ON THE EXISTENCE OF RATIONAL EXPECTATIONS EQUILIBRIUM

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Abstract: We prove a theorem on the existence of general economic equilibrium under uncertainty when agents learn from prices and expectations are required to satisfy a rationality condition.

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1. INTRODUCTION

We shall present a theorem concerning the existence of competitive equilibrium for markets in which agents use both private information and prices to estimate the uncertain environment of an economy. Because each agent's demand reflects his own private information, private information influences market prices. As a result, each agent, when he sees prices, obtains information beyond his own private information about the economic environment. The theory of rational expectations equilibrium departs from the standard Arrow-Debreu-McKenzie competitive theory in that agents are assumed to form a model of the relationship between prices, their own private information, and the particular state of the environment. In the words of Roy Radner, equilibrium requires "that their models not be obviously controverted by the observations of the market." We will prove the following result. For each n-agent economy in a very general class, there exists an n-tuple of models such that, when each agent believes his model and maximizes expected utility, the joint distributions of prices, private information, and states that emerges supports the n-tuple of models.

The theorem requires that there be a random variation in demand that is naturally modeled as unrelated to private information, prices, and the environment. Other than this special feature, the result is extremely general: it covers essentially all strictly concave utility functions, arbitrary distributions of the environment variable, and situations in which the dimension of the environment variable is large relative to the number of commodities. One of the happy consequences of this generality is that it extends the existence theory for rational expectations general equilibrium to cases in which prices cannot hope to convey the union of the private information possessed by the agents in the economy. This fits nicely with our intuition. Although prices are useful indicators of other agents' private information, one feels that the set of possible values of private information is larger than the set of prices, and so each equilibrium price is typically associated with a variety of private information values.

Let's now be more precise about the meaning of the statement that the joint distribution of prices, private information, and states that emerges supports an n-tuple of models. Suppose agent j has a model, which is formalized as a probability measure μ_j on the product of the price simplex, the set of states of the environment, and the jth person's signal set S_j. He observes a signal s_j from S_j, which is interpreted as his private information, and he is presented with an opportunity to trade at a price p. The agent should then compute the conditional distribution (relative to the model μ_j) of states, given s_j and p, and maximize expected utility with respect to this conditional distribution. If an intelligent agent sees his model controverted, then he will change his model. In the present context, the change in an agent's model is accompanied by a change in his demand function. As a consequence, stationary preferences can only be assured to generate stationary demands when each agent's model is supported. Since demand functions must be stationary for prices and allocations to be stationary, an economy can only be in steady state equilibrium when each agent's model is supported. This is the defining characteristic of rational expectations equilibrium.

But note that the foundation of statistical inference depends on the idea that agents will reject correct models a nonzero fraction of the time, since they can observe only a finite number of samples. Furthermore, agents (as statisticians) understand this. The standard definition of rational expectations equilibrium requires that there be models $\mu_1,\dots,\mu_n$ such that, if agents believe them, then the actual distribution of observed outcomes, call it $\tau(\mu)$, will be equal to μ. This ignores the difficulty that correct models will be rejected based on finite samples, and it is equivalent to saying that the "expected value" of the m observation finite samples of $\tau(\mu)$ will be equal to μ.[1] Theorem 2 is based on an alternative, but very closely related formulation.

As before, each agent is endowed with a model. They use their models to formulate their demands during a possibly large but finite number (say m) of "plays" of the economy. After m plays an agent will almost surely not see any price twice (especially given the postulated exogenous randomness in demand), and there will be an infinite number of price-signal pairs for which he has no experience concerning the conditional distribution of states. If he were asked to formulate a model of the joint distribution of signals, prices, and states based on an empirical distribution υ_j, then he would most likely do so by averaging the distribution of observations over neighborhoods. In this spirit, for each j, we take (as a given of the economy) a function g_j that is used to smooth m-observation empirical distributions. It is natural to think of g_j's as dependent on m. Since m is finite, the

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[1] By "expected value" of the m observation sample, we mean the measure whose value on each Borel set A is the expected value of the empirical distributions on A.

m-observation empirical distributions υ_j are random, and thus their smoothings, denoted by $\sigma_j(\upsilon_j)$ are random as well. Formally, $\sigma_j(\upsilon_j)$ is taken to be a convolution. The convolution process is a natural way to draw a continuous model that is arbitrarily close to υ_j and is defined for all prices and signals. But in fact, any continuous and linear method for drawing continuous models defined for all prices and signals is adequate for our purpose.[2]

With or without smoothing, finite samples cannot be expected to reproduce the model exactly, because they are random. The standard idealization requires that the distribution of outcomes of the economy reproduce the model on average. Our idealization requires instead that the smoothed empirical distributions reproduce the model on average. Specifically, Theorem 2 establishes the existence of models $\mu_1,\ldots,\mu_n$ such that if agents believe them, then the expected value of the smoothed m-observation finite samples of $\tau(\mu)$ will be equal to μ . This formally captures the idea that if agents use the given models and formulate new models from m-period empirical samples in a reasonable manner, then the given models will be reproduced.

