ON THE INCONSISTENCY OF CERTAIN AXIOMS ON SOLUTION CONCEPTS FOR NON-COOPERATIVE GAMES

Dilip Abreu
David G. Pearce

Econometric Research Program
Research Memorandum No. 288

August 1981

Econometric Research Program
Princeton University
207 Dickinson Hall
Princeton, New Jersey

ABSTRACT

The mutual compatibility of four recently discussed axioms on solution concepts for extensive form games is explored. Various subsets of the axioms are shown to be inconsistent, and it is demonstrated that the results are robust to certain weakenings of the axioms.

I. Introduction

Recent contributions by Selten (1965 and 1975), Harsanyi (1976), Myerson (1978), and others have discussed the problem of Nash equilibria (Nash, 1951) that specify intuitively implausible behavior for some or all agents. The disquiet felt about the Nash solution concept has led to the suggestion that there are various axiomatic requirements that a satisfactory solution concept ought to satisfy. At a recent conference,[1] E. Kohlberg discussed a set of four such axioms. While Kohlberg has conjectured that there exists a solution concept satisfying these properties generically, the possibility of satisfying these everywhere is apparently an open question.[2] This note demonstrates that the four axioms are mutually inconsistent. Indeed, we show that a much weaker set of requirements is also inconsistent.

II. Notation and Definitions

We restrict ourselves to finite n-person non-cooperative games in extensive form. We provide below a very brief description of such a game. A more detailed treatment may be found in Selten (1975), whose notation we have adopted.

---

[1] 1981, N.B.E.R. Conference on Mathematical Economics, (University of California at Berkeley).

[2] We wish to thank Hugo Sonnenschein for bringing this question to our attention.

An extensive form game Γ may be written as

$$\Gamma = (K, P, U, C, h)$$

where the elements of Γ are as described below. K is a topological tree. It is defined by a set of <u>vertices</u> X, and a set of <u>edges</u> or <u>alternatives</u> A, which connect certain pairs of vertices. The set of endpoints of K is denoted by Z. $P = (P_1,\ldots,P_n)$ is the player partition. It partitions the non-terminal vertices of X into player sets. $U = (U_1,\ldots,U_n)$ is the information partition; it partitions P_i into information sets. A_u is the set of all alternatives at vertices $x \in u$, where $u \in U$. A choice c at $u \in U$ is a subset of A_u that contains exactly one alternative at x for every $x \in u$. C is the choice partition. It partitions A into choices c. C_u is the set of all choices at the information set u; it partitions A_u.

A pure strategy π_i of player i is a function assigning to each information set $u \in U_i$ a choice $c \in C_u$. Π_i denotes the set of all pure strategies of player i. A mixed strategy q_i is a probability distribution over Π_i. It assigns a probability $q_i(\pi_i)$ to each $\pi_i \in \Pi_i$. Q_i denotes the set of all mixed strategies of player i. The payoff function h assigns to each endpoint $z \in Z$

an n-tuple of payoffs. $H(q_1,\ldots,q_n)$ is the n-tuple of expected payoffs associated with the profile of mixed strategies $(q_1,\ldots,q_n)$. A strategy profile $(q_1,\ldots,q_n) \in Q_1 x\ldots x\, Q_n$ is a Nash equilibrium if $H_i(q_1,\ldots,q_{i-1},q_i, q_{i+1},\ldots,q_n) \geq H_i(q_1,\ldots,q_{i-1},q_i',q_{i+1},\ldots,q_n)$ for all $q_i' \in Q_i$, for each i. More generally, a solution concept F associates with the game Γ, a set of strategy profiles in Γ.

A vertex x is said to come after a choice c if the path from the origin to x contains an edge in c. An edge a is said to come after a choice c if $a \notin c$ and the path from the origin to a contains an edge in c. A vertex x is said to come after a vertex y if $x \neq y$ and the path from the origin to x contains y. An edge a is said to come after a vertex y if a is contained in any path from y to any terminal node.

We say that game $\hat{\Gamma} = (\hat{K}, \hat{P}, \hat{U}, \hat{C}, \hat{h})$ is obtained from game $\Gamma = (K, P, U, C, h)$ by removing some choice $c^* \in C_{u^*}$ at some player's information set $u^* \in U_k$ if

(1) $\hat{K}$, $\hat{P}$, $\hat{U}$, and $\hat{C}$ are identical to K, P, U, and C but for the deletion of:

  (a) vertices that come after the choice c^*

  (b) edges that are part of, or come after, the choice c^*.

