RATIONAL EXPECTATIONS EQUILIBRIUM WITH LINEAR MODELS

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Abstract: We prove the existence of general economic equilibrium under uncertainty when agents form linear models of the relation between their private information, prices, and the state of the environment. Equilibrium requires not only that markets clear, but also that each agent's linear model is, modulo a truncation, the least squares best fit to the data that is generated by the workings of the economy when agents adhere to their models.

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1. INTRODUCTION

We present a model of rational expectations equilibrium in which the state of the environment is random and agents receive private information that is correlated with the state. Because each agent's demand reflects his private information, private information influences market prices. Furthermore, since prices are public, each agent, when he comes to market, obtains information beyond his own private information about the value of the state of the environment. From this perspective, prices play more than their usual allocative role; they become a conduit for the transfer of private information. An equilibrium price vector p must guide the decentralized allocation of commodities when consumers use, in addition to their private information, the information that is communicated by the price vector p .

In this paper we assume that agents form linear models of the relation between their private information, prices, and the state of the environment. Rational expectations equilibrium requires that these models not be controverted. This is taken to mean that each agent's model is the linear least squares fit to the data that is generated by the workings of the economy when agents adhere to their models. We will show that for each n-agent economy in a very general class, there is at least one rational expectations equilibrium.

Two special features of the analysis should be noted. First, the theorem requires that there be some (possibly small) random variation in demand or noise that is unrelated to private information, prices, and the state of the environment. Second, for technical reasons, it is necessary that the slope coefficients of the least squares regression be adjusted to lie in an arbitrary compact interval that is specified in advance. Other than these special features, the result admits a wide range of economies; it covers a general class of preferences and essentially any specification of private information. Furthermore, it applies when the dimension of the environment variable is large relative to the number of commodities. This is the case in which one does not expect prices to be as informative as the union of private information.

The decision to model the beliefs of agents by models that are determined by a finite number of parameters, in this case linear models, leads to a substantial simplification in the rational expectations theory. In the absence of such an assumption, the space of models of the joint distribution of private information values, prices, and states of the environment is quite complex, and both computation and comparative statics analysis are not a serious prospect. (For example, one might take the models of agents to be the joint distributions themselves.) On the other hand, when agents are confined to linear models, the problem of solving for an equilibrium is naturally put into a familiar form. Observe that a linear model L_j can be viewed as a finite dimensional vector. If agents adhere to the models $L_1,\ldots,L_n$, then the workings of the economy will produce joint distributions of private information, prices, and states denoted $\tau_1(L),\ldots,\tau_n(L)$, where $L = (L_1,\ldots,L_n)$. Denote the linear fits of these distributions by $(\sigma_1(\tau_1(L)),\ldots,\sigma_n(\tau_n(L)))$. Then, the finite dimensional vector L is an equilibrium n-tuple of models if any only if it is a fixed point of the map $\sigma \circ \tau$. All that is necessary to prove the existence of an equilibrium is to make sure that $\sigma \circ \tau$ takes a compact convex subset of Euclidean space into itself and that $\sigma \circ \tau$ is continuous on its domain. This is what the special assumptions are about. The result is a simple, but quite general theory; one that should be useful for analyzing the role of prices as conduits for private information.

Let's now explain in somewhat greater detail our notion of rational expectations equilibrium. The set of signals for the jth agent is taken to be S^j, a Euclidean space, prices are points in the usual simplex Δ, and the set of states of the environment is a Euclidean space Ω. A linear model for agent j is an affine function $L_j: \Delta \times S_j \to \Omega$. Suppose that agent j, with consumption set $\mathbb{R}^k_+$ and linear model L_j, observes $s^j \in S^j$ and is permitted to trade at price p. If he has the state-dependent utility function $u^j: \mathbb{R}^k_+ \times \Omega \to R$, and if he adheres to his model L_j, then he will maximize $u_j(\cdot, L_j(p,s_j))$ subject to his budget constraint.

