frac{1}{2}, \frac{1}{2})\zeta_\alpha(\frac{1}{2}, \frac{1}{2}) = 8$.

If the α -th agent announces s_α and is allocated the nonmonetary net trade vector, θ_α , then he earns $-p_\alpha \cdot \theta_\alpha^-$ and must pay $v(\theta_\alpha^+, s_\alpha)$. We can express an agent's net receipt of money as a real valued function from strategies and nonmonetary net trades. Thus, for $s_\alpha \in \mathcal{S}$ and $\theta \in \theta(s)$,

$$\theta_0(\theta_\alpha, s_\alpha) = - \left[v(\theta_\alpha^+, s_\alpha) + p_\alpha \cdot \theta_\alpha^- \right]. \quad (5.18)$$

We are now in a position to define the arbitrage profit function, $\pi: (R^l \times \mathcal{S})^{\#A} \rightarrow R$ by, for all $s \in \mathcal{S}^{\#A}$ and $\theta \in \theta(s)$,

$$\pi(\theta, s) = - \sum_\alpha \theta_0(\theta_\alpha, s_\alpha) \quad (5.19)$$

We next define the correspondence $\Pi : \mathcal{S}^{\#A} \rightarrow (R^l)^{\#A}$, which associates to each list of strategies the arbitrage profit-maximal set of allocations. For each $s \in \mathcal{S}^{\#A}$,

$$\Pi(s) = \{ \theta \in \Theta(s) : \theta \text{ is } \pi\text{-maximal on } \Theta(s) \} .^* \quad (5.20)$$

Finally, we define the correspondence $F : \mathcal{S}^{\#A} \rightarrow (R^l)^{\#A}$ by, for all $s \in \mathcal{S}^{\#A}$,

$$F(s) = \{ \theta \in \Pi(s) : \theta \text{ is } \|\cdot\|_1\text{-maximal on } \Pi(s) \} . \quad (5.21)$$

We can now define the strategic outcome correspondence for our game. Given a selection $s \in S$, the outcome of our game is a random vector, $f(s) = (f_\alpha(s))_{\alpha \in A}$, the support of whose distribution is $F(s)$: that is, $f(s)$ is a measurable function from some measure space into $\Theta(s)$; let the probability measure $\mu(s)$ be the distribution of $f(s)$; then $F(s) \subset T$, for all closed sets $T \subset \Theta(s)$ such that $\mu(s)(T) = 1$. Thus our SOF assigns to each selection, $s \in S$, a random vector whose range is the volume-maximal subset of the profit-maximal subset of the set of s -allowable allocations.

Before suggesting an interpretation of the SOF, we digress with a methodological comment about Strategic Market Games. By appropriately specifying agents' strategy sets, it is possible to analyze, explicitly, one or two aspects of some economic process. There will be many other details which will require attention, if the Market Game is to be a logically complete representation of the particular market process.

*Given a function $f: X \rightarrow R$, $x \in X$ is f -maximal on X if $x' \in X \Rightarrow f(x) \geq f(x')$.

An important role of the SOF in the analysis is to take care of these complications, hopefully in an intuitively reasonable way. The SOF, in short, is a "black box" which "ties up the loose ends" of the analysis, it is a mechanism by which selections are transformed, via some unmodelled economic process, into outcomes. We may take Cournot's classic study (4) as an illustration. Cournot's concern was to model "competition between producers." In his model producers announced quantities and received profits. The problem of which quantity announcement to make was carefully modelled; the process by which quantity announcements were transformed into profits was not explicitly analyzed. Cournot, in fact, blackboxed the entire consumption sector of the economy, as well, of course, as the process of price formation.

Our model, on the other hand, is quite explicit about both the behavior of consumers and the process of price formation. What we "black-box" is the process of intermediation, by which sale-offers are coalesced into package deals, which are then transmitted to buyers. We can, however, associate with our black-box an institutional story which, while not completely rigorous, does provide some economic rationale for the specification of our SOF. The institution which we invent is a hypothetical arbitrage sector, whose business is to buy commodities from sellers (paying the sellers, in money, at the prices they specify) and sell back commodity bundles to buyers (receiving

money from the buyers at the per unit rates which they implicitly specify).

A selection, $s \in \mathcal{S}^{\#A}$, defines for the arbitrage sector an "industry production possibility set," which is just the set of s -allowable allocations, together with an array of "factor" and "product" prices. The "factor prices" are the ask-prices announced by sellers; the "product prices" are the per unit rates for the alternative package deals, which buyers implicitly specify. The arbitrage profit correspondence, Π , associates to each selection those allocations which maximize industry profits over the industry production possibility set.

It will be recalled that our arbitrage sector selects from only a subset of the profit-maximal allocations, that is, those allocations which are volume-maximal on the profit-maximal set. We need this additional, and rather counter-intuitive, requirement because there are many selections, in particular, the Nash selections, for which arbitrage profits are at most zero, so that allocations such that minimal trading occurs will be profit maximal along with, say, the constrained Walrasian allocations.

By restricting the range of the outcome, $f(s)$, to $F(s)$, rather than $\Pi(s)$, we ensure that allocations which are dominated by the criterion of volume of trade are never realized. [It is not easy to provide a fully convincing rationale for volume-maximizing behavior. We could, however, have finessed the problem in the following way: consider the following sequence

of games, $(\Gamma(\epsilon))$, where $\epsilon > 0$ tends to zero. The game $\Gamma(\epsilon)$ is identical to ours except that the arbitrage firms act as " ϵ -satisficers": they will arbitrage a transaction if and only if marginal arbitrage profits accrue at a rate of at least ϵ per unit. For $\Gamma(\epsilon)$, $\epsilon > 0$, the second stage of the arbitrageurs' maximization program can be shown to be redundant: any element of the arbitrage-profit maximal set of allocations is a volume-maximal element of that set. The equilibria of our game can be thought of as the limit, as ϵ approaches zero, of the equilibrium of the sequence of games, $(\Gamma(\epsilon))$. In the Nash equilibria for $\Gamma(\epsilon)$, the per unit rates, which buyers implicitly specify for the commodity bundles they receive, will exceed the per unit costs of these bundles by ϵ . It can be shown that, under the conditions of the Theorem (part (ii) below, the Nash allocations for $\Gamma(\epsilon)$ converge, as ϵ tends to zero, to constrained Walrasian allocations.]

To conclude our institutional story, the hypothetical arbitrage firms are owned by the agents, to whom arbitrage profits are distributed as dividend checks. Profits are thus ultimately "consumed" by the traders themselves. We assume, informally, that agents act myopically in the sense that they do not perceive the relationship between their actions and their dividend earnings: agents do not take into consideration what arbitrage profits they will receive when they choose which strategies to announce. Since arbitrage profits from Nash selections are zero, this assumption is rather harmless.]

V.4 The Payoff Function and Equilibrium Notions

The payoff function, $P : S \rightarrow R^{\#A}$ associates to each selection a vector of numerical payoffs, one for each agent. The α -th agent's payoff, given a selection, s , is simply α 's expected utility from the lottery, $f_\alpha(s)$, which is the outcome of the game:

$$P_\alpha(s) = Eu_\alpha(\langle \theta_0(f_\alpha(s), s_\alpha), f_\alpha(s) \rangle). \quad (5.22)$$

Let $N(E)$ denote the set of Nash selections for E , that is:

$$N(E) = \{s \in S : \text{for all } \alpha \in A \text{ and } s' \in S_\alpha, P_\alpha(s) \geq P_\alpha(s/s'_\alpha)\} \quad (5.23)$$

[The following notation will be utilized extensively: given $x = (x_1, \dots, x_n)$ define $x/y_k = (x_1, \dots, x_{k-1}, y_k, x_{k+1}, \dots, x_n)$. Further, suppose that $x_k = (x_{k1}, \dots, x_{km})$ then define $x/y_{kh} = x/y_k$ where $y_k = (x_{k1}, \dots, x_{k,h-1}, y_{kh}, x_{k,h+1}, \dots, x_{km})$. $x/y_{k_1 h_1}, y_{k_2 h_2}$ and $x/(y_{k_q h_q})_{q \in Q}$ are defined analogously.]

We now define precisely our benchmark notion of "competitive behavior." Given a selection, s , a trader is said to be s-active on a market if he has a positive probability of trading on that market. Formally, given $s \in S$, α is said to be an s-active buyer of λ (s-active seller of λ) if there exists $\theta \in F(s)$ such that $\theta_{\alpha\lambda} > 0$ ($\theta_{\alpha\lambda} < \theta$). [Since $F(s)$ is the support of $f(s)$, the existence of just one such allocation is sufficient to ensure that α has a positive probability of trading.] A selection, s , is said to be constrained Walrasian (i.e., competitive) if all s-active traders announce constrained Walrasian prices and if these prices, together with any possible

realization of the outcome, $f(s)$, constitute a constrained Walrasian pair for E . Formally, we define the set of constrained Walrasian selections for E , $W(E)$, by

$$W(E) = \{s \in S: \text{there exists } \tilde{p} \in R_+^{\ell}, \tilde{p} \gg 0, \text{ s.t., for all } \theta \in F(s)$$

$$\theta_{\alpha\lambda} < 0 \Rightarrow p_{\alpha\lambda} = \tilde{p}_{\lambda} \text{ for all } \lambda \in L, \alpha \in A \quad (5.24a)$$

$$\theta_{\alpha}^+ > 0 \Rightarrow p_{\alpha}(\eta(\theta_{\alpha}^+)) = \tilde{p} \cdot \eta(\theta_{\alpha}^+) \text{ for all } \alpha \in A \quad (5.24b)$$

$$(\tilde{p}, \theta) \text{ is a constrained Walrasian pair for } E \}. \quad (5.24c)$$

Condition (5.24b) states that the per unit price which α bids, $p_{\alpha}(\eta(\theta_{\alpha}^+))$, for the commodity bundle he receives, θ_{α}^+ , equals the cost, evaluated at prices \tilde{p} , of the package deal, $\eta(\theta_{\alpha}^+)$.

Note that from the Proposition above, we know that $W(E)$ is nonempty, if E satisfies A1-A5.

Our main result is the Theorem below, which states that, firstly, constrained Walrasian selections are also Nash selections, and, secondly, a Nash selection, s , "such that all markets are s -thick" is a constrained Walrasian selection. Given $s \in S$, a market λ is said to be s -thick if there are at least two s -active sellers of λ and at least two s -active buyers of λ .