Before beginning the formal analysis we should point out the close relationship between equilibrium defined here and the definition of rational expectations equilibrium that has usually been employed. In Corollary 3, we show that for any economy (with

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[2] The convolution process involves averaging over the available data. This procedure is used in drawing contour maps based on a finite number of observations.

exogenous random variation in demand), and for any positive ϵ, there exists an n-tuple of assigned models, so that the average unsmoothed m-period empirical distributions are within ϵ of the assigned models. In Note 2 an equilibrium of this type is illustrated for an Edgeworth Box type example due to David Kreps [5] that is known to have no equilibrium in the standard idealization.

2. RESULTS

There are k commodities, indexed by i. Thus, a commodity bundle is a vector $x \in \mathbb{R}_+^k$. There are n individuals, indexed by j. They have initial endowments $e_1, \ldots, e_n \in \mathbb{R}_+^k$. We assume there is in society a positive amount of each commodity, i.e., $\sum_{j=1}^{n} e_j >> 0$.

Individuals possess state-dependent utility functions. Thus, there is a state space Ω, a compact metric space. Individual j has a utility function $u_j : \mathbb{R}_+^k \times \Omega \to \mathbb{R}$ which is (i) jointly continuous, (ii) for each $\omega \in \Omega$, $u_j(x,\omega)$ is strictly concave in x, and (iii) $\partial u_j / \partial x$ is jointly continuous in x and ω, and $\partial u_j / \partial x >> 0$. Observe that, for each $x \in \mathbb{R}_+^k$, $\sup_\omega |u_j(x,\omega)| < \infty$ by continuity and the compactness of Ω.

Next, we must specify the information structure available to each agent. Agent j observes a signal in his signal space S_j, which is assumed to be a compact metric space. $S = S_1 \times \ldots \times S_n$ is a compact metric space, with the obvious metric. The specification of the information is completed by specifying a Borel probability measure P' on $\Omega \times S$, which gives the (objective) joint distribution of the states and signals. Thus, the information structure is totally exogenous.

Each agent makes use of the information in his signal. In addition, the information possessed by the other agents will be reflected to some extent in the Walras equilibrium price that emerges. Agents use their information by referring to a model of the joint distribution of prices, states, and signals. The set of models for person j is M_j = {Borel probability measures on $\Delta \times \Omega \times S_j$} where Δ is the price simplex $\{p \in \mathbb{R}^k_+ : p \geq 0 , \sum_{i=1}^{k} p_i = 1\}$, and $M = M_1 \times \ldots \times M_n$. If agent j has a model $\mu_j \in M_j$, if he observes a signal $s_j \in S_j$, and if he is presented with an opportunity to trade at a price p , then he is assumed to computer the conditional distribution (relative to the model μ_j) of ω , given s_j and p , and maximize expected utility with respect to this conditional distribution.

Next we define the smoothing process that is used to draw new models from empirical distributions. Let P be the product of normalized Lebesgue measure on Δ with P' . Thus, P is a Borel probability measure on $\Delta \times \Omega \times S$. Let P_j be the marginal probability of P on $\Delta \times \Omega \times S_j$. Let $g_j : \Delta^2 \times \Omega^2 \times S_j^2 \to (0,\infty)$ be jointly continuous in its arguments, and satisfy for all $\bar{p} \in \Delta$, $\bar{s}_j \in S_j$, $\bar{\omega} \in \Omega$,

$$\int_{\Delta\times\Omega\times S_j} g_j(p,\bar{p},\omega,\bar{\omega},s_j,\bar{s}_j)\, dP_j(p,\omega,s_j) = 1 .$$ We then form a new

model $\sigma_j(\mu_j) \in M_j$ whose density function (with respect to P_j) is defined to be $g_j * \mu_j(p,\omega,s_j) = \int_{\Delta\times\Omega\times S_j} g_j(p,\bar{p},\omega,\bar{\omega},s_j,\bar{s}_j)d\mu_j(\bar{p},\bar{\omega},\bar{s}_j)$.

The construction of $g_j * \mu_j$ is essentially a convolution, a familiar operation in analysis. We shall see that $g_j * \mu_j$ is continuous. As we argued in the introduction, the convolution process is a natural way to draw a close continuous model, defined for all prices and signals, from an empirical sample (obtained from a finite number of observations in $\Delta\times\Omega\times S$).

Suppose now each agent has a model $\mu_j \in \sigma_j(M_j)$, and sees a signal $s_j \in S_j$. Let $d_j(p,\mu_j,s_j)$ be the vector in $\mathbb{R}^k_+$ maximizing expected utility over the budget set $\{x: p\cdot x \leqq p \cdot e_j\}$, with respect to the conditional distribution of ω given p and s_j , where the conditional distribution is calculated using the model μ_j . We shall show that d_j is uniquely defined. Let $d: \Delta\times M\times S \to \mathbb{R}^k$ be defined by $d(p,\mu,s) = \sum_{j=1}^{n} d_j(p,\mu_j,s_j) - e_j$, the market excess function, and let $\Phi = \{$continuous functions $f: \mathrm{Int}(\Delta) \to \mathbb{R}^k$ satisfying, (i) $p\cdot f(p) = 0$, and (ii) $p_i \to 0$ for some $i \Rightarrow |f(p)| \to \infty\}$. We shall show that $d(\cdot,\mu,s) \in \Phi$ for all $\mu \in \sigma(M)$, $s \in S$.