We assume that the uncertainty in the economy and the private information structure is specified by a measure P on $\Omega \times S^1 \times S^2 \times \ldots \times S^n$, with the interpretation that nature picks a point from a Borel subset Q of $\Omega \times S^1 \times S^2 \times \ldots \times S^n$ with probability $P(Q)$. If nature picks the point $(\omega, s^1, s^2, \ldots, s^n)$, then the jth agent receives the private signal s^j and after his participation in the market learns that the state is ω. Since

s^j is random, for each value of p, the excess demand of the jth agent (with the model L_j) depends on $u^j(\cdot, L^j(p, s_j))$, and is thus random. To this is added the random variation in demand. Therefore, the market excess demand function is a random variable that is determined by the models, the vector $(s^1, s^2, \ldots, s^n)$, and the noise; it is correlated with ω. For any realization of market excess demand, equilibrium prices are defined, and we will assume that the economy settles on one of these (or possibly a mixture of them) in a manner that is sufficiently continuous. By this process a price vector (to be more accurate, a distribution of prices) is associated with each draw $(\omega, s^1, s^2, \ldots, s^n)$ by nature. Thus, the measure P, the distribution of agent's characteristics, and the models $(L_1, \ldots, L_n)$ determine for each j a joint distribution of ω, s^j, and p.

We now assume that each agent computes the least squares linear regression of ω on s_j and p

(followed by an adjustment that places the regression coefficients in a prespecified compact interval.) Our theorem shows that there exist linear models $(L_1, \ldots, L_n)$ that are reproduced under the above process. These models form a rational expectations equilibrium.

It is worth noting a significant difference between our notion of rational expectations equilibrium and the notion of a price function that is used in most of the literature. A price function is a map from states to prices such that, if a state ω occurs and the Walrasian auctioneer announces the corresponding price p, then markets will clear. This is a weaker notion of equilibrium because it seems to suggest that there is an auctioneer who learns the state and then announces the price. One example that highlights the difference is the following: Suppose an agent knows the state exactly, but his utility is independent of the state. Then his demand would not depend on his signal, and hence there is no market mechanism which could encode his information into the price. However, the price function could well convey the information; indeed, price functions generated by the Full Communications Equilibrium will always convey the information!

Before beginning the formal analysis, we will take a moment to point out the relation between the result presented here and those of an earlier paper [1]. In that paper we establish the existence of a rational expectations equilibrium when agents are assumed to form models by applying a convolution to the empirical distributions that are generated when agents adhere to their models. The convolution process represents a non-parametric method for drawing models from data, and equilibrium requires that the convolutions of the empirical distribution generated when agents adhere to their models reproduce those models. For the class of convolutions we considered, the convolution of an arbitrary model, which is itself a model, has two nice properties. First, for each value of the signal variable, it generates a continuous excess demand function. Second, it lies in a compact set of models that can be specified in advance. These properties are also guaranteed to hold if each agent draws a linear model in the above prescribed way, and they play a major role in preparing the way for the application of a suitable fixed point theorem. The advantage of the present parametric treatment is that it leads to a theory that is a good deal simpler and perhaps more useful for applied analysis.

2. RESULTS

A commodity bundle is a vector $x \in \mathbb{R}^k_+$. There are n individuals, indexed by j. They have initial endowments $e_1, \ldots, e_n \in \mathbb{R}^k_+$. We assume there is in society a positive amount of each commodity, i.e., $\sum_{j=1}^{n} e_j >> 0$.

Individuals possess state-dependent utility functions. Thus, there is a state space Ω, a Euclidean space. Individual j has a utility function $u_j : \mathbb{R}^k_+ \times \Omega \to \mathbb{R}$ which (i) is jointly C^2 in x and ω, (ii) has negative definite Hessian with respect to x, for each ω, (iii) $\partial u_j / \partial x >> 0$ and $u_j(x) = u_j(0)$ if $x_i = 0$ for some i.

Next, we must specify the information structure available to each agent. Agent j observes a signal in his signal space S_j, which is assumed to be a Euclidean space. Let $S = S_1 \times \ldots \times S_n$. The specification of the information is completed by specifying a Borel probability measure P on $\Omega \times S$, which gives the (objective) joint distribution of the states and signals. It is assumed that the convex hull of the support of P is full dimensional, and that the distribution possesses finite second moments. Thus, the information structure is totally exogenous.

Each agent makes use of the information in his signal. In addition, the information possessed by the other agents will be reflected to some extent in the Walras equilibrium price that emerges. Agents use their information by referring to a model of the joint distribution of prices, states, and signals.