Theorem: Let E satisfy A1-A5, then

$$(i) \quad \emptyset \neq W(E) \subset N(E)$$

$$(ii) \quad s \in N(E) \text{ and all markets } s\text{-thick} \Rightarrow s \in W(E).$$

V.6 Corollary to the Main Result

Part (ii) of this theorem is not completely satisfactory. We cannot impose conditions on the primitives of the analysis which will ensure that a Nash selection will always be constrained Walrasian. [This problem is, in fact, generic to the NAE. Postlewaite-Schmeidler (21) prove that, under very weak conditions, there exists no Strategic Market Game such that the Nash and Walrasian allocations coincide.] We can, however, prove a weaker result without invoking the requirement that all markets be thick. The Corollary states that, under somewhat stronger assumptions, a Nash selection will be semi-Walrasian, that is, the outcome will range over allocations which are "constrained Walrasian with respect to the set of open markets." [The term "semi-Walrasian" was, we believe, coined by Mas-Colell (15).]

To make this statement precise, we need some more definitions. Given a selection $s \in S$, a market λ is said to be s-open if there is a positive probability that trade will occur on the λ -th market, that is, if there exists $\theta \in F(s)$ and $\alpha \in A$ such that $\theta_{\alpha\lambda} \neq 0$. [Since $F(s)$ is the support of $f(s)$, the / of existence of one such allocation is sufficient, once again, to ensure that trade occurs with positive probability.] Given $s \in S$, let $L(s)$ denote the set of s-open markets. The concept of a "constrained Walrasian pair with respect to the set of s-open markets" is

analogous to the concept of a constrained Walrasian pair, except that condition (5.8b) is replaced by the condition that agent α 's net trade vector is u_α -maximal on the subset of α 's budget set, defined by the condition that net trades on all but s -open markets are zero. Formally, given $K \subset L$ and $p \in R_+^L$, define

$$\beta_\alpha(p, K) = \{x \in \beta_\alpha(p) : \lambda \in L \sim K \Rightarrow x_\lambda = 0\} \quad (5.6')$$

and

$$\begin{aligned} \xi_\alpha(p, K) = \{x \in \beta_\alpha(p, K) : x' \in \beta_\alpha(p, K) \Rightarrow u_\alpha(\langle -p \cdot x, x \rangle) \\ \geq u_\alpha(\langle -p \cdot x', x' \rangle)\} \end{aligned} \quad (5.7')$$

A price allocation pair (\tilde{p}, \tilde{x}) , where $\tilde{p} \in R_+^L$ and $\tilde{x} = (\tilde{x}_\alpha)_{\alpha \in A}$, $\tilde{x}_\alpha \in R^L$, is said to be a constrained Walrasian pair with respect to K for E if

$$\tilde{p} \gg 0 \quad (5.8a)$$

$$\tilde{x} \in \xi_\alpha(\tilde{p}, K), \text{ for all } \alpha \in A \quad (5.8')$$

$$\sum_\alpha \tilde{x}_\alpha = 0 \quad (5.8c)$$

Finally, given $s \in S$, we define a semi-Walrasian selection analogously to a constrained Walrasian selection; the only difference is that price allocation pairs are, not constrained Walrasian, but constrained Walrasian with respect to the set of s -open markets, $L(s)$. Formally, we define the set of semi-Walrasian selections for E , $W(E, L(s))$, by

$$W(E, L(s)) = \{s \in S \text{ s.t. there exists}$$

$$\tilde{p} \in R_+^L, \tilde{p} \gg 0, \text{ s.t. for all } \theta \in F(s):$$

$$\theta_{\alpha\lambda} < 0 \Rightarrow p_{\alpha\lambda} = \tilde{p}_\lambda \text{ for all } \alpha \in A, \lambda \in L. \quad (5.24a)$$

$$\theta_\alpha^+ > 0 \Rightarrow p_\alpha(\eta(\theta_\alpha^+)) = \tilde{p} \cdot \eta(\theta_\alpha^+), \text{ for all } \alpha \in A \quad (5.24b)$$

(\tilde{p}, θ) is a constrained Walrasian pair w.r.t. $L(s)$ for E (5.24c')

Our corollary applies to what we call "twin economies."

Informally, a twin economy is an economy in which each agent has at least one "identical twin." More precisely, let

$A = (A_i)_{i \in I}$ be a partition by type of A , defined by the condition, for all $i \in I$:

$$\text{for all } \alpha, \alpha' \in A_i, u_\alpha = u_{\alpha'}, \text{ and } \langle w_{\alpha 0}, w_\alpha \rangle = \langle w_{\alpha' 0}, w_{\alpha'} \rangle \quad (5.25)$$

A twin economy satisfies the condition:

$$(A6) \quad \#A_i \geq 2, \text{ for all } i \in I \quad (5.26)$$

We will also need to strengthen one of our initial assumptions on utility functions:

$$(A2') \quad u_\alpha \text{ is strictly quasi-concave, for } \alpha \in A. * \quad (5.2')$$

We can now state the Corollary:

Corollary: If E satisfies $A1, A2', A3-A6$, then

$$s \in N(E) \Rightarrow s \in W(E, L(s)) .$$

* A function $f: X \rightarrow R$ is strictly quasi-concave if for all $x, x' \in X$ s.t. $f(x) \geq f(x')$, then $f(\alpha x + (1-\alpha)x') > f(x')$ for all $\alpha \in (0, 1]$.

VI. WHY SHOULD BUYERS ANNOUNCE ACCEPTANCE SETS?"

This section is a summary of Section VI of Simon (27): the reader is referred to the original for an elaboration of the ideas outlined here. Our objective is to compare two alternative ways of generalizing Bertrand's model. We have adopted the first approach (which we call our Original Specification), while Dubey (8) chose the second (the Alternative Specification).* In the Original Specification, buyers announce "acceptance sets" and sets of associated prices, while in the Alternative Specification, buyers announce "acceptance sets" and sets of associated prices.

Our argument is that Bertrand's partial equilibrium model of "competition-by-price" is ultra-competitive in spirit: a seller can, by shaving his ask-price, increase his market share at the expense of other suppliers. An extension of his model to a general equilibrium context should preserve this competitive spirit. A seller should, by sufficiently discounting his price, be able to induce buyers to purchase more of his commodity, and less of the commodities sold in neighboring markets. We endeavor to show that the obvious way to generalize Bertrand

*This is, in fact, one of two more of the radical differences between Dubey's paper and mine.

(the Alternative Specification) is unsatisfactory, because it precludes the possibility of this kind of competition across markets. Our Original Specification, on the other hand, neither precludes the possibility of competition across markets nor ensures that it will occur. To capture the essence of Bertrand in a general equilibrium Game, then, we must further restrict agents' strategy sets, so that buyers are obliged to convey, "reasonably accurately," the extent to which they regard the commodities sold in different markets as substitutes. These additional restrictions are introduced in the sequel to this paper (Simon (28)).

VII. CONCLUSION

In this paper we have developed a game-theoretic model of pure exchange, in which the process of price formation is fully decentralized: the players themselves dictate the prices at which they will trade. Our work can thus be viewed as an extension to a general equilibrium context of the spirit of Bertrand's classic critique of Cournot.

The paper addresses two questions which arise from modern Walrasian analysis, in which prices are treated by agents as parametric. Firstly, if all agents take prices as given, where do prices come from in the first place? In models based on the Cournot paradigm, prices are determined by the "impersonal forces of the market." In our model, prices emerge in the most natural way: agents choose them.

The second question which concerns us is: When is the Walrasian model appropriate as an idealization of actual economic situations? We focus in particular on the Walrasian hypothesis of price-taking behavior and seek conditions under which this hypothesis can be regarded as innocuous. We construct a Market Game, in which agents are permitted to act strategically with respect to prices, and compare the Nash equilibrium allocations for this Game to the allocations which arise in equilibrium, when agents act as price-takers. The main result of our paper is that, when there are at least two buyers and

at least two sellers active in every market, then the Nash allocations coincide with the price-taking allocations for the underlying economy.

This result leads us to an interpretation of Walrasian analysis which contrasts sharply with the conventional wisdom. Research based on the Cournot paradigm reinforces the widespread view that a key explanatory variable determining the extent to which strategic models yield "competitive" outcomes is the size of agents relative to the market. "Competitiveness," in this view, has come to be equated with "powerlessness": agents act "competitively" (that is, as "price-takers") if they have no "market power." Our purpose has been to focus attention upon the ultra-competitive aspect of the Walrasian equilibrium concept (using "competitive" as it is conventionally understood). This aspect was emphasized by Bertrand (3), but the NAEF, shackled by the Cournot paradigm, fails to elucidate it. Our model is an attempt to demonstrate that Walrasian outcomes result when the logic of "competition-by-price" is taken to its extreme conclusion.

VIII. PROOFS

We begin by defining some additional notation. In Section VIII.2 we state and prove some properties of the strategic outcome function and the auxiliary mappings, which will be used extensively in the subsequent sections. Sections VIII.3-VIII.6 are devoted to proofs of the Proposition, the two parts of the Theorem and the Corollary.

VIII.1 Notation

Given a list of strategies, $s \in \mathcal{S}^{\#A}$, we define the vector $\bar{p}(s) = (\bar{p}_1(s), \dots, \bar{p}_\ell(s))$, where $\bar{p}_\lambda(s)$ is the "maximum realized sale price for λ ," that is, the maximum ask-price for λ submitted by an s-active seller of λ . [Note that $\bar{p}(s) \gg 0$, since we have constrained agents to announce strictly positive ask-price vectors (5.12).] If no s-active seller exists, then we define $\bar{p}_\lambda(s)$ as infinity; that is, $\bar{p}_\lambda(s) < \infty \Rightarrow \lambda$ is s-open. Formally, define

$$\bar{p}: \mathcal{S}^{\#A} \rightarrow ([0, \infty])^\ell \text{ by, for all } s = ((p_{\alpha\lambda}, \dots, p_{\alpha\ell}), z_\alpha, r_\alpha, \zeta_\alpha)_{\alpha \in A} \in \mathcal{S}^{\#A},$$

$$\bar{p}_\lambda(s) = \begin{cases} \max_{\alpha} \{p_{\alpha\lambda} : \alpha \text{ is an s-active seller of } \lambda\} & \lambda \text{ is s-open} \\ \infty & \text{otherwise} \end{cases} \quad (8.1)$$

Let $A_\lambda(s)$ denote the set of s-active sellers of λ whose ask-price for λ is $\bar{p}_\lambda(s)$: for all $\lambda \in L$,

$$A_\lambda(s) : \{\alpha \in A : p_{\alpha\lambda} = \bar{p}_\lambda(s); \text{ there exists } \theta \in F(s) \text{ s.t. } \theta_{\alpha\lambda} < 0\} \quad (8.2)$$

We next distinguish two special classes of strategy lists, the second class being a subset of the first. We say that a strategy list has "the common ask-price property" if for all λ ,

all s -active sellers of λ announce the same ask-price for λ .