We assume that there is a random disturbance term in the demand. This is formalized as a continuous function $C: \{d(\cdot,\mu,s): \mu\in M, s\in S\} \to M_\Phi$, where Φ is endowed with the topology of uniform convergence on compact sets and M_Φ is the set of Borel probability measures on Φ, endowed with the topology of weak convergence. Let $\Phi' = \bigcup_{\mu,s}$ support $(C(d(\cdot,\mu,s)))$.

Any $f \in \Phi$ has a Walras equilibrium price, i.e., $p \in \Delta$ such that $f(p) = 0$. We suppose that the economy settles on equilibrium prices in a consistent fashion. In other words, there is a measurable map $W: \Phi' \to M_\Delta$, where M_Δ is the set of Borel probability measures on Δ, endowed with the topology of weak convergence with the property that $W(\varphi)(\{p: \varphi(p) = 0\}) = 1$. A special case of this is a measurable selection from the equilibrium correspondence, i.e., a map $\bar{W}: \Phi \to \Delta$ such that $\varphi(\bar{W}(\varphi)) = 0$. We assume that C and W satisfy the following richness condition:

for all $\mu, s, C(d(\cdot,\mu,s))(\{f \in \Phi': W$ is discontinuous at $f\}) = 0$.

In other words, the disturbed demand avoids the points of discontinuity of W, with probability one. A simple example of C and W satisfying this condition is given in Note 7.

Suppose we are given a profile of models $\mu \in \sigma(M)$. The process defined above associates with each play of the economy a probability distribution on equilibrium prices, and hence the process generates a joint distribution on $\Delta \times \Omega \times S_j$, for each j. This joint distribution, denoted $\tau_j(\mu)$, is the true model relating prices, states and j's signal, assuming that every trader $\bar{j}$ believes the model $\mu_{\bar{j}}$. We have thus determined a map $\tau: \sigma(M) \to M$ defined by $\tau(\mu) = (\tau_1(\mu), \dots, \tau_n(\mu))$.

The above material has really been an extended definition of the process by which the economy determines prices. Some of the steps of the definition need to be justified by lemmas which we shall give below. But we can now state the results.

<u>Proposition 1</u>: There exists $\mu \in M$ such that $\sigma(\tau(\mu)) = \mu$.

In other words, if each agent j believes the model μ_j, then the smoothed joint distribution of the outcomes of the economy on $\Delta \times \Omega \times S_j$ will be μ_j.

<u>Theorem 2</u>: There exists $\mu \in M$ such that, for each j, μ_j is the expected value of the smoothing of the m-sample empirical distribution from $\tau(\mu)_j$. Formally, let $\Lambda_j = (\Sigma \times \Omega \times S_j)^m$, $\gamma_j = (\tau(\mu)_j)^m$. Given $\lambda_j \in \Lambda_j$, define the associated empirical measure $\upsilon_j^{\lambda_j}(B) = |\{i:(\lambda_j)_i \in B\}|/m$, for B Borel, $B \subset \Delta \times \Omega \times S_j$. For any Borel set B,

$$\int_{\Lambda_j} \sigma_j(\upsilon_j^{\lambda_j})(B)\,d\gamma_j = \mu_j(B) .$$

<u>Corollary 3</u>: Let ρ_j be the Prohorov metric [1] on M_j, $\rho(\mu,\upsilon) = \sum_{j=1}^{n} \rho_j(\mu_j,\upsilon_j)$ for $\mu,\upsilon \in M$. Given any $\varepsilon > 0$, there is a profile of models $\mu \in M$ such that $\rho(\tau(\mu),\mu) < \varepsilon$.

Before we begin the proofs, a few words about the role of some key assumptions are in order. First, it is important to note that τ is not defined for arbitrary $\mu \in M$, since the conditional distribution has no natural definition on prices outside the support of μ. When such conditional distributions are defined they may not be continuous in ρ. Consequently, an excess demand function generated by an arbitrary μ may have no equilibrium price. The mathematical importance of the smoothing operation σ derives from the fact that τ is defined for each $\mu \in \sigma(M)$, i.e., excess demand functions generated by $\mu \in \sigma(M)$ always have at least one equilibrium price. Since M is a convex and compact subset of a topological vector space, it is a fixed point space; i.e., every continuous function on M has at least one fixed point. The idea of the proof of Proposition 1 is to show that $\tau \circ \sigma: M \to M$ is continuous. Unfortunately this requires an additional assumption, since for given signals, small changes in a model can lead to a large change in the equilibrium price, with the result that $\tau \cdot \sigma$ is not continuous. The discontinuity is a consequence of the fact that the Walras correspondence is not lower hemi-continuous, and it is remedied here by the introduction of a random disturbance in demand, which is most naturally modeled as unrelated to prices, private information, and the state of the environment.