Agents have linear models of the relationship between prices, signals, and states. Thus, the model of agent j is of the form $L_j(p,s_j) = \overline{\omega} + M_j(p-\overline{p}, s_j-\overline{s}_j)$, where $\overline{\omega} \in \Omega$, $\overline{p} \in \Delta$, $\overline{s}_j \in S_j$ are constants, M_j is a $(\dim\Omega) \times (\dim\Delta + \dim S_j)$ matrix, and (p,s_j) denotes the vector whose components are the components of p followed by the components of s_j. As we noted above, it is necessary to restrict the (ρ,σ) entry to lie in a prespecified compact interval $[a_{\rho\sigma j}, b_{\rho\sigma j}]$. Since L_j is a linear least squares fit, we may take $\overline{\omega}$ and $\overline{s}_j$ to be the expected values of ω and s_j with respect to their (exogenous) distribution P, while $\overline{p}$ is the expected value of p with respect to the endogenous distribution of p which the agent observes. Since all agents observe the same prices, we need not employ a subscript j on $\overline{p}$.

Thus, the set of models for person j can be represented as $\mathcal{M}_j = \{(M_j,\overline{p}): M_j$ is a $\dim\Omega \times (\dim\Delta + \dim S_j)$ matrix with (ρ,σ) entry in $[a_{\rho\sigma j}, b_{\rho\sigma j}]$, and $\overline{p} \in \Delta\}$. The set of profiles of models is $\mathcal{M} = \{(M,\overline{p}): M = (M_1,\ldots,M_n)$ and $(M_j,\overline{p}) \in \mathcal{M}_j\}$. $\mathcal{M}$ is a compact convex subset of a Euclidean space.

Suppose now each agent has a model $(M_j,\overline{p}) \in \mathcal{M}_j$ and sees a signal $s_j \in S_j$. Let $d_j(p,(M_j,\overline{p}),s_j)$ be the unique vector in $\mathbb{R}^k_+$ maximizing $u_j(x, L_j(p,s_j))$ over the budget set $\{x: p\cdot x \leqq p\cdot e_j\}$.

We assume that there is a random disturbance term in the demand. Given $y \in \mathbb{R}^k$, define

$$d(p,(M,\overline{p}),s,y) = \sum_{j=1}^{n} (d_j(p,(M_j,\overline{p}),s_j) - e_j) + y - \frac{p\cdot y}{p\cdot p}\,p\,;$$

d is essentially the market excess demand, translated by y, and adjusted to satisfy Walras' Law. We suppose that the disturbance term y is distributed according to γ, a probability measure absolutely continuous with respect to Lebesque measure.

We will show that, given $(M,\overline{p}),s,$ and y, (i) $p \cdot d(p,(M,\overline{p}),s,y) = 0$, (ii) $d(p,(M,\overline{p}),s,y)$ is bounded below, and (iii) $p_i \to 0$ for some i implies $|d(p,(M,\overline{p}),s,y)| \to \infty$. Hence, there exists $p \in \Delta$ such that $d(p,(M,\overline{p}),s,y) = 0$. We need to assume that the economy settles on such prices in a consistent fashion. For our purposes, it is convenient to suppose that, whenever there are only a finite number of such p, the economy settles randomly (with equal probabilities) on one such p; however, this specific form is inessential (see Anderson-Sonnenschein [1]). We shall show that, given $M,\overline{p},$ and s, there will be only a finite number of such p for γ--almost all y.

Suppose we are given a profile of models $(M,\overline{p}) \in \mathcal{M}$. The process defined above associates with each play of the economy a probability distribution on equilibrium prices, and hence the process generates a joint distribution on $\Delta \times \Omega \times S$, denoted $\tau(M,\overline{p})$. The marginal distribution on $\Delta \times \Omega \times S_j$, denoted $\tau_j(M,\overline{p})$, is the true model relating prices, states and j's signal, assuming that every trader $\overline{j}$ believes the model $(M_j,\overline{p})$.

Given a probability measure μ_j on $\Delta \times \Omega \times S_j$, we can form the linear least squares approximation, of the form

$$\omega = \overline{\omega} + N_j(p - \int p d\mu_j;\ s_j - \overline{s}_j) ,$$

where N is a matrix. This is uniquely defined, provided that the convex hull of the support of the marginal of μ_j on $\Delta \times S_j$ is full-dimensional; we shall show that this is always the case if μ_j is taken to be $\tau_j(M,\overline{p})$ for $(M,\overline{p}) \in \mathcal{M}$. Observe further that $\int p d\tau_j(M,\overline{p})$ does not depend on j. Hence, we have defined a new model $\sigma(\mu_1,\ldots,\mu_n) = ((N_1',\ldots,N_n'), \int p d\tau_j(M,\overline{p}))$, where N_j' is obtained by truncating the coefficients of N_j to lie in the intervals prespecified in the definition of $\mathcal{M}_j$.