Formally, $s \in \mathcal{S}^{\#A}$ has the common ask-price property if for all $\theta \in F(s)$, for all $\lambda \in L$, $\theta_{\alpha\lambda} < 0 \Rightarrow p_{\alpha\lambda} = \bar{p}_\lambda(s)$. (8.3)

A subset of the above defined class of strategies is the class satisfying the "zero arbitrage profit property": loosely,

$s \in \mathcal{S}^{\#A}$ has this property if it satisfies (8.3), and if the per unit price announced by any buyer for any purchase he may

realize equals the unit cost of the package deal associated with that purchase, evaluated at the common ask-price vector,

$\bar{p}(s)$. Formally, $s \in \mathcal{S}^{\#A}$ has the zero arbitrage profit property if

(i) s has the common ask-price property

(ii) for all $\theta \in F(s)$, for all $\alpha \in A$, $\theta_\alpha^+ > 0 \Rightarrow p_\alpha(\eta(\theta_\alpha^+))$

$$= \bar{p}(s) \cdot \eta(\theta_\alpha^+) \quad (8.4)$$

[The reader will note that constrained and semi-Walrasian selections satisfy the "zero arbitrage profit property" (5.24a and 5.24b). A major part of the proof of the Theorem (part (ii)) is devoted to showing that Nash selections also have this property.]

VIII.2 Properties of the SOF and Auxiliary Mappings

Given a $\#A$ -dimensional list of strategies, $s \in \mathcal{S}^{\#A}$,

the following properties hold:

p1: $\Theta(s)$ is a closed, convex set.

p2: $\pi(\cdot, s)$ is concave on $\Theta(s)$.

p3: $\Pi(s)$ and $F(s)$ are convex sets.

p4: For all $\theta \in \Pi(s)$, $\pi(\theta, s) \geq 0$.

p5: For all $\theta \in \Pi(s)$, $\sum_{\alpha} \theta_{\alpha} = 0$.

p6: For all $\theta \in \Pi(s)$ and for all $\alpha \in A$,

$p_{\alpha\lambda} < \bar{p}_{\lambda}(s) < \infty \Rightarrow \theta_{\alpha\lambda}^{-} = z_{\alpha\lambda}$. (that is, if two s-active sellers of λ announce different ask-prices for λ , then the cheaper seller will never be rationed.)

Properties p7 and p8-p9 hold for classes of strategy lists satisfying, respectively, (8.3) and (8.4).

p7: If $s \in \mathcal{S}^{\#A}$ has the common ask-price property then for all $\theta \in \Pi(s)$ and $\alpha \in A$, $v(\theta_{\alpha}^{+}, s) \geq \bar{p}(s) \cdot \theta_{\alpha}^{+}$.

p8: If $s \in \mathcal{S}^{\#A}$ has the zero arbitrage profit property then $\max \{\pi(\theta, s) : \theta \in \Theta(s)\} = 0$ (i.e. s yields at most zero arbitrage profits!).

p9: If $s \in \mathcal{S}^{\#A}$ has the zero arbitrage profit property then for all $\alpha \in A$ and for all $x \in R_{+}^{\ell}$, $v(x, s_{\alpha}) \leq \bar{p}(s) \cdot x$. [p9 ensures that, for such selections, the iso-value surfaces, $\{x \in R_{+}^{\ell} : \rho_{\alpha}(\eta(x)) \|x\|_1 = c\}$ are supported by the hyperplanes, $H(\bar{p}(s), c)$, i.e. the situation illustrated in Figure VIII.1 cannot arise. (Note that this figure does not violate (8.4) if, say, $F(s) = \{\tilde{\theta}\}$, since $v(\tilde{\theta}_{\alpha}^{+}, s_{\alpha}) = \bar{c} = \bar{p}(s) \cdot \tilde{\theta}_{\alpha}^{+}$].

Proofs of p1-p9.

p1 is obvious, from the definition of $\Theta(s)$ (5.15).

p2 is obvious, since $\pi(\theta, s) = \sum_{\alpha} [v(\theta_{\alpha}^{+}, s_{\alpha}) + p_{\alpha} \cdot \theta_{\alpha}^{-}]$, and $v(\cdot, s_{\alpha})$ is the minimum of two concave functions and hence is itself concave.

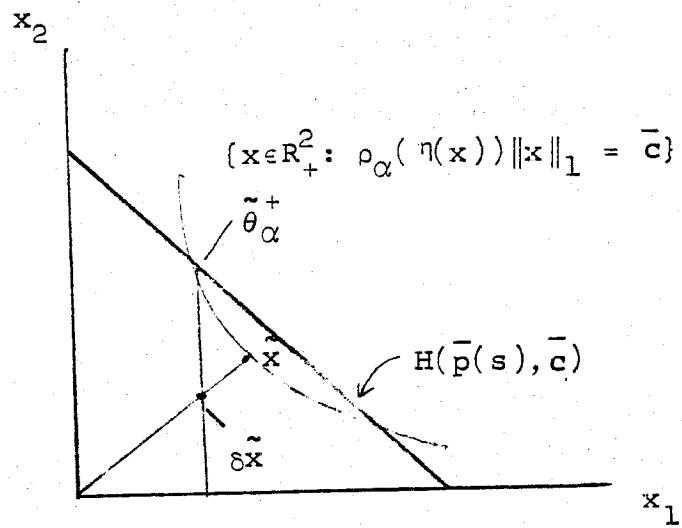


Figure VIII.1.

p3: is obvious, since $\pi(\cdot, s)$ and $\|\cdot\|_1$ are concave functions.

p4: obvious, since the zero allocation is s-allowable

p5: obvious, since ask-prices are constrained to be strictly positive and because sellers can be rationed (5.13a).

p6: choose $\tilde{\theta} \in \Theta(s)$ such that $\theta_{a\lambda}^- > z_{a\lambda}$, for $a \in A$ such that $p_{a\lambda} < \bar{p}_\lambda(s)$. Since $\bar{p}_\lambda(s) < \infty$, there exists $b \in A_\lambda(s)$ and $\hat{\theta} \in \Pi(s)$ such that $\hat{\theta}_{b\lambda} < 0$. Let $\bar{\theta} = \frac{1}{2}\tilde{\theta} + \frac{1}{2}\hat{\theta}$. $\bar{\theta}_{a\lambda} > z_{a\lambda}$ and $\bar{\theta}_{b\lambda} < 0$. Define $\theta(\epsilon) = \bar{\theta}/\bar{\theta}_{b\lambda}^{+\epsilon}, \bar{\theta}_{a\lambda}^{-\epsilon}$, $\epsilon > 0$. For ϵ small enough, $\theta(\epsilon) \in \Theta(s)$. Clearly, $\pi(\theta(\epsilon), s) > \pi(\bar{\theta}, s) \geq \pi(\tilde{\theta}, s)$, so $\tilde{\theta} \notin \Pi(s)$.

p7: obvious, since any allocation such that the inequality were violated would be arbitrage-profit dominated by the allocation which assigned to α the zero commodity bundle.

p8: follows immediately from the definition of the zero arbitrage profit property (8.4).

p9: suppose that p9 were not true, i.e. there exists $a \in A$ and $\tilde{x} \in R_+^l$ s.t. $v(\tilde{x}, s_a) > \bar{p}(s) \cdot \tilde{x}$. Clearly $\tilde{x}_\lambda > 0 \Rightarrow \lambda$ is s-open (otherwise $\bar{p}(s) = \infty$), so that $\tilde{x}_\lambda > 0 \Rightarrow$ there exists $\theta \in \Pi(s)$ s.t. $0 > \sum_{\alpha} \theta_{\alpha\lambda}^- \geq \sum_{\alpha} z_{\alpha\lambda}$. For $\delta > 0$ small enough we can find $\tilde{\theta} \in \Theta(s)$ such that $\tilde{\theta}_a^+ = \delta x$. $\pi(\tilde{\theta}, s) > 0$, violating p8, a contradiction, (see figure VIII.1).

VIII.3 Proof of the Proposition

Proposition: If E satisfies A1-A5, then a constrained Walrasian pair exists.

Proof: In order to apply standard techniques, we first renormalize prices to lie in the l -dimensional unit simplex, i.e. $\Delta^l = \{ \langle q_0, q \rangle \in \mathbb{R} \times \mathbb{R}_+^l : q_0 + \sum q_\lambda = 1 \}$. Define, for each $\alpha \in A$, the relation $\langle \beta_0, \beta \rangle_\alpha : \Delta^l \rightarrow \mathbb{R} \times \mathbb{R}^l$ by, for $\langle q_0, q \rangle \in \Delta^l$,

$$\langle \beta_0, \beta \rangle_\alpha(\langle q_0, q \rangle) = \{ \langle x_0, x \rangle \in \mathbb{R} \times \mathbb{R}^l : \langle x_0, x \rangle \geq \langle w_{\alpha 0}, w_\alpha \rangle;$$

$$x_0 = -q \cdot x / q_0; q \cdot x^+ \leq q_0 w_{\alpha 0} \}$$

Define, for each $\alpha \in A$, the relation $\langle \xi_0, \xi \rangle_\alpha : \Delta^l \rightarrow \mathbb{R} \times \mathbb{R}^l$ by, for $\langle q_0, q \rangle \in \Delta^l$,

$$\langle \xi_0, \xi \rangle_\alpha(\langle q_0, q \rangle) = \{ \langle x_0, x \rangle \in \langle \beta_0, \beta \rangle_\alpha(\langle q_0, q \rangle) :$$

$$\langle x_0, x \rangle \text{ is } u_\alpha\text{-maximal in } \langle \beta_0, \beta \rangle_\alpha(\langle q_0, q \rangle) \} .$$

$$\text{Let } \langle \xi_0, \xi \rangle = \sum_\alpha \langle \xi_0, \xi \rangle_\alpha .$$

Existence follows from Lemma 1, Section 2.2 of Hildenbrand (13, p. 150). To apply this Lemma, we need to check that $\langle \xi_0, \xi \rangle$ satisfies the following conditions

(i) for all $\langle q_0, q \rangle \in \Delta^l$ and $\langle x_0, x \rangle \in \langle \xi_0, \xi \rangle(\langle q_0, q \rangle)$,
 $q_0 x_0 + q \cdot x = 0$.