Once it is established that $\tau \circ \sigma: M \to M$ is continuous, we know that there exists υ such that $\tau(\sigma(\upsilon)) = \upsilon$, and hence there exists μ such that $\sigma(\tau(\mu)) = \mu$, which is the conclusion of Proposition 1. Theorem 2 and Corollary 3 follow easily.

We now begin the proof. Each lemma is quite routine. In outline, we need to verify that M is a fixed point space and that everything is continuous in everything else; in particular, $\tau \sigma: M \to M$ is continuous.

Lemma 4: (M, ρ) is a fixed point space, i.e., if $f: M \to M$ is continuous, then there exists $\mu \in M$, $f(\mu) = \mu$.

Proof: Let F_j = {finite signed Borel measures on $\Delta \times \Omega \times S_j$}, with the weak-star topology. All $\mu_j \in F_j$ are regular [1]. By Alaoglu's Theorem [3], M_j is a compact subset of F_j. It is clearly convex. Hence M is a compact convex subset of $F_1 \times \cdots \times F_n$, and so is a fixed point space by Schauder's Fixed Point Theorem [3].

Lemma 5: Fix g_j satisfying the assumptions listed above. Then (i) for all $\mu_j \in M_j$ and all $(p,\omega,s_j) \in \Delta\times\Omega\times S_j$,

$$0 < g_j * \mu_j(p,\omega,s_j) \le \sup|g_j(p,\bar{p},\omega,\bar{\omega},s_j,\bar{s}_j)|,$$

(ii) $\sigma_j(\mu_j) \in M_j$, and

(iii) for all $\epsilon > 0$, there exists $\delta > 0$ such that for all $\mu_j \in M_j$, $|g_j*\mu_j(p,\omega,s_j) - g_j*\mu_j(p',\omega',s_j')| <$ whenever p' is within δ of $\bar{p}$, ω within δ of $\bar{\omega}$, and s_j within δ of $\bar{s}_j$.

Proof

(i) is trivial. To show (ii), we need only show $\sigma_j(\mu_j)(\Delta\times\Omega\times S_j) = 1$, which follows from Fubini's Theorem. To prove (iii), fix $\epsilon > 0$. Since g_j is continuous on a compact metric space, we can find $\delta > 0$ such that

$$|g_j(p,\bar{p},\omega,\bar{\omega},s_j,\bar{s}_j) - g_j(p',\bar{p},\omega',\bar{\omega},s_j',\bar{s}_j)| < \epsilon,$$

if p,ω,s_j are all within δ of p',ω',s_j'. When this condition is satisfied,

$$|g_j*\mu_j(p,\omega,s_j) - g_j*\mu_j(p',\omega',s_j')|$$

$$\le \int_{\Delta\times\Omega\times S_j} |g_j(p,\bar{p},\omega,\bar{\omega},s_j,\bar{s}_j)-g_j(p',\bar{p},\omega',\bar{\omega},s_j',\bar{s}_j)|\,d\mu_j(\bar{p},\bar{\omega},\bar{s}_j) < \epsilon$$

Lemma 6: $\sigma : M \to M$ is continuous.

Proof: It is enough to show that $\sigma_j : M_j \to M_j$ is continuous. Since M_j is metric, it is enough to consider sequences $\mu_j^\ell \to \mu_j \in M_j$. For fixed $p,s_j,\omega,g_j(p,\bar{p},\omega,\bar{\omega},s_j,\bar{s}_j)$ is a bounded continuous function of $\bar{p}$, $\bar{\omega}$, $\bar{s}_j$. Thus, $g_j*\mu_j^\ell(p,\omega,s_j) \to g_j*\mu_j(p,\omega,s_j)$. Hence for any $h \in C(\Delta\times\Omega\times S_j)$, $h(p,\omega,s_j)g_j*\mu_j^\ell(p,\omega,s_j) \to h(p,\omega,s_j)g_j*\mu_j^\ell(p,\omega,s_j)$ for all p,ω,s_j.

$$\int_{\Delta\times\Omega\times S_j} h(p,\omega,s_j)g_j*\mu_j^{\ell}(p,\omega,s_j)dP_j \to \int_{\Delta\times\Omega\times S_j} h(p,\omega,s_j)g_j*\mu_j(p,\omega,s_j)dP_j$$

by the Dominated Convergence Theorem. Thus, $\sigma_j(\mu_j^{\ell}) \to \sigma_j(\mu_j)$.

<u>Lemma 7</u>: Let $\bar{u}_j(x,p,\mu_j,s_j) = \int_{\Omega} u_j(x,\omega)g_j*\mu_j(p,\omega,s_j)dP\,(\omega)$,

where P_{Ω} is the marginal of P on Ω .

Then $\bar{u}_j$ is jointly continuous in its arguments.

For fixed p , μ_j, s_j, $\bar{u}_j$ is strictly concave and C^1 in x . For any compact set $K \subset \mathbb{R}^k_+$, there is a compact set $L \subset \mathrm{Int}(\mathbb{R}^k_+)$ such that $\frac{\partial\bar{u}_j}{\partial x} / \left|\frac{\partial\bar{u}_j}{\partial x}\right| \in L$ for all $x \in K$ and all p, μ_j, s_j .