The above material has really been an extended definition of the process by which the economy determines prices. Some of the steps of the definition need to be justified by lemmas which we shall give below. But we can now state the results.

<u>Theorem 1</u>: There exists $(M,\overline{p}) \in \mathcal{M}$ such that $\sigma(\tau(M,\overline{p})) = (M,\overline{p})$.

In other words, if each agent j believes the model $(M_j,\overline{p})$, then the truncation of the best fit model is $(M_j,\overline{p})$.

The idea of the proof of Theorem 1 is to show that $\sigma \circ \tau$: $\mathcal{M} \to \mathcal{M}$ is continuous. Since the slope restrictions guarantee that $\mathcal{M}$ is a compact convex subset of a Euclidean space, Brouwer's Fixed Point Theorem then shows that $\sigma \circ \tau$ has a fixed point. For given signals, small changes in a model can lead to a large change in the equilibrium price, with the result that $\tau \circ \sigma$ is not continuous. The discontinuity is a consequence of the fact that the Walras correspondence is not lower hemi-continuous, and it is remedied here by the introduction of the random disturbance in demand.

We can now begin the proof.

Lemma 2: (i) d_j is uniquely defined and gives a C^1 mapping from $(\text{Int } \Delta) \times \mathcal{M}_j \times S_j$ to $\mathbb{R}^k_+$,

(ii) if $p_i \to 0$ for some component i, then $|d_j(p,(M_j,\bar{p}),s_j)| \to \infty$, and

(iii) for all $(M,\bar{p}) \in \mathcal{M}$, $s \in S$, and $y \in \mathbb{R}^k$, there exists $p \in \Delta$ such that $d(p,(M,\bar{p}),s,y) = 0$.

Proof: $d_j(p,(M_j,\bar{p}),s_j)$ maximizes $u_j(x,L_j(p,s_j))$ subject to $p \cdot x \leq p \cdot e_j$. Hence, (i) follows from the linearity of L_j in p and the assumptions on u_j. Fix $(M_j,\bar{p})$ and s_j, and let x range over a compact set K. Then $L_j(p,s_j)$ ranges over a compact set in Ω, and so $\partial u_j/\partial x$ ranges over a compact set of strictly positive vectors, and so $(\partial u_j/\partial x)/|\partial u_j/\partial x|$ also varies over a compact set of strictly positive vectors; (ii) then follows by standard arguments. Since $d(p,(M,\bar{p}),s,y) \geq -\sum_{j=1}^{n} e_j - (|y|,\ldots,|y|)$ and $p \cdot d(p,(M,\bar{p}),s,y) = 0$, it is well known that there exists p such that $d(p,(M,\bar{p}),s,y) = 0$.

Lemma 3: Let $W((M,\bar{p});s,y) = \{p : d(p,(M,\bar{p}),s,y) = 0\}$ denote the Walras price correspondence. Given $(M,\bar{p}) \in \mathcal{M}$ and $s \in S$, then for γ-almost all y, $W((M,\bar{p}),s,y)$ is a finite set which varies continuously as $(M,\bar{p})$ and s are moved over a neighborhood.

Proof: We have shown in Lemma 2 that $d(\cdot,(M,\bar{p}),s,y)$ is C^1. Following Debreu [2], define $F : \Delta \times \mathbb{R} \to \mathbb{R}^k$ by $F(p,w) = -d(p,(M,\bar{p}),s,0) + \frac{w}{p \cdot p}p$. $F(p,p \cdot y) = y$ if and only if $d(p,(M,\bar{p}),s,y) = 0$. Moreover, 0 is a regular value of $d(\cdot,(M,\bar{p}),s,y)$ if and only if y is a regular value for F.

By Sard's Theorem, 0 is a regular value of $d(\cdot,(M,\bar{p}),s,y)$, for γ-almost all y; the conclusions then follow as in Debreu [2].