(ii) $\langle \xi_0, \xi \rangle$ is compact-valued, bounded from below and u.h.c.

(iii) If the sequence $(\langle q_0^n, q^n \rangle)$ of strictly positive price vectors converges to $\langle q_0, q \rangle$ which is not strictly positive, then

$$\inf \{x_0 + \sum_{\lambda} x_{\lambda} : \langle x_0, x \rangle \in \langle \xi_0, \xi \rangle (\langle q_0^n, q^n \rangle)\} > 0,$$

for n large enough.

(i) is true by definition of $\langle \xi_0, \xi \rangle$; (ii) follows from standard arguments; (iii) follows by an argument which is a special case of that used in the proof of Lemma 3 of Anderson (1, p. 11).

VIII.4 Proof of the Theorem (part (i))

Theorem (i): If E satisfies A1-A5, then $\emptyset \neq W(E) \subset N(E)$.

Proof: Suppose $\tilde{s} \in W(E)$, that is, there exists $\tilde{p} \in \mathbb{R}_+^L$,

s.t. for all $\theta \in F(\tilde{s})$:

$$\theta_{\alpha\lambda} < 0 \Rightarrow p_{\alpha\lambda} = \tilde{p}_{\lambda}, \text{ for all } \lambda \in L, \alpha \in A \quad (5.24a)$$

$$\theta_{\alpha}^+ > 0 \Rightarrow p_{\alpha}(\eta(\theta_{\alpha}^+)) = \tilde{p}_{\alpha} \cdot \eta(\theta_{\alpha}^+) \text{ for all } \alpha \in A \quad (5.24b)$$

$$(\tilde{p}, \theta) \text{ is a c.w. pair for } E \quad (5.24c)$$

Clearly, $\bar{p}_{\lambda}(\tilde{s}) < \infty \Rightarrow \bar{p}_{\lambda}(\tilde{s}) = \tilde{p}_{\lambda}$. Note that $\theta \in F(\tilde{s}) \Rightarrow \pi(\theta, \tilde{s}) = \tilde{p} \cdot \sum_{\alpha} \theta_{\alpha} = 0$, since \tilde{s} has the zero arbitrage profit property.

Pick $a \in A$, $\hat{s}_a = (\hat{p}_a, \hat{z}_a, \hat{\rho}_a, \hat{\zeta}_a) \in S_a$ and define $\hat{s} = \tilde{s}/\hat{s}_a$. We must show that $P_a(\hat{s}) \leq P_a(\tilde{s})$. Our approach is to show that $\theta \in F(\hat{s}) \Rightarrow \theta_a \in \beta_a(\tilde{p})$ and $\theta_0(\theta_a, \hat{s}_a) \leq -\tilde{p} \cdot \theta_a$. This is sufficient to prove the theorem, since $P_a(\hat{s}) \leq u_a(\langle \theta_0(\theta_a, \hat{s}_a), \theta_a \rangle) \leq u_a(\langle -\tilde{p} \cdot \theta_a, \theta_a \rangle) \leq u_a(\langle -\tilde{p} \cdot \theta'_a, \theta'_a \rangle) = P_a(\tilde{s})$,

where $\theta' \in F(\tilde{s})$ (the second inequality holds by monotonicity (A3) and the third since $\theta'_a \in \xi_a(\tilde{p})$).

We first show that agent a cannot buy any commodity bundle for less than its cost, evaluated at the common ask-price, \tilde{p} , because such a transaction would yield negative arbitrage profits.

Pick $\hat{\theta} \in F(\hat{s})$. To show $\hat{\theta}_a \in \beta_a(\tilde{p})$ we need only show that $\tilde{p} \cdot \hat{\theta}_a^+ \leq w_{a0}$. Suppose for some $\lambda \in L$, $\hat{\theta}_a > 0$. From (5.13c) and (p7) we know that $z_{a\lambda} = 0$. Also, for $\alpha \neq a$, $p_{\alpha\lambda} < \tilde{p}_\lambda \Rightarrow z_{\alpha\lambda} = 0$ (using (5.24a) and p6)). Therefore $\tilde{p}_\lambda \leq \bar{p}_\lambda(\hat{s})$. Also $v(\hat{\theta}_a^+, \hat{s}_a) \geq \tilde{p}_\lambda \cdot \hat{\theta}_a^+$ (p7). Therefore $\tilde{p} \cdot \hat{\theta}_a^+ \leq \bar{p}(\hat{s}) \cdot \hat{\theta}_a^+ \leq v(\hat{\theta}_a^+, \hat{s}_a) \leq w_{a0}$.

We next show that agent a will be unable to earn more than the value, at prices \tilde{p} , of the goods which he sells, because buyers have committed themselves to trade only at the price vector \tilde{p} , that is, $-\tilde{p} \cdot \hat{\theta}_a \geq \theta_0(\hat{\theta}_a, \hat{s}_a) = -[v(\hat{\theta}_a^+, \hat{s}_a) + \tilde{p} \cdot \hat{\theta}_a^-]$.

$$\begin{aligned} \pi(\hat{\theta}, \hat{s}) &= \sum_{\alpha} [v(\hat{\theta}_{\alpha}^+, \hat{s}_{\alpha}) + \hat{p}_{\alpha} \cdot \hat{\theta}_{\alpha}^-] \leq \sum_{\alpha \neq a} [\tilde{p} \cdot \hat{\theta}_{\alpha}^+ + \tilde{p} \cdot \hat{\theta}_{\alpha}^-] - \theta_0(\hat{\theta}_a, \hat{s}_a) \\ &\leq \tilde{p} \cdot \sum_{\alpha} \hat{\theta}_{\alpha} - \tilde{p} \cdot \hat{\theta}_a - \theta_0(\hat{\theta}_a, \hat{s}_a) \leq 0 - \tilde{p} \cdot \hat{\theta}_a - \theta_0(\hat{\theta}_a, \hat{s}_a). \quad [\text{The first inequality holds by (p9), that is,} \end{aligned}$$

$$v(\hat{\theta}_{\alpha}^+, \hat{s}_{\alpha}) = v(\hat{\theta}_{\alpha}^+, \tilde{s}_{\alpha}) \leq \tilde{p} \cdot \hat{\theta}_{\alpha}^+, \text{ for } \alpha \neq a].$$

Finally, since $\pi(\hat{\theta}, \hat{s}) \geq 0$, we have $-\tilde{p} \cdot \hat{\theta}_a \geq \theta_0(\hat{\theta}_a, \hat{s}_a)$.

This completes the proof of the first part of the Theorem.

VIII.5 Proof of the Theorem (part ii)

Theorem (ii): If E satisfies A1-A5, then $s \in N(E)$ s.t. all markets are s -thick $\Rightarrow s \in W(E)$.

Proof: The proof is organized into six parts.

(A): $s \in N(E) = s$ has the common ask-price property (7.3).

Proof of (A): suppose $\tilde{s} \in S$ and there exists $\tilde{\theta} \in F(\tilde{s})$ such that $\tilde{\theta}_{a\lambda} < 0$, for some $a \in A$ s.t. $\tilde{p}_{a\lambda} < \bar{p}_\lambda(s)$. We will show that there exists $s'_a = \tilde{s}_a/p'_{a\lambda}$, $p'_{a\lambda} > p_{a\lambda}$, such that $p_a(\tilde{s}/s'_a) > p_a(\tilde{s})$, establishing that $\tilde{s} \notin N(E)$.

It is sufficient to show that, for $(p'_{a\lambda} - \tilde{p}_{a\lambda})$ small enough, $F(\tilde{s}/s'_a) = F(\tilde{s})$. To establish this we first show that as we move away from $F(\tilde{s})$, by rationing a 's sales of λ , arbitrage profits fall at a rate bounded away from zero. Therefore, for a small enough increase in a 's ask-price for λ , profits will still fall as a 's sales of λ are rationed. See Figure VIII.2.

Pick $\tilde{\theta} \in \Pi(\tilde{s})$ and $b \in A_\lambda(\tilde{s})$ s.t. $\tilde{\theta}_{b\lambda} < 0$. By (p6), we know that $\tilde{\theta}_{a\lambda} = \tilde{z}_{a\lambda} < 0$. Let $\epsilon = (\tilde{p}_{a\lambda} - \bar{p}_\lambda(\tilde{s}))\tilde{z}_{a\lambda} > 0$, and let $\gamma = \tilde{\theta}_{b\lambda}/\tilde{z}_{a\lambda} > 0$. Let $\hat{s}_a = \tilde{s}_a/(1+\gamma)\tilde{z}_{a\lambda}$, and define $\hat{s} = \tilde{s}/\hat{s}_a$. [Note that $\hat{s} \in \mathcal{S}^{\#A}$ though, possibly, $\hat{s} \notin S$, since, possibly $(1+\gamma)\tilde{z}_{a\lambda} < w_{a\lambda}$]. We will show that, for $\theta \in \Theta(\tilde{s})$ such that $\theta_{a\lambda} = (1-\delta)\tilde{z}_{a\lambda}$, arbitrage profits for θ are less than maximal by at least $\delta\epsilon$, establishing that, as a is rationed, profits fall at a rate bounded away from zero.