<u>Proof</u>: Suppose $x^{\ell} \to x$, $p^{\ell} \to p$, $\mu_j^{\ell} \to \mu_j$ and $s_j^{\ell} \to s_j$. By Lemma 5 (iii),

$$\sup_{\omega}|g_j*\mu_j^{\ell}(p^{\ell},\omega,s_j^{\ell}) - g_j*\mu_j^{\ell}(p,\omega,s_j)| \to 0 .$$

By Lemma 6, for each ω

$$|g_j*\mu_j^{\ell}(p,\omega,s_j) - g_j*\mu_j(p,\omega,s_j)| \to 0 .$$

Combining these, we get, for each ω

$$|g_j*\mu_j^{\ell}(p^{\ell},\omega,s_j^{\ell}) - g_j*\mu_j(p,\omega,s_j)| \to 0 .$$

Also, $\sup_{\ell,\omega} |u_j(x^{\ell},\omega)| < \infty$, and $\sup_{\omega}|u_j(x^{\ell},\omega) - u_j(x,\omega)| \to 0$. Therefore, since the integrand is bounded and converging everywhere, $\bar{u}_j(x^{\ell},p^{\ell},\mu_j^{\ell},s_j^{\ell}) \to \bar{u}_j(x,p,\mu_j,s_j)$, proving joint continuity.

Fix p, μ_j, s_j . Then

$$\partial\bar{u}_j/\partial x = \partial/\partial x \int_{\Omega} u_j(x,\omega)\, g_j*\mu_j(p,\omega,s_j)dP_{\Omega}(\omega)$$

$$= \int_{\Omega} \partial/\partial x\, u_j(x,\omega)\, g_j*\mu_j(p,\omega,s_j)dP_{\Omega}(\omega).$$

Hence $\partial\bar{u}_j/\partial x$ exists and is continuous. Strict concavity follows from Jensen's Inequality.

Now suppose x ranges over a compact set K. Because $\partial u_j/\partial x$ is positive and continuous in x and ω, it is contained in a compact set L' of strictly positive vectors by the compactness of $K\times\Omega$. Hence $(\partial\bar{u}_j/\partial x)/|\partial\bar{u}_j/\partial x|$ is also contained in a compact set.

<u>Proposition 8</u>: (i) d_j is uniquely defined and gives a continuous mapping from $(\text{Int }\Delta)\times M_j\times S_j$ to $\mathbb{R}^k_+$,

(ii) if $p_i \to 0$ for some component i, then $|d_j(p,\mu_j,s_j)| \to \infty$, and

(iii) for all $\mu\in M$, $s\in S$, $d(\cdot,\mu,s)\in\Phi$.

<u>Proof</u>: $d_j(p,\mu_j,s_j)$ maximizes expected utility, which is given by

$$\bar{u}_j(x,p,\mu_j,s_j)/\int_\Omega g_j*\mu_j(p,\omega,s_j)dP_\Omega(\omega) .$$

Since the denominator is finite and independent of x, $d_j(p,\mu_j,s_j)$ is that x in the budget set which maximizes $\bar{u}_j(x,p,\mu_j,s_j)$. Hence, (i) is a standard result. Conclusion (ii) follows from the last sentence of Lemma 7, by standard arguments. Conclusion (iii) follows from (i), (ii), and the definition of Φ.

<u>Lemma 9</u>: Define $\Psi: M\times S \to \Phi$ by $\Psi(\mu,s) = d(\cdot,\mu,s)$. Then Ψ is jointly continuous.

<u>Proof</u>: It is enough to show that, if K is a compact subset of Int(Δ), then $\sup_{p\in K} |d_j(p,\mu_j^\ell,s_j^\ell) - d_j(p,\mu_j,s_j)| \to 0$ whenever $\mu_j^\ell \to \mu_j$ and $s_j^\ell \to s_j$. But d_j is jointly continuous on the compact metric space $K\times M_j\times S_j$, hence uniformly continuous, so the statement follows.

We are now in a position to define carefully the map τ, which gives the distribution of the outcomes of the economy when agents use the profile μ.

τ will be defined on the range of σ; i.e., $\tau:\sigma(M) \to M$. $\tau(\sigma(\mu))_j$ is the joint distribution on $\Delta \times \Omega \times S_j$ obtained when the economy is run over all the states, signals, and disturbances. Formally, $\tau(\sigma(\mu))_j$ is the Borel measure on $\Delta \times \Omega \times S_j$ whose values on cylinder sets $A \times B \times D$ (where $A \subset \Delta$, $b \subset \Omega$, $d \subset S_j$ are Borel sets) is

$$\tau(\sigma(\mu))_j(A \times B \times D) = \int_{B \times \hat{D}} C(\Psi(\mu,s))(W^{-1}(A))\, dP'(\omega,s),$$

where $\hat{D} = \{s \in S : s_j \in D\}$. Note that this is well-defined, because Ψ really depends on $\sigma(\mu)$, not μ.

<u>Proposition 10</u>: $\tau \circ \sigma : M \to M$ is continuous.