We are now in a position to define carefully the map τ, which gives the distribution of the outcomes of the economy when agents use the profile $(M,\bar{p})$. $\tau(M,\bar{p})_j$ is the joint distribution on $\Delta \times \Omega \times S_j$ obtained when the economy is run over all the states, signals, and disturbances. The probability that the equilibrium price lies in a set $A \subset \Delta$ (given $(M,\bar{p}),s,y)$ is thus $\frac{|W((M,\bar{p}),s,y) \cap A|}{|W((M,\bar{p}),s,y)|}$, at least provided $W((M,\bar{p}),s,y))$ is finite; since this is the case for almost all y, the probability that the price lies in A (given $(M,\bar{p})$ and s, but allowing y to vary) is thus

$$\int_{\mathbb{R}^k} \frac{|W((M,\bar{p}),s,y) \cap A|}{|W((M,\bar{p}),s,y)|} \, d\gamma(y) .$$

Thus, $\tau(M,\bar{p})_j$ is the Borel measure on $\Delta \times \Omega \times S_j$ whose values on cylinder sets $A \times B \times D$ (where $A \subset \Delta$, $B \subset \Omega$, $D \subset S_j$ are Borel sets) is

$$\tau(M,\bar{p})_j(A\times B\times D) = \int_{B\times\hat{D}} \int_{\mathbb{R}^k} \frac{|W((M,\bar{p}),s,y) \cap A|}{|W((M,p),s,y)|} \, d\gamma(y)\, dP(\omega,s),$$

where $\hat{D} = \{s \in S;\ s_j \in D\}$. Agent j now forms a new model in $\mathcal{M}_j$ by first taking the linear least squares approximation to $\tau(M,\bar{p})_j$, which has the form $L''(p,s_j) = \bar{\omega} + M''_j(p-\bar{p}';s_j-\bar{s}_j)$, as will be shown in a moment. He then truncates the coefficients of M''_j to lie in their specified intervals; in other words, he defines the new model by specifying $(M'_j,\bar{p}')$, where the (ρ,σ)entry of M'_j is given by

$$\begin{cases} b_{\rho\sigma j} & \text{if the } (\rho,\sigma) \text{ entry of } M''_j > b_{\rho\sigma j} \\ a_{\rho\sigma j} & \text{if the } (\rho,\sigma) \text{ entry of } M''_j < a_{\rho\sigma j} \\ \text{the } (\rho,\sigma) \text{ entry of } M''_j \text{ otherwise.} \end{cases}$$

Let $\sigma_j(\tau(M,\overline{p})_j) = (M'_j,\overline{p}')$ and $\sigma(\tau(M,\overline{p})) = ((M'_1,\ldots,M'_n),p')$.

Lemma 4: Given any $(M,\overline{p}) \in \mathcal{M}$, the linear least squares approximation to $\tau(M,\overline{p})_j$ is well defined and has the form claimed in the preceding paragraph. Thus, $\sigma(\tau(M,\overline{p}))$ is defined, and $\sigma \circ \tau\colon \mathcal{M} \to \mathcal{M}$ is continuous.

Proof: Let $\mu_j = \tau(M,\overline{p})_j$. Since the marginal ν_j of μ_j on $\Omega \times S_j$ is the same as the marginal of P, and Δ is compact, all the second moments of μ_j exist and the convex hull of the support of ν_j is full-dimensional on $\Omega \times S_j$.

Fix $(M,\overline{p})$ and s. We will show that the resulting distribution of prices p is full-dimensional. This, together with the previous paragraph will establish that $\tau(M,\overline{p})_j$ has full-dimensional distribution. Suppose the support of the price distribution is contained in H, an affine subset of Δ with $\dim H < \dim\Delta$. Define $\varphi\colon \Delta \times \mathbb{R} \to \mathbb{R}^k$ by $\varphi(p,w) = wp - \sum_{j=1}^{n} (d_j(p,(M,\overline{p}),s_j) - e_j)$. φ is C^1, and hence $\varphi(H \times \mathbb{R})$ has Lebesque measure 0, by the Transformation Theorem. If $d(p,(M,\overline{p}),s,y) = 0$, then $\varphi(p,\frac{p\cdot y}{p\cdot p}) = -\,d(p,(M,\overline{p}),s,y) - \frac{p\cdot y}{p\cdot p}\,p + y + \frac{p\cdot y}{p\cdot p}\mathbf{p} = \mathbf{y}$. Therefore $\gamma(\{y : d(p,(M,\overline{p}),s,y) = 0 \text{ for some } p \in H\}) \leq \gamma(\phi(H \times \mathbb{R})) = 0$, since γ is absolutely continuous with respect to Lebesque measure. Thus, for fixed s, the support of distribution of prices p has full-dimensional convex hull.

This shows that the linear least squares fit of ω in terms of p and s_j is uniquely defined, and has the given form, so that $\sigma(\tau(M,\overline{p}))$ is defined.