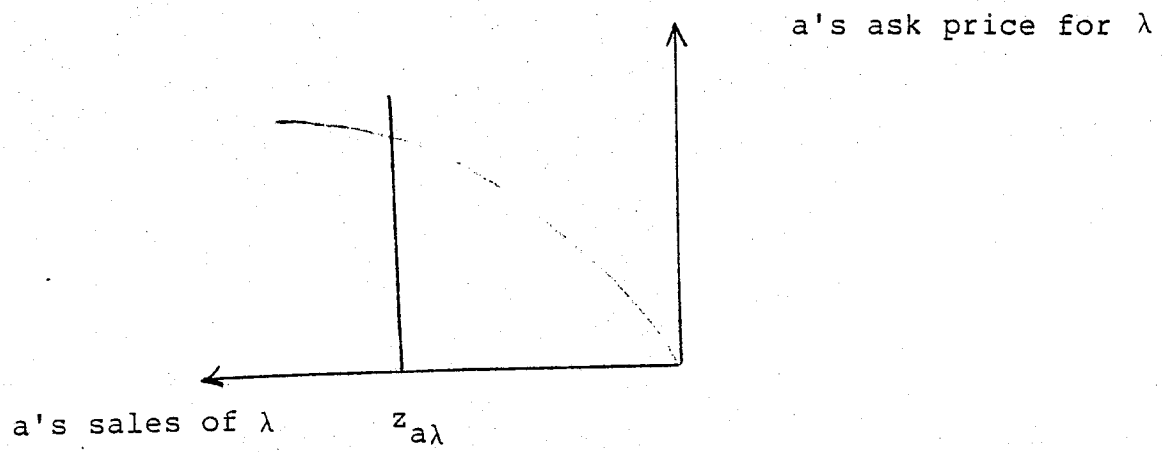


Figure VIII.2

Define $\hat{\theta} = \bar{\theta}/\hat{\theta}_{a\lambda}$, where $\hat{\theta}_{a\lambda} = (1+\gamma)\bar{z}_{a\lambda}$ and $\hat{\theta}_{b\lambda} = 0$. Note that $\sum \hat{\theta}_\alpha = 0$, so $\theta \in \Theta(\bar{s}) \subset \Theta(\hat{s})$. $\pi(\hat{\theta}, \hat{s}) = \pi(\bar{\theta}, \bar{s}) + \gamma\epsilon$. Now pick $\theta^{\delta\alpha} \in \Theta(s)$ such that $\theta^{\delta}_{a\lambda} = (1-\delta)\bar{z}_{a\lambda}$, where $\delta \in (0, 1]$, and define $\bar{\theta} = \frac{\gamma}{\delta+\gamma}\delta + \frac{\delta}{\delta+\gamma}\hat{\theta}$. $\bar{\theta} \in \Theta(\bar{s})$, since $\bar{\theta}_{a\lambda} = z_{a\lambda}$, so that using the concavity of $\pi(\cdot, \bar{s})$ (p2), $\pi(\bar{\theta}, \bar{s}) \geq \pi(\theta^{\delta}, \bar{s}) \geq \frac{\gamma}{\delta+\gamma}\pi(\theta^{\delta}, \bar{s}) + \frac{\delta}{\delta+\gamma}\pi(\hat{\theta}, \hat{s}) = \frac{\gamma}{\delta+\gamma}\pi(\theta^{\delta}, \bar{s}) + \frac{\delta}{\delta+\gamma}[\pi(\bar{\theta}, \bar{s}) + \gamma\epsilon]$. This establishes that $\pi(\bar{\theta}, \bar{s}) \geq \pi(\theta^{\delta}, \bar{s}) + \delta\epsilon$.

Define $s'_a = \bar{s}_a / (\bar{p}_{a\lambda} - \frac{1}{2}\epsilon / \bar{z}_{a\lambda})$ and let $s' = \bar{s}/s'_a$. Clearly $\Theta(s') = \Theta(\bar{s})$. Pick $\theta^{\delta} \in \Theta(s)$ such that $\theta_{a\lambda} = (1-\delta)\bar{z}_{a\lambda}$. Now $\pi(\theta^{\delta}, s') = \pi(\theta^{\delta}, \bar{s}) - \frac{1}{2}(1-\delta)\epsilon$. $\pi(\bar{\theta}, \bar{s}) - \delta\epsilon - \frac{1}{2}(1-\delta)\epsilon \leq \pi(\bar{\theta}, s') + \frac{1}{2}\epsilon - \delta\epsilon - \frac{1}{2}(1-\delta)\epsilon < \pi(\bar{\theta}, \bar{s}')$ for $\delta > 0$. Therefore $\theta^{\delta} \in \Pi(s') \Rightarrow \delta = 0$, so that $\Pi(s') = \Pi(\bar{s})$ and $F(s') = F(\bar{s})$.

Finally, $\theta_0(f_a(s'), s') = \theta_0(f_a(\bar{s}), \bar{s}) + \frac{1}{2}\epsilon$, so that $P_a(s'_a) > P_a(\bar{s})$

(B) $s \in N(E) \Rightarrow s$ has the zero arbitrage profit property.

Proof of (B): Suppose there exists $\bar{s} \in S$, such that \bar{s} has the common ask-price property (7.3), and $\bar{\theta} \in F(\bar{s})$ such that, for some $a \in A$, $\bar{p}_a(\eta(\bar{\theta}_a^+)) > \bar{p}(\bar{s}) \cdot \eta(\bar{\theta}_a^+)$. (The reverse inequality cannot hold, by (p7)). We will show that \bar{s} cannot be a Nash selection because either agent a can lower his bid-price for the package deal $\eta(\bar{\theta}_a^+)$, or some seller can raise his ask-price for some good, without changing the set of arbitrage-profit-maximal allocations. In either case some agent can thus increase his payoff by defecting from his part of \bar{s} .

The proof of (B) involves the following thought experiment: we pick an s-active seller, b_λ , of each commodity whose market is s-open and consider the effect on arbitrage profits if each b_λ doubles the quantity of λ he offers for sale. There are two possible classes of outcomes: either arbitrage profits would increase (Case I) or they would not (Case II). (See Figure VIII.3.) If case I obtains then, using an argument parallel to the preceding proof of (A), we show that there exists λ such that, as we move away from $F(s)$, by rationing b_λ 's sales of λ , arbitrage profits fall at a rate bounded away from zero. Therefore, once again, for a small enough increase in b_λ 's ask-price for λ , profits will still fall as b_λ 's sales of λ are rationed. If case II obtains, then we show that agent a can lower his bid price for $\eta(\tilde{\theta}_a^+)$ without risking being rationed. Our reasoning is as follows: a's action will lead to rationing only if the resources thus released can profitably be diverted to satisfy the purchase offer of some other buyer. If such a competing offer existed, then arbitrage profits would have risen when the thought experiment was conducted; since case II obtained, we conclude that no such competing offer exists.

For each $\lambda \in L$ such that λ is s-open pick $b_\lambda \in A_\lambda(\tilde{s})$; otherwise pick b_λ arbitrarily from A. For each $\lambda \in L$ define

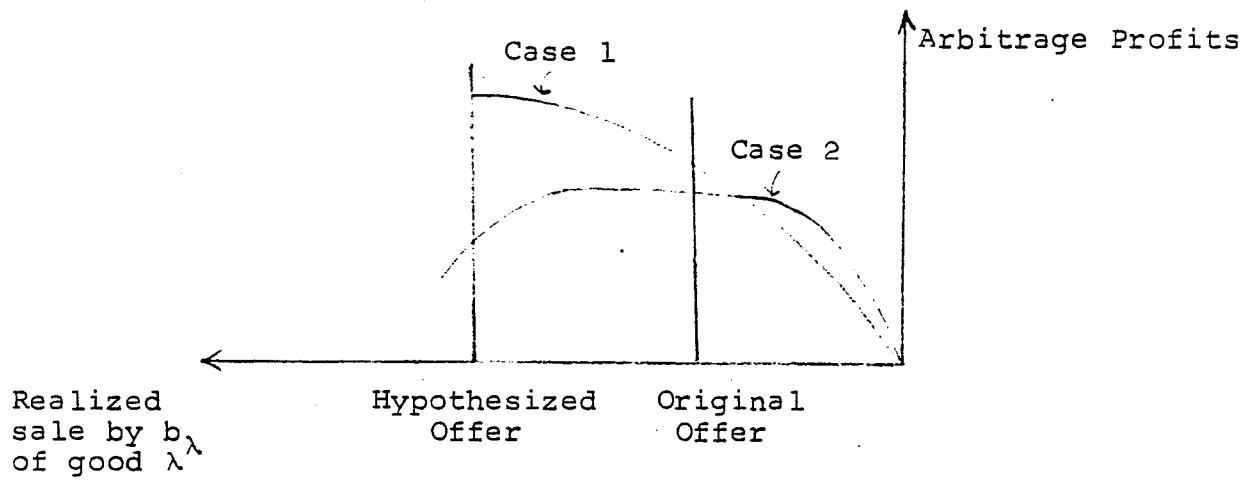


Figure VIII.3

$\hat{s}_{b\lambda} = \tilde{s}_{b\lambda} / 2\tilde{z}_{b\lambda}$ and define $\hat{s} = \tilde{s} / (\hat{s}_{b\lambda})_{\lambda \in L}$. Note that $\hat{s} \in \mathcal{S}^{\#A}$ (though, possibly, $\hat{s} \notin S$). Either case I: $\Pi(\hat{s}) \cap \Theta(\tilde{s}) = \emptyset$, or case II: $\Pi(\hat{s}) \cap \Theta(\tilde{s}) \neq \emptyset$, is true.