<u>Proof</u>: Suppose $\mu^\ell \to \mu$. We need to show $\tau(\sigma(\mu^\ell)) \to \tau(\sigma(\mu))$. Suppose $F \in C(\Delta \times \Omega \times S_j)$. Then

$$\int_{\Delta \times \Omega \times S_j} F d\tau(\sigma(\mu^\ell))_j$$

$$\int_{\Omega \times S} \int_{\Phi} \int_{\Delta} F(p,\omega,s_j) dW(\varphi) dC(\Psi(\mu^\ell,s)) dP'(\omega,s).$$

Fix s. By Lemma 9, $\Psi(\mu^\ell,s) \to \Psi(\mu,s)$. Since C is continuous, $C(\Psi(\mu^\ell,s)) \to C(\Psi(\mu,s))$. $\int_\Delta F(p,\omega,s_j) dW(\varphi)$ is a continuous function of $W(\varphi)$, which in turn is continuous at almost all φ (with respect to the measure $C(\Psi(\mu,s))$. Hence (Billingsley [1]),

$$\int_{\Phi} \int_{\Delta} F(p,\omega,s_j) dW(\varphi) dC(\Psi(\mu^\ell,s))$$

$$\to \int_{\Phi}\int_{\Delta} F(p,\omega,s_j)dW(\varphi)dC(\Psi(\mu,s)).$$

Since these integrals are bounded by the supremum of F,

$$\int_{\Omega\times S}\int_{\Phi}\int_{\Delta} F(p,\omega,s_j)dW(\varphi)dC(\Psi(\mu^{\ell},s))dP'(\omega,s)$$
$$\to \int_{\Phi}\int_{\Delta} F(p,\omega,s_j)dW(\varphi)dC(\Psi(\mu,s)).$$

by the Dominated Convergence Theorem.

Thus, $\tau(\sigma(\mu^{\ell}))_j \to \tau(\sigma(\mu))_j$, so $\tau(\sigma(\mu^{\ell})) \to \tau(\sigma(\mu))$.

<u>Proof of Proposition 1</u>: $\tau\ \sigma$ is continuous and M is a fixed point space, so there exists $\upsilon \in M$ such that $\tau(\sigma(\upsilon)) = \upsilon$. Let $\mu = \sigma(\upsilon)$.

<u>Proof of Theorem 2</u>: Let

$$\lambda_j = ((\overset{\lambda}{p}{}_1^j, \overset{\lambda}{\omega}{}_1^j, (\overset{\lambda}{s}{}_j^j)_1), \ \ldots\ , (\overset{\lambda}{p}{}_m^j, \overset{\lambda}{\omega}{}_m^j, (\overset{\lambda}{s}{}_j^j)_m)) .$$

$$\int_{\Lambda_j} \sigma_j(\overset{\lambda}{\upsilon}{}_j^j)(B)d\gamma_j$$
$$= \int_{\Lambda_j}\int_B d\sigma_j(\overset{\lambda}{\upsilon}{}_j^j)d\gamma_j$$
$$= \int_{\Lambda_j}\int_B\int g_j(p,\bar{p},\omega,\bar{\omega},s_j,\bar{s}_j)d\overset{\lambda}{\upsilon}{}_j^j(\bar{p},\bar{\omega},\bar{s}_j)dP_j(p,\omega,s_j)d\gamma_j$$
$$= \int_{\Lambda_j}\int_B \frac{1}{m}\sum_{i=1}^{m} g_j(p,\overset{\lambda}{p}{}_i^j,\omega,\overset{\lambda}{\omega}{}_i^j,s_j(\overset{\lambda}{s}{}_j^j)_i)dP_j(p,\omega,s_j)d\gamma_j$$
$$= \int_B\int_j \frac{1}{m}\sum_{i=1}^{m} g_j(p,\overset{\lambda}{p}{}_i^j,\omega,\overset{\lambda}{\omega}{}_i^j,s_j,\overset{\lambda}{s}{}_j^j)d\gamma_j\ dP_j(p,\omega,s_j)$$

$$= \int_B \frac{1}{m} \sum_{i=1}^{m} \int_{\Delta\times\Omega\times S_j} g_j(p,\bar{p},\omega,\bar{\omega},s_j,\bar{s}_j)\,d\tau(\mu)_j \; dP_j(p,\omega,s_j)$$

$$= \int_B g_j * \tau(\mu)_j \; dP_j(p,\omega,s_j)$$

$$= \sigma_j(\tau(\mu))(B)$$

$= \mu_j(B)$, by Proposition 1.

<u>Proof of Corollary 3 (Sketch)</u>: Let A_j be the support of P_j in $\Delta\times\Omega\times S_j$, and let D be the diagonal of A_j, $D = \{(p,p,\omega,\omega,s_j,s_j): (p,\omega,s_j) \in A_j\}$. Thus, for any $\delta > 0$, the P_j-measure of the ball with center $(\bar{p},\bar{\omega},\bar{s}) \in A_j$ and radius δ is uniformly bounded away from 0. Thus, given $\varepsilon > 0$, we can construct a positive continuous g_j such that for all $\bar{p}$, $\bar{s}_j$, $\bar{\omega}$,

$$\int_{\Delta\times\Omega\times S_j} g_j(p,\bar{p},\omega,\bar{\omega},s_j,\bar{s}_j)\; dP_j(p,\omega,s_j) = 1 ,$$

and such that, for all $(\bar{p},\bar{\omega},\bar{s}_j) \in A_j$, $g_j(p,p,\omega,\omega,s_j,s_j) < \varepsilon/n$ unless p,ω,s are all within ε/n of $\bar{p}$, $\bar{\omega}$, $\bar{s}_j$.