It remains to prove continuity. Suppose $(M^n, \overline{p}^n) \to (M,\overline{p})$. Fix $s \in S$, $\omega \in \Omega$ . . By Lemma 3, for γ-almost all y, $W((M^n,\overline{p}^n),s,y) \to W((M,\overline{p}),s,y)$ in the sense that, for sufficiently large n, $W((M^n,p^n),s,y)$ has the same number of points as $W((M,\overline{p}),s,y)$, and these points converge. Thus, if F is any bounded continuous function, $F : \Delta \times \Omega \times S \to \mathbb{R}$,

$$\int_{\mathbb{R}^k} \frac{\sum_{p\in W((M^n,\overline{p}^n),s,y)} F(p,\omega,s)}{|W((M^n,\overline{p}^n),s,y)|} d\gamma(y) \to \int_{\mathbb{R}^k} \frac{\sum_{p\in W((M,\overline{p}),s,y)} F(p,\omega,s)}{|W((M,\overline{p}),s,y)|} d\gamma(y),$$

by the Dominated Convergence Theorem.

Hence

$$\int_{\Delta\times\Omega\times S} F(p,\omega,s)\,d\tau(M^n,\overline{p}^n)$$

$$= \int_{\Omega\times S} \int_{\mathbb{R}^k} \frac{\sum_{p\in W((M,\overline{p}),s,y)} F(p,\omega,s)}{|W((M,\overline{p}),s,y)|} d\gamma(y)\,dP(\omega,s)$$

$$\to \int_{\Omega\times S} \int_{\mathbb{R}^k} \frac{\sum_{p\in W((M,\overline{p}),s,y)} F(p,\omega,s)}{|W((M,\overline{p}),s,y)|} d\gamma(y)\,dP(\omega,s)$$

by the Dominated Convergence Theorem,

$$= \int_{\Delta\times\Omega\times S} F(p,\omega,s)\,d\tau(M,\overline{p}) \quad . \qquad (*)$$

The coefficients of the linear least squares fit are, given the full dimensionality of the distribution of $\tau(M,\bar{p})$, continuous functions of terms of the form $\int \alpha\, d\tau(M,\bar{p})$ or $\int \alpha\beta d\tau(M,\bar{p})$, where α and β are components of $p, \omega,$ or s. If α and β are components of ω or s, then their distributions under $\tau(M,\bar{p})$ or $\tau(M^n,\bar{p}^n)$ are their marginal distribution under P, and so $\int \alpha d\tau(M^n,\bar{p}^n) = \int \alpha d\tau(M,\bar{p})$ and $\int \alpha\, \beta d\tau(M^n,\bar{p}^n) = \int \alpha\, \beta d\tau(M,\bar{p})$. If α and β are components of p, $\int \alpha d\tau(M^n,\bar{p}^n) \to \int \alpha d\tau(M,\bar{p})$ and $\int \alpha\, \beta d\tau(M^n,\bar{p}^n) \to \int \alpha\, \beta d\tau(M,\bar{p})$ by (*). Finally, suppose α is a component of p and β is a component of ω or s. For $k \in \mathbb{N}$ let

$$\beta^k = \begin{cases} k & \text{if } \beta > k \\ \beta & \text{if } |\beta| \le k \\ -k & \text{if } \beta < -k \end{cases} .$$

Then $\int \alpha\, \beta^k d\tau(M^n,\bar{p}^n) \to \int \alpha\, \beta^k d\tau(M,\bar{p})$ by (*). But

$$\left| \int \alpha\, \beta d\tau(M^n,\bar{p}^n) - \int \alpha\, \beta^k\, d\tau(M^n,\bar{p}^n) \right|$$

$$\le \int |\alpha\, (\beta - \beta^k)| d\tau(M^n,\bar{p}^n) \le \int |\beta - \beta^k|\, d\tau(M^n,\bar{p}^n)$$

$= \int |\beta - \beta^k|\, dP \to 0$ as $k \to \infty$. Using the same argument for $\tau(M,\bar{p})$, we get $\int \alpha\, \beta d\tau(M^n,\bar{p}^n) \to \int \alpha\, \beta\, d\tau(M,\bar{p})$.

Proof of Theorem 1:

Since $\sigma \circ \tau$ is continuous, and M is a closed convex subset of a Euclidean space, $\sigma \circ \tau$ has a fixed point by Brouwer's Fixed Point Theorem.

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