Suppose case I obtains. Pick $\hat{\theta} \in \Pi(\hat{s})$. Let $\tilde{\pi} = \pi(\hat{\theta}, \tilde{s})$. (recall $\tilde{\theta} \in F(\tilde{s})$) and $\hat{\pi} = \pi(\tilde{\theta}, \tilde{s})$. Let $\hat{\pi} - \tilde{\pi} = \varepsilon > 0$. Define $\Lambda(\tilde{s}) = \{\lambda \in L: \tilde{\theta}_{b\lambda} = \tilde{z}_{b\lambda}\}$. Clearly, from (p2) $\Lambda(\tilde{s}) \neq \emptyset$. We claim that there exists $k \in \Lambda(\tilde{s})$ such that for all $\delta \in (0, 1]$ and for all $\theta \in \Theta(\tilde{s})$ satisfying $\theta_{b_k k} = (1 - \delta^\ell)\tilde{z}_{b_k k}$, $\tilde{\pi} - \pi(\theta, \tilde{s}) \geq \delta\varepsilon$. Suppose, to the contrary that, for all $\lambda \in \Lambda(\tilde{s})$, there exists $\delta^\lambda \in (0, 1]$ and $\theta^\lambda \in \Theta(\tilde{s})$, satisfying $\theta^\lambda_{b\lambda} = (1 - \delta^\lambda)\tilde{z}_{b\lambda}$, such that $\hat{\pi} - \pi(\theta^\lambda, \tilde{s}) < \delta^\lambda$. Pick $\tilde{\delta} > 0$ small enough such that, for $\lambda \in \Lambda(\tilde{s})$, $\tilde{\delta} \leq \delta^\lambda$, and, for $\lambda \in L \setminus \Lambda(\tilde{s})$, $\tilde{\theta}_{b\lambda} \geq (1 - \tilde{\delta})\tilde{z}_{b\lambda}$. For $\lambda \in \Lambda(\tilde{s})$, define $\bar{\theta}^\lambda = ((\delta^\lambda - \tilde{\delta})/\delta^\lambda)\tilde{\theta} + (\tilde{\delta}/\delta^\lambda)\theta^\lambda$; otherwise let $\bar{\theta}^\lambda = \tilde{\theta}$. Note that, for all λ , $\bar{\theta}^\lambda_{b\lambda} \geq (1 - \tilde{\delta})\tilde{z}_{b\lambda}$. Also, by (p2), for $\lambda \in \Lambda(\tilde{s})$, $\pi(\bar{\theta}^\lambda, \tilde{s}) > \tilde{\pi} - \tilde{\delta}\varepsilon$. Define $\bar{\theta} = 1/\ell \sum \bar{\theta}^\lambda$. Note that $\pi(\bar{\theta}, \tilde{s}) > \tilde{\pi} - \tilde{\delta}\varepsilon$ and for all $\lambda \in L$, $\bar{\theta}_{b\lambda} \geq (1 - \tilde{\delta})\tilde{z}_{b\lambda}$. Define $\theta^* = (\tilde{\delta}/(1 + \tilde{\delta}))\hat{\theta} + (1/(1 + \tilde{\delta}))\bar{\theta}$. $\theta^* \in \Theta(\tilde{s})$ and $\pi(\theta^*, \tilde{s}) > \pi(\tilde{\theta}, \tilde{s})$, by (p2), a contradiction, since $\tilde{\theta} \in \Pi(\tilde{s})$. This establishes the existence of $k \in \Lambda(\tilde{s})$, satisfying the above condition.

Let $b = b_k$. Define $s'_b = \tilde{s}_b / (\tilde{p}_{bk} - \varepsilon/2\tilde{z}_{bk})$ and $s' = \tilde{s}/s'_b$. Note that $\Theta(s') = \Theta(\tilde{s})$ and that, for all $\theta \in \Theta(s')$, $\pi(\theta, s') = \pi(\theta, \tilde{s}) - \varepsilon\theta_{bk}/2\tilde{z}_{bk}$.

Pick $\theta^\delta \in \Theta(s')$ such that $\theta^\delta_{bk} = (1 - \delta)\tilde{z}_{bk}$, $\delta \in (0, 1]$. $\tilde{\pi} - \pi(\theta^\delta, \tilde{s}) \geq \delta\varepsilon$. Therefore $\pi(\tilde{\theta}, s') \geq \pi(\theta^\delta, s') + \delta\varepsilon/2$, establishing that $\Pi(s') = \Pi(\tilde{s})$ and $F(s') = F(\tilde{s})$. Hence $P_b(s') > P_b(\tilde{s})$, so $\tilde{s} \notin N(E)$.

Suppose now that case II obtains, i.e., for all $\theta \in \Pi(\hat{s})$, $\pi(\theta, \hat{s}) \leq \tilde{\pi}$. Clearly then, for all $\theta \in \Pi(\tilde{s})$, for all $\alpha \in A$ and for all $x \in R_+^l$, $v(\theta_\alpha^+, \tilde{s}_\alpha) - \bar{p}(\tilde{s}) \cdot \theta_\alpha^+ \geq v(x, \tilde{s}_\alpha) - \bar{p}(\tilde{s}) \cdot x$. Pick $\delta > 0$ such that $\tilde{\rho}_a(\eta(\tilde{\theta}_a^+)) - \delta/\tilde{\zeta}_a(\eta(\tilde{\theta}_a^+)) > \bar{p}(\tilde{s}) \cdot \eta(\tilde{\theta}_a^+)$. Define ρ'_a by, for all $\eta \in \Delta_+^l$, $\rho'_a(\eta) = \tilde{\rho}_a(\eta) - \delta/\tilde{\zeta}_a(\eta)$. Let $s'_a = \tilde{s}_a/\rho'_a$ and $s' = \tilde{s}/s'_a$. Note that $\theta(s') = \theta(\tilde{s})$. Also note that, for all $\theta \in \Pi(\tilde{s})$, $v(\theta_a^+, s'_a) - \bar{p}(\tilde{s}) \cdot \theta_a^+ = v(\theta_a^+, \tilde{s}_a) - \delta - \bar{p}(\tilde{s}) \cdot \theta_a^+ \geq v(x, s'_a) - \bar{p}(\tilde{s}) \cdot x$, for all $x \in R_+^l$. Therefore $\Pi(s') = \Pi(\tilde{s})$, $F(s') = F(\tilde{s})$ and $P_a(s') > P_a(\tilde{s})$, so that $s \notin N(E)$.

(C) Suppose $s \in N(E)$. Then for all $\theta \in \Pi(s)$ and all $\alpha \in A$, $u_\alpha(\langle \theta_0, \theta_\alpha, s_\alpha \rangle, \theta_\alpha) \leq P_\alpha(s)$.

Proof: Suppose that $\tilde{s} \in N(E)$ and $\tilde{\theta} \in \Pi(\tilde{s})$. \tilde{s} has the zero arbitrage profit property, i.e. $\pi(\tilde{\theta}, \tilde{s}) = 0$ and, for all $\alpha \in A$, $v(\tilde{\theta}_\alpha^+, \tilde{s}_\alpha) = \bar{p}(\tilde{s}) \cdot \tilde{\theta}_\alpha^+$. Also $\sum_\alpha \tilde{\theta}_\alpha = 0$, by (p5). Suppose that, for some $a \in A$, $u_a(\langle \theta_0, \tilde{\theta}_a, \tilde{s}_a \rangle, \tilde{\theta}_a) > P_a(\tilde{s})$. We will show that there exists a strategy for a , \hat{s}_a , such that a is guaranteed to realize a net trade arbitrarily close to $\tilde{\theta}_a$, so that, by playing this alternative strategy, agent a can increase his payoff.

Since, by assumption (A1), u_a is continuous, there exists $\varepsilon > 0$ such that $\langle x_0, x \rangle \in B(\langle \theta_0, \tilde{\theta}_a, \tilde{s}_a \rangle, \tilde{\theta}_a, \varepsilon) \Rightarrow u_a(\langle x_0, x \rangle) > P_a(\tilde{s})$.* Define $x^\delta = \tilde{\theta}_a^- + (1-\delta)\tilde{\theta}_a^+$ and pick $\hat{\delta} > 0$ small

*Given $x \in R^n$, we define the ε -ball, $B(x, \varepsilon) = \{y \in R^n: d(x, y) > \varepsilon\}$, where $d(x, y) = \max\{|x_k - y_k|: k \in \{1, \dots, n\}\}$.

enough that $\langle - (v(\tilde{\theta}_a^+, \tilde{s}_a) + (1-\delta)\bar{p}(s) \cdot \tilde{\theta}_a^-), x \rangle \in$
 $B(\langle \theta_0(\tilde{\theta}_a, \tilde{s}_a), \tilde{\theta}_a \rangle, \epsilon)$. Let $\hat{x} = x^{\hat{\delta}}$ and pick $\hat{\zeta}_a$ **CONVEX**
satisfying $\hat{\zeta}_a(\eta(\tilde{\theta}_a^+)) = (1-\hat{\delta})\|\tilde{\theta}_a^+\|$ and, for $\eta \neq \eta(\tilde{\theta}_a^+)$,
 $\hat{\zeta}_a(\eta)\bar{p}(\tilde{s}) \cdot \eta > \hat{\zeta}_a(\eta(\tilde{\theta}_a^+))\bar{p}(\tilde{s}) \cdot \eta(\tilde{\theta}_a^+)$. Define \hat{s}_a as follows:
 $\hat{p}_a = (1-\hat{\delta})\bar{p}(\tilde{s})$; $\hat{z}_a = \hat{x}^-$; $\hat{\zeta}_a$; $\hat{\rho}_a = \bar{p}(\tilde{s}) \cdot \tilde{\theta}_a^+ / \hat{\zeta}_a$. Observe that
 $\hat{s}_a \in S_a$ and that $v(\hat{x}^+, \hat{s}_a) - \bar{p}(\tilde{s}) \cdot \hat{x}^+ = \hat{\delta}\bar{p}(\tilde{s}) \cdot \tilde{\theta}_a^+ > v(y, \hat{s}_a)$
 $- \bar{p}(\tilde{s})y$, for all $y \in R_+^l$. Let $\hat{s} = \tilde{s} / \hat{s}_a$.