Form σ using these g_j's. By Proposition 1, there exists μ^{ε} such that $\tau(\sigma(\mu^{\varepsilon})) = \mu^{\varepsilon}$. But observe that the support of μ^{ε}_j is contained in A_j (indeed, the marginals of μ^{ε}_j and P_j on $\Omega\times S_j$ are equal). It follows that $\rho(\mu^{\varepsilon}_j,\sigma(\mu^{\varepsilon}_j)) < \varepsilon/n$. Letting $\upsilon = \sigma(\mu^{\varepsilon})$, we see that $\rho(\tau(\upsilon),\upsilon) = \rho(\mu^{\varepsilon},\sigma(\mu^{\varepsilon})) < \varepsilon$.

3. NOTES

The first two notes are based on the following example, which is due to David Kreps [5].

Example: There are two agents and two commodities. $\Omega = \{H,T\}$, the two states occurring with equal probability 1/2. The endowments are (1,0) and (0,1). The utility functions are

$$u_1((x,y),\omega) = \begin{cases} x^{1/4}y^{1/2} & \text{if } \omega = H \\ x^{1/4}y^{1/4} & \text{if } \omega = T \end{cases}$$

$$u_2((x,y),\omega) = \begin{cases} x^{1/2}y^{1/4} & \text{if } \omega = H \\ x^{1/4}y^{1/4} & \text{if } \omega = T \end{cases}$$

S_1 is a one-point space, so agent 1 receives no information from his signal. $S_2 = \{H,T\}$. 2's signal is perfectly correlated with the state ω.

There is no standard rational expectations equilibrium, either fully or partially revealing, for this economy. Assume that 1 has a model of the economy. Given this model, 1's excess demand depends only on price; 2's demand however depends also on the state. If p clears the market when $\omega = H$, then p can't clear the market when $\omega = T$. Hence, the sets of prices clearing the market in the two states are disjoint, and the equilibrium would of necessity be completely revealing.

If 1 now knows the state, the only market clearing price is (1/2,1/2), independent of whether $\omega = H$ or $\omega = T$, a contradiction.

Note 1: One can introduce exogenous noise into the above example and still have non-existence of standard rational expectations equilibrium. Observe that there exists $\delta > 0$ such that, for all p, 2's demand shifts by at least δ when the state changes. Hence, if the noise is modeled as an independent shift in the demand for the first commodity, which is picked from $(-\delta/2, \delta/2)$ with a continuous density that is positive at zero, then the sets of prices clearing the market in the two states are again disjoint, and a contradiction is obtained as before.

Note 2: Here we observe that there exist equilibria for the above example, in the sense promised by Corollary 3, even without the exogenous noise. Let 1 have the following model:

if $p_x \geq p_y$, then the conditional probability of H is $1-\varepsilon$;

if $p_x < p_y$, the conditional probability of T is $1-\varepsilon$.

A simple calculation shows that for any small positive ε, this leads to an equilibrium. When 1 sees $p_x > p_y$, he believes the probability of H is $1-\varepsilon$, when in fact it is 1. It is clear that, for small ε, it would take agent 1 a very long time to reject his model.

Note 3: A sequel to this paper will study the existence of rational expectations equilibrium when the models that agents draw from their observations are restricted to some prespecified set, such as the class of linear models.

NOTE 4:

The set of models that generate, for each signal, a continuous excess demand function is not compact, and this fact is important for understanding the problems associated with obtaining a general existence theorem for rational expectations equilibrium. Our approach depends upon using a smoothing procedure to guarantee that the models that agents draw from empirical distributions are a compact subset of the models that generate continuous excess demand functions. Another method for obtaining a compact set of models to work with is to restrict prices to the vertices of a simplicial subdivision of the price simplex. We have explored a formulation in which, for each play of the economy, the economy settles on a simplex σ of the subdivision, such that 0 belongs to the convex hull of the excess demands at the vertices of σ. The existence of such a simplex is guaranteed by the vector labelling version of the Scarf fixed point algorithm [7]. Agents are then assigned different vertices, with the interpretation that there is some price dispersion. With a continuum of agents, we can assign the prices in such a way that markets exactly clear; with a finite number, markets will only approximately clear. We assume there is exogenous noise; the richness condition is simpler than the one we've used in this paper. In this framework, one gets an exact rational expectations equilibrium, without smoothing. However, there are two serious drawbacks to this approach. First, the process by which agents are assigned different prices is not modeled, and to make markets clear this assignment must be chosen with great care. Second, the method depends on the assumption that agents see only their own price, but not the distribution of prices that other agents are receiving concurrently. Agents could learn more by surveying other agents to obtain information on this distribution.