(2) $\hat{h}$ is the restriction of h to $\hat{Z}$, where $\hat{Z} \subset Z$ is the set of terminal nodes of $\hat{\Gamma}$.

A formal definition of the removal of a choice is given in Note 1.

Let Γ and $\hat{\Gamma}$ be defined as above. A pure strategy $\pi_i \in \Pi_i$ for player i is said to be admissible with respect to $\hat{\Gamma}$ if

$$i \neq k \quad \text{or}$$

$$\pi_k(u^*) \neq c^* .$$

A mixed strategy $q_i \in Q_i$ for player i is said to be admissible with respect to $\hat{\Gamma}$ if

$$i \neq k \quad \text{or}$$

$$q_k(\pi_k) > 0 \text{ implies } \pi_k \text{ is admissible with respect to } \hat{\Gamma} .$$

The projection $\eta(\pi_i)$ of an admissible pure strategy $\pi_i \in \Pi_i$ of Γ on $\hat{\Gamma}$ is $\hat{\pi}_i \in \hat{\Pi}_i$ such that $\hat{\pi}_i(\hat{u}) = \pi_i(u) \wedge \hat{A}_{\hat{u}}$, for each $\hat{u}$, where u satisfies $\hat{u} = u \wedge \hat{X}$. The projection $\eta(q_i)$ of an admissible mixed strategy $q_i \in Q_i$ of Γ on $\hat{\Gamma}$ is $\hat{q}_i \in \hat{Q}_i$ such that $\hat{q}_i(\eta(\pi_i)) = q_i(\pi_i)$, for all admissible π_i.

Let $(\pi/\bar{\pi}_k) = (\pi_1, \ldots, \pi_{k-1}, \bar{\pi}_k, \pi_{k+1}, \ldots, \pi_n)$.

A pure <u>strategy</u> $\pi_k^* \in \Pi_k$ is said to be <u>dominated</u> if there exists $\pi_k' \in \Pi_k$ such that

$$H_i(\pi/\pi_k') \geq H_i(\pi/\pi_k^*) \quad \text{for all } \pi \in \Pi_1 \times \ldots \times \Pi_n$$

and $H_i(\pi/\pi_k') > H_i(\pi/\pi_k^*)$ for some $\pi \in \Pi_1 \times \ldots \times \Pi_n$.

A choice $c^* \varepsilon C_{u^*}$ at $u^* \varepsilon U_k$ is dominated if $\pi_k \varepsilon \Pi_k$ and $\pi_k(u^*) = c^*$ imply that π_k is a dominated strategy.

Let τ be a subgame[3] of Γ and x_τ be its origin. Let τ have a unique Nash equilibrium $r = (r_1,\ldots,r_n)$. We say that $\tilde{\Gamma}$ is obtained from Γ by replacing the subgame τ, if $\tilde{\Gamma}$ is obtained from Γ by removing all edges and vertices that come after x_τ, and by assigning to x_τ (the new terminal node thus created in the reduced game $\tilde{\Gamma}$) the payoff vector $H^\tau(r)$.[4] Again, a formal definition is provided in a note (see Note 2).

Let Γ, τ, and $\tilde{\Gamma}$ be as defined above. The restriction $\psi(\pi_i)$ of a pure strategy $\pi_i \varepsilon \Pi_i$ of Γ to $\tilde{\Gamma}$ is $\tilde{\pi}_i \varepsilon \tilde{\Pi}_i$ such that $\tilde{\pi}_i(\tilde{u}) = \pi_i(\tilde{u})$ for every $\tilde{u} \varepsilon \tilde{U}_i$. The restriction $\psi(q_i)$ of a mixed strategy $q_i \varepsilon Q_i$ of Γ on $\tilde{\Gamma}$ is $\tilde{q}_i \varepsilon \tilde{Q}_i$ such that $\tilde{q}_i(\psi(\pi_i)) = q_i(\pi_i)$ for all $\pi_i \varepsilon \Pi_i$. Let $U_i^\tau = \{u|u \varepsilon U_i \text{ and } u \notin \tilde{U}_i\}$. The extension $\Phi(\tilde{\pi}_i)$ of a pure strategy $\tilde{\pi}_i \varepsilon \tilde{\Pi}_i$ of $\tilde{\Gamma}$ to Γ is $\pi_i \varepsilon \Pi_i$ such that

$$\pi_i(u) = \tilde{\pi}_i(u) \quad \text{for every} \quad u \varepsilon \tilde{U}_i$$

and $$\pi_i(u) = r_i(u) \quad \text{for every} \quad u \varepsilon U_i^\tau .$$

The extension $\Phi(\tilde{q}_i)$ of a mixed strategy $\tilde{q}_i \varepsilon \tilde{Q}_i$ of $\tilde{\Gamma}$

---

[3]For a definition of a subgame, see Selten (1975).