We now show that $\theta \in F(\hat{s}) \Rightarrow \theta_a = \hat{x}$. Pick $\theta \in \Theta(\hat{s})$ such
that $\theta_a \neq \hat{x}$. $\pi(\theta, \hat{s}) =$

$$\begin{aligned} & - \sum_{\alpha \neq a} \theta_0(\theta_\alpha, \hat{s}_\alpha) + \bar{p}(\tilde{s}) \cdot \theta_a^+ - \bar{p}(\tilde{s}) \cdot \theta_a^- + v(\theta_a^+, \hat{s}_a) + \hat{p}_a \cdot \theta_a^- \\ & = - \sum_{\alpha \neq a} \theta_0(\theta_\alpha, \tilde{s}_\alpha) + \bar{p}(\tilde{s}) \cdot \theta_a^+ + (1-\hat{\delta})\bar{p}(\tilde{s}) \cdot \theta_a^- + v(\theta_a^+, \hat{s}_a) - \bar{p}(\tilde{s}) \cdot \theta_a^+ \\ & \leq - \sum_{\alpha \neq a} \bar{p}(\tilde{s}) \cdot \theta_\alpha + \bar{p}(\tilde{s}) \cdot \theta_a - \hat{\delta}\bar{p}(\tilde{s}) \cdot \theta_a^- + v(\theta_a^+, \hat{s}_a) - \bar{p}(\tilde{s}) \cdot \theta_a^+ , \end{aligned}$$

[using (p9), which establishes that $v_\alpha(\theta_\alpha, \tilde{s}_\alpha) \leq \bar{p}(\tilde{s}) \cdot \theta_\alpha^+$, $\alpha \neq a$]
 $< 0 - \hat{\delta}\bar{p}(\tilde{s}) \cdot \hat{x}^- + v(\hat{x}^+, \hat{s}_a) - \bar{p}(\tilde{s}) \cdot \hat{x}^+$. To show that
 $\theta \notin \Pi(\hat{s})$, we construct $\hat{\theta} \in \Theta(\hat{s})$ such that $\pi(\hat{\theta}, \hat{s}) =$
 $- \hat{\delta}\bar{p}(\tilde{s}) \cdot \hat{x}^- + v(\hat{x}^+, \hat{s}_a) - \bar{p}(\tilde{s}) \cdot \hat{x}^+$. Define $\hat{\theta}$ as follows:

$$\begin{aligned} \hat{\theta}_a & = \hat{x}_a; \hat{\theta}_\alpha^+ = \tilde{\theta}_\alpha^+, \alpha \neq a \text{ and, for all } \lambda \in L \text{ and } \alpha \neq a, \text{ define} \\ \hat{\theta}_{\alpha\lambda}^- & = \left(\sum_{\alpha \neq a} \tilde{\theta}_{\alpha\lambda}^- + \hat{\delta}\tilde{\theta}_{a\lambda}^+ \right) \tilde{\theta}_{\alpha\lambda}^- / \sum_{\alpha \neq a} \tilde{\theta}_{\alpha\lambda}^- . \text{ Note that } \sum_{\alpha} \hat{\theta}_\alpha^+ = (1-\hat{\delta})\tilde{\theta}_a^+ + \\ \sum_{\alpha \neq a} \tilde{\theta}_\alpha^+ + \sum_{\alpha} \tilde{\theta}_\alpha^- + \hat{\delta}\tilde{\theta}_a^+ & = 0 . \quad \pi(\hat{\theta}, \hat{s}) = \sum_{\alpha \neq a} [(\hat{\theta}_\alpha^+, \hat{s}_\alpha) + \bar{p}(\tilde{s}) \cdot \hat{\theta}_\alpha^-] \\ & + \bar{p}(\tilde{s}) \cdot \hat{\theta}_a^+ - \bar{p}(\tilde{s}) \cdot \hat{\theta}_a^- + v(\hat{\theta}_a^+, \hat{s}_a) + \hat{p}_a \cdot \hat{\theta}_a^- = \\ \sum_{\alpha \neq a} \bar{p}(\tilde{s}) \cdot \hat{\theta}_\alpha^+ + \bar{p}(\tilde{s}) \cdot \hat{\theta}_a^- & - \hat{\delta}\bar{p}(\tilde{s}) \cdot \hat{\theta}_a^- + v(\hat{\theta}_a^+, \hat{s}_a) - \bar{p}(\tilde{s}) \cdot \hat{\theta}_a^+ \end{aligned}$$

(using the fact that $v(\hat{\theta}_\alpha, \hat{s}_\alpha) = \bar{p}(\tilde{s}) \cdot \hat{\theta}_\alpha$, for $\alpha \neq a$) = 0
 - $\hat{\delta} \bar{p}(\tilde{s}) \cdot \hat{x}^- + v(\hat{x}_a^+, \hat{s}_a) - \bar{p}(\tilde{s}) \cdot \hat{x}^+$. This establishes that
 $\theta \in \Pi(\hat{s}) = \theta_a = \hat{x}$, so that $P_a(\hat{s}) > P_a(\tilde{s})$, a contradiction,
 since $\tilde{s} \in N(E)$.

(D) $s \in N(E)$, $\theta \in F(s) =$ for all $\alpha \in A$, $u_\alpha(\theta_0(\theta_\alpha, s_\alpha), \theta_\alpha) = P_\alpha(s)$.

Proof of (D): (D) is an immediate consequence of (C). Without loss of generality, suppose that, to the contrary, $\tilde{s} \in N(E)$, $\tilde{\theta} \in F(\tilde{s})$ and, for some $a \in A$, $u_a(\theta_0(\tilde{\theta}_a, \tilde{s}_a), \tilde{\theta}_a) < P_a(\tilde{s})$. Since $F(\tilde{s})$ is the support of $f(\tilde{s})$ and u_a is continuous (A1), there exists $\hat{\theta} \in F(\tilde{s})$ such that $u_a(\theta_0(\hat{\theta}_a, \tilde{s}_a), \hat{\theta}_a) > P_a(\tilde{s})$. But this is a contradiction of (C), since $\hat{\theta} \in \Pi(\tilde{s})$.

(E) Suppose $\tilde{s} \in N(E)$ and that, for some $a \in A$, there exists $\tilde{\theta} \in F(\tilde{s})$ such that $-\sum_{\alpha \neq a} \tilde{\theta}_\alpha^+ \ll \tilde{\theta}_a^- \leq \tilde{\theta}_a^+ \ll -\sum_{\alpha \neq a} \tilde{\theta}_\alpha^-$. Then $\tilde{\theta}_a \in \xi_a(\bar{p}(\tilde{s}))$.

Proof: since u_a is quasi-concave (assumption A2), it suffices to establish that there exists $\epsilon > 0$ such that $u_a(\langle -\bar{p}(s) \cdot \tilde{\theta}_a, \tilde{\theta}_a \rangle) \geq u_a(\langle -\bar{p}(\tilde{s}) \cdot x, x \rangle)$, for all $x \in B(\tilde{\theta}_a, \epsilon) \cap \beta_a(\bar{p}(\tilde{s}))$. For each $\lambda \in L$, let $\delta^\lambda = \min(\sum_{\alpha \neq a} \tilde{\theta}_{\alpha\lambda}^+, -\sum_{\alpha \neq a} \tilde{\theta}_{\alpha\lambda}^-)$ and let $\delta = \frac{1}{2} \min \{ \delta^\lambda : \lambda \in L \}$.

Pick $\tau \in (0, 1)$ and $\epsilon \in (0, \delta)$ such that for all $\lambda \in L$, if $\tilde{\theta}_{a\lambda} \neq 0$, then $|\tilde{\theta}_{a\lambda}| > \epsilon$ and, also, $\sum_{\alpha \neq a} \tilde{\theta}_{\alpha\lambda}^+ - \delta < \tau \sum_{\alpha \neq a} \tilde{\theta}_{\alpha\lambda}^+ < \sum_{\alpha \neq a} \tilde{\theta}_{\alpha\lambda}^+ - \epsilon$.

Pick $\tilde{x} \in B(\tilde{\theta}_a, \epsilon) \cap \beta_a(\bar{p}(\tilde{s}))$. We will show that there exists a strategy, \hat{s}_a , for a which would guarantee that agent a realized a net trade arbitrarily close to \tilde{x} . Since $\tilde{s} \in N(E)$, we can conclude that $u_a(\langle -\bar{p}(\tilde{s}) \cdot \tilde{x}, \tilde{x} \rangle) \leq u_a(\langle -\bar{p}(\tilde{s}) \cdot \tilde{\theta}_a, \tilde{\theta}_a \rangle)$, since otherwise, using part (D), $P_a(\tilde{s}/\hat{s}_a) > P_a(\tilde{s})$. The argument closely parallels the argument of part (C).

We assume, to the contrary, that $u_a(\langle -\bar{p}(\tilde{s}) \cdot \tilde{x}, \tilde{x} \rangle) > u_a(\langle -\bar{p}(\tilde{s}) \cdot \tilde{\theta}_a, \tilde{\theta}_a \rangle)$, and reach a contradiction. Define $\hat{x} = \tilde{x}^- + (1-\beta)\tilde{x}^+$, where $\beta > 0$ is small enough that $u_a(\langle -(\bar{p}(\tilde{s}) \cdot \tilde{x}^+ + (1-\beta)\bar{p}(\tilde{s}) \cdot \tilde{x}^-), \hat{x} \rangle) > u_a(\langle -\bar{p}(\tilde{s}) \cdot \tilde{\theta}_a, \tilde{\theta}_a \rangle)$ and pick $\hat{\zeta}_a$ convex satisfying $\hat{\zeta}_a(\eta(\tilde{x}^+)) = (1-\beta)\|\tilde{x}^+\|$ and for $\eta \neq \eta(\tilde{x}^+)$, $\hat{\zeta}_a(\eta)\bar{p}(\tilde{s}) \cdot \eta > \hat{\zeta}_a(\eta(\tilde{x}^+))\bar{p}(\tilde{s}) \cdot \eta(\tilde{x}^+)$.

Define \hat{s}_a as follows: $\hat{p}_a = (1-\beta)\bar{p}(\tilde{s})$; $\hat{z}_a = \hat{x}^-$; $\hat{\zeta}_a$; $\hat{\rho}_a = \bar{p}(\tilde{s}) \cdot \tilde{x}^+ / \hat{\zeta}_a$. Observe that $\hat{s}_a \in S_a$ (since $\tilde{x} \in \beta_a(\bar{p}(\tilde{s}))$) and that $v(\hat{x}^+, \hat{s}_a) - \bar{p}(\tilde{s}) \cdot \hat{x}^+ = \beta\bar{p}(\tilde{s})\tilde{x}^+ > v(y, \hat{s}_a) - \bar{p}(\tilde{s}) \cdot y$, for all $y \in R_+$. Let $\hat{s} = \tilde{s} / \hat{s}_a$.