Note 5: A significant feature of our formulation is that the information an agent possesses will not be conveyed unless it alters his demand function. This is in contrast to most of the rational expectations literature, in which such information could be (indeed, typically is) passed on. As an example, suppose an agent knows the state exactly, but his utility is independent of the state. Then his demand would not depend on his signal, and hence there is no market mechanism which could encode his information into the price. We feel our formulation models this situation correctly, by ensuring the information is not conveyed through the prices.

Note 6: The literature on rational expectations equilibria is quite extensive. For a recent survey, see Radner [6].

Note 7: In this note, we give a simple example of a disturbance C and price selection W satisfying the assumptions in the paper.

We suppose that $u_j(x,\omega)$ is C^2 in x, and that the Hessian in x is negative definite and jointly continuous in x and ω. We suppose further that all g_j's are C^1 in p_j and $\bar{p}_j$, with these derivatives jointly continuous in all the arguments. Under these hypotheses, it is not hard to show that $d(p,\mu,s)$ is C^1 in p, for all μ and s, and that $\frac{\partial d}{\partial p}$ is jointly continuous in p, μ, s.

We now define a uniform translation of an excess demand function. Specifically, $T:\Phi\ \mathbb{R}^k \to \Phi$ is defined by

$$T(\varphi,y)(p) = \varphi(p)+y - \frac{p\cdot y}{p\cdot p}\,p\ .$$

Note that $(y - \frac{p\cdot y}{p\cdot p}\,p)$ is the orthogonal projection of y onto $\{x: p\cdot x = 0\}$. $T(\varphi,y) \in \Phi$. Suppose γ is any probability measure on $\mathbb{R}^k$ which is absolutely continuous with respect to Lebesgue measure. Our dispersion term will be composed of translations of φ by y, with γ determining the probability of translating by y. In other words,

$$C(\varphi)(A) = \gamma(\{y: T(\varphi,y)\in A\}) \text{ when } A \text{ is a Borel subset of } \Phi\ .$$

Note that $\Phi \subset \{f \in \Phi:\ f \text{ is } C^1\}$.

We now define W . First, fix a measurable selection $Z:\Phi \hookrightarrow \Delta$ from the Walras price correspondence. Such a selection exists because the Walras price correspondence is a closed subset of $\Phi'\times\Delta$ which is in turn a closed subset of $\{f\in\Phi:\ f \text{ is } C^1\}\times\Delta$ ([4]). Now define $W:\Phi'\to\Delta$ as follows: if φ has only finitely many equilibrium prices, $W(\varphi)$ puts equal weight on them. Otherwise, $W(\varphi)$ puts point mass on $Z(\varphi)$. We'll see that "typically", there will be finitely many equilibrium prices.

Fix $\varphi \in \{f\in\Phi:\ f \text{ is } C^1\}$. Following an idea of Debreu [2], define $F: \Delta\times\mathbb{R} \to \mathbb{R}^k$ by $F(p,w) = -\varphi(p) + \frac{w}{p\cdot p}\,p$. $F(p,p\cdot y) = y$ if and only if p is a Walras price for $T(\varphi,y)$. Moreover, 0 is a regular value for $T(\varphi,y)$ if and only if y is a regular value for F . By Sard's Theorem, $\gamma(\{y: 0 \text{ is a regular value of } T(\varphi,y)\})=1$.

We claim that, if 0 is a regular value of $T(\varphi,y)$, W is continuous at $T(\varphi,y)$. To see this, suppose $\varphi^\ell \in \Phi'$, $y^\ell \in \mathbb{R}^k$, and $T(\varphi^\ell,y^\ell)$ converges to $T(\varphi,y)$ uniformly on compact sets. But $d(\cdot,\mu,s)$ is continuous in the topology of C^1-

convergence on compact sets, so $\{d(\cdot,\mu,s): \mu\in M,\ s\in S\}$ is compact in this topology. If we fix p, we see that $y^{\ell} - \frac{p\cdot y^{\ell}}{p\cdot p}\,p$ must stay bounded, so $\{T(\varphi^{\ell},y^{\ell})\}$ is contained in a compact subset in the topology of C^1-convergence on compact sets. Hence, $T(\varphi^{\ell},y^{\ell})$ must converge to $T(\ell,y)$ C^1 on compact sets. We may find a compact $K \subset \mathrm{Int}(\Delta)$ such that, for all ℓ, any equilibrium price of $T(\varphi^{\ell},y^{\ell})$ or of $T(\varphi,y)$ is contained in K . Since 0 is a regular value of $T(\varphi,y)$, $T(\varphi^{\ell},y^{\ell})$ has the same finite number of equilibria as $T(\varphi,y)$ (for ℓ sufficiently large), and these equilibria converge to the equilibria of $T(\varphi,y)$. Thus, $W(T(\varphi^{\ell},y^{\ell})) \to w(T(\varphi,y))$, so W is continuous at $T(\varphi,y)$. Thus $C(\varphi)$ $(\{\bar{\varphi}$: W is continuous at $\bar{\varphi}\}) = 1$.

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