We need to show that $\theta \in F(\hat{s}) \Rightarrow \theta_a = \hat{x}$. As in part (C), for $\theta \in \Theta(\hat{s})$ such that $\theta_a \neq \hat{x}$, $\pi(\theta, \hat{s}) < 0 - \beta\bar{p}(\tilde{s})\tilde{x}^- + v(\hat{x}^+, \hat{s}_a) - \bar{p}(\tilde{s}) \cdot \hat{x}^+$. To show that $\theta \notin \Pi(\hat{s})$, we need to construct $\hat{\theta} \in \Theta(\hat{s})$ such that $\pi(\hat{\theta}, \hat{s}) = -\beta\bar{p}(\tilde{s}) \cdot \hat{x}^- + v(\hat{x}^+, \hat{s}_a) - \bar{p}(\tilde{s}) \cdot \hat{x}^+$. Define $\hat{\theta}$ as follows: $\hat{\theta}_a = \hat{x}$; for $\alpha \neq a$, $\hat{\theta}_\alpha^+ = \tau\tilde{\theta}_\alpha^+$ and, for all $\lambda \in L$, $\hat{\theta}_{\alpha\lambda}^- = \gamma^\lambda \tilde{\theta}_{\alpha\lambda}^-$, where $\gamma^\lambda = (\hat{x}_\lambda + \tau \sum_{\alpha \neq a} \tilde{\theta}_{\alpha\lambda}^+) / - \sum_{\alpha \neq a} \tilde{\theta}_{\alpha\lambda}^-$. $\sum_{\alpha \neq a} \hat{\theta}_\alpha^- = -(\hat{x} + \sum_{\alpha \neq a} \hat{\theta}_\alpha^+)$, so that $\hat{\theta}_\alpha^- = 0$; also note that for $\alpha \neq a$, $v(\hat{\theta}_\alpha^+, \hat{s}_\alpha) = \bar{p}(\tilde{s}) \cdot \hat{\theta}_\alpha^+$. By a manipulation identical to that in part (C), $\pi(\hat{\theta}, \hat{s}) = -\beta\bar{p}(\tilde{s}) \cdot \hat{x}^- + v(\hat{x}^+, \hat{s}_a) - \bar{p}(\tilde{s}) \cdot \hat{x}^+$. It remains to be shown that $\hat{\theta} \in \Theta(\hat{s})$; it suffices to show that

$$\gamma^\lambda \in [0, 1], \text{ for all } \lambda \in L, \text{ that is, } 0 \leq \hat{x}_\lambda + \tau \sum_{\alpha \neq a} \tilde{\theta}_{\alpha\lambda}^+ \leq - \sum_{\alpha \neq a} \tilde{\theta}_{\alpha\lambda}^-.$$

There are four cases to be checked:

$$\text{Case (i): } \tilde{\theta}_{\alpha\lambda} \geq \hat{x}_\lambda \geq 0: 0 \leq \hat{x}_\lambda + \sum_{\alpha \neq a} \tilde{\theta}_{\alpha\lambda}^+ \leq \tilde{\theta}_\lambda + \sum_{\alpha \neq a} \tilde{\theta}_{\alpha\lambda}^+ \\ = \sum_{\alpha \neq a} \tilde{\theta}_{\alpha\lambda}^- \quad (\text{since } \tilde{\theta} \in \Pi(\tilde{s}) \text{ and hence } \sum_{\alpha} \tilde{\theta}_{\alpha\lambda} = 0).$$

$$\text{Case (ii): } \hat{x}_\lambda > \tilde{\theta}_{a\lambda} \geq 0: 0 \leq \hat{x}_\lambda + \tau \sum_{\alpha \neq a} \tilde{\theta}_{\alpha\lambda}^+ < \tilde{\theta}_{a\lambda} + \epsilon + \sum_{\alpha \neq a} \tilde{\theta}_{\alpha\lambda}^- - \epsilon \\ \leq - \sum_{\alpha \neq a} \tilde{\theta}_{\alpha\lambda}^-.$$

For cases (iii) and (iv), both $\tilde{\theta}_{a\lambda}$ and \hat{x}_λ are nonpositive, so that $\hat{x}_\lambda + \tau \sum_{\alpha \neq a} \tilde{\theta}_{\alpha\lambda}^+ \leq - \sum_{\alpha \neq a} \tilde{\theta}_{\alpha\lambda}^-$, trivially. We need only check the left inequality.

$$\text{Case (iii): } 0 \geq \hat{x}_\lambda \leq \tilde{\theta}_{a\lambda}: \hat{x}_\lambda + \tau \sum_{\alpha \neq a} \tilde{\theta}_{\alpha\lambda}^+ > \tilde{\theta}_{a\lambda} + \sum_{\alpha \neq a} \tilde{\theta}_{\alpha\lambda}^+ - \delta \\ \geq \tilde{\theta}_{a\lambda} + \sum_{\alpha \neq a} \tilde{\theta}_{\alpha\lambda}^+ - \delta + \sum_{\alpha \neq a} \tilde{\theta}_{\alpha\lambda}^- + 2\delta > 0 \quad (\text{since } \sum_{\alpha} \tilde{\theta}_{\alpha\lambda} = 0 \text{ and, by construction of } \delta, \sum_{\alpha \neq a} \tilde{\theta}_{\alpha\lambda}^- + 2\delta < 0).$$

Case (iv): $0 \geq \tilde{\theta}_{a\lambda} \geq \hat{x}_\lambda: \hat{x}_\lambda + \tau \sum_{\alpha \neq a} \tilde{\theta}_{\alpha\lambda}^+ > \tilde{\theta}_{a\lambda} - \epsilon + \sum_{\alpha \neq a} \tilde{\theta}_{\alpha\lambda}^+ - \delta + \sum_{\alpha \neq a} \tilde{\theta}_{\alpha\lambda}^- + 2\delta > 0$ (since $\delta - \epsilon > 0$). This establishes that $\hat{\theta} \in \Theta(\hat{s})$, that $\theta \in \Pi(\hat{s}) \Rightarrow \theta_a = \hat{x}$, and that $p_a(\hat{s}) > p_a(\tilde{s})$, a contradiction, since $s \in N(E)$.

(F) Suppose that $s \in N(E)$ and all markets are s -thick.

Then $s \in W(E)$.

Proof of (F): To establish (F), we need only show that for all agents, there exists an allocation satisfying the condition of part (E). Let $\tilde{s} \in N(E)$ and pick $a \in A$ arbitrarily. For each $\lambda \in L$, there exists $b_\lambda, c_\lambda \in A$ and $\theta^\lambda, \theta^{\ell+\lambda} \in F(\tilde{s})$ such that $\theta_{b_\lambda}^\lambda > 0$ and $\theta_{c_\lambda}^{\ell+\lambda} > 0$ (from the definition of thickness). Define

$$\tilde{\theta} = \sum_{\lambda} (\theta^\lambda + \theta^{\ell+\lambda}) / 2\lambda. \quad \tilde{\theta} \in F(\tilde{s}). \quad \text{For each } \lambda \in L \quad \tilde{\theta}_{b_\lambda} > 0 \text{ and } \tilde{\theta}_{c_\lambda} < 0 \text{ so that } - \sum_{\alpha \neq a} \tilde{\theta}_\alpha^+ \ll \tilde{\theta}_a^- \leq \tilde{\theta}_a^+ \ll - \sum_{\alpha \neq a} \tilde{\theta}_\alpha^-. \quad \text{Therefore,}$$

by (E), $\tilde{\theta}_a^+ \in \xi_a(\bar{p}(\tilde{s}))$. Since, for all $\theta \in F(\tilde{s})$, $u_a(\langle \theta_0(\theta_a, \tilde{s}_a), \theta_a \rangle) = u_a(\langle \theta_0(\tilde{\theta}_a^+, \tilde{s}_a), \tilde{\theta}_a^+ \rangle)$, using (D), it follows that $\theta_a \in \xi_a(\bar{p}(\tilde{s}))$, for all $\theta \in F(\tilde{s})$. Since a was chosen arbitrarily, the same holds for all $\alpha \in A$. By (p5), $\sum_{\alpha}^{\theta} \alpha = 0$, for all $\theta \in F(\tilde{s})$. Finally, by part (B), \tilde{s} has the zero arbitrage profit property. Therefore $\tilde{s} \in W(E)$.

VIII.6 Proof of the Corollary

Corollary: If E satisfies A1, A2', A3-A6, then

$$s \in N(E) \Rightarrow s \in W(E, L(s)).$$

Proof of the Corollary: Parts (A)-(D) of VIII.5 hold independently of the thickness assumption. We can, further, modify the statements of parts (E) and (F), to (E') and (F'), stated below. The proofs of (E') and (F') exactly parallel the proofs of (E) and (F).

(E') Suppose $s \in N(E)$ and that, for some $a \in A$, there exists $\tilde{\theta} \in F(\tilde{s})$ such that, for all $\lambda \in L(s) - \sum_{\alpha \neq a} \tilde{\theta}_{\alpha\lambda}^+ \ll \tilde{\theta}_{a\lambda}^- \leq \tilde{\theta}_{a\lambda}^+ \ll - \sum_{\alpha \neq a} \tilde{\theta}_{\alpha\lambda}^-$. Then $\tilde{\theta}_a \in \xi_a(\bar{p}(s), L(s))$.

(F') Suppose $s \in N(E)$ and all s -open markets are s -thick. Then $s \in W(E, L(s))$.

Let $\tilde{s} \in N(E)$. From part (D) and assumption (A2') we conclude that $F(\tilde{s})$ is a singleton set. Let $F(\tilde{s}) = \{\tilde{\theta}\}$. We first show that "twins" realize the same net trade vector in the

Nash equilibrium. Let $a, b \in A_1$, for some $i \in I$ (i.e., a, b are "twins" see (5.25)), and suppose that $\tilde{\theta}_a \neq \tilde{\theta}_b$. Let $x = 1/2 \tilde{\theta}_a + 1/2 \tilde{\theta}_b$. We can assume, without loss of generality, that $u_a(\langle \theta_0(x, \tilde{s}_a), x \rangle) > p_a(\tilde{s})$, using (A2'). Define $\hat{\theta}$ as follows. $\hat{\theta}_a = x$, $\hat{\theta}_b = 0$, $\hat{\theta}_\alpha = 1/2 \tilde{\theta}_\alpha$, for $\alpha \neq a, b$. In the usual way (see the proofs of parts (C) and (E)), we can construct $\hat{s}_a \in S_a$ such that the singleton set, $F(\tilde{s}/\hat{s}_a)$ is arbitrarily close to $\{\hat{\theta}\}$, so that $p_a(\tilde{s}/\hat{s}_a) > p_a(\tilde{s})$, a contradiction, since $\tilde{s} \in N(E)$.

The proof is completed by noting that if a market is s -open, it is also s -thick. This is true since if there is one s -active seller (buyer) on either side of a market, then there are at least two (including the "twin" of the first).

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