[4]$H^\tau(\cdot)$ is the expected payoff function of the subgame τ.

to Γ is $q_i \in Q_i$ satisfying

$$q_i(\Phi(\tilde{\pi}_i)) = \tilde{q}_i(\tilde{\pi}_i) \quad \text{for all} \quad \tilde{\pi}_i \in \tilde{\Pi} .$$

III. The Proposed Axioms

The four axioms are stated, somewhat informally, below.

(A1) Dependence on Normal Form.

If Γ_1 and Γ_2 are extensive form games having the same normal form G, then

$$F(\Gamma_1) = F(\Gamma_2) .$$

This axiom is motivated by the belief that while information about the structure of a game is lost in moving from the extensive form to the normal form, all strategically relevant information is preserved. Underlying this view is the fact that there is, by definition, an isomorphism between the strategy space of an extensive form game and the strategy space of the associated normal form.

(A2) Nonemptiness.

$F(\Gamma) \neq \phi$ for all Γ, where ϕ denotes the empty set.

(A2) simply requires that every game must have at least one solution. It is, however, a nontrivial requirement. One might, for instance, only be willing to regard a strategy profile as a "solution" if the strategies comprising it were the only ones that could plausibly be chosen. It seems that many games will not be predictable in this sense, and thus would be assigned empty solution sets, in violation of (A2). The following axiom (A2'), is less vulnerable to this objection. Substantially weaker than (A2), it requires that a solution exist only if the game in question has a unique Nash equilibrium.

(A2') Weak Nonemptiness.

If the game Γ has a unique Nash equilibrium, then $F(\Gamma) \neq \phi$.

We note that (A2') is not one of the four basic axioms in question.

(A3) Dominance.

If $\hat{\Gamma}$ is obtained from Γ by removing a set of dominated choices, then

(i) $(q_1,\ldots,q_n) \in F(\Gamma)$ implies that (a) q_i is admissible with respect to $\hat{\Gamma}$, for all i, and (b) $(\eta(q_1),\ldots,\eta(q_n)) \in F(\hat{\Gamma})$.

(ii) $(\hat{q}_1,\ldots,\hat{q}_n) \in F(\hat{\Gamma})$ implies that there exists $(q_1,\ldots,q_n) \in F(\Gamma)$ such that q_i is admissible with respect to $\hat{\Gamma}$ for all i , and $\eta(q_i) = \hat{q}_i$ for all i .

This axiom may be motivated as follows. Any strategy that involves a dominated choice is a dominated strategy. Hence a player who makes a dominated choice is exposing himself to unnecessary risk, in the sense that there are strategies available to other players that result in avoidable losses to the player in question.

(A4) (Weak) Subgame Replacement (See Note 3)

Let τ be a subgame of Γ and let τ have a unique Nash equilibrium $r = (r_1, \ldots, r_n)$. Let $\tilde{\Gamma}$ be obtained from Γ by replacing the subgame τ. Then subgame replacement requires that

(i) $\{q_i\} \in F(\Gamma)$ implies:

(a) $q_i(u) = r_i(u)$ for all $u \in U_i^\tau$, for each i,

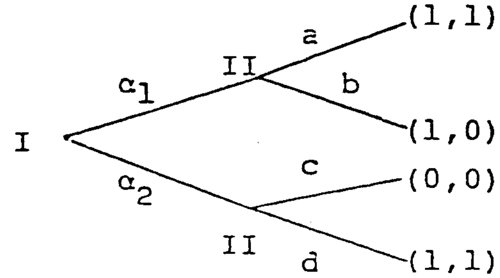
(b) $\{\psi(q_i)\} \in F(\tilde{\Gamma})$.

(ii) $\{\tilde{q}_i\} \in F(\tilde{\Gamma})$ implies $\{\Phi(\tilde{q}_i)\} \in F(\Gamma)$.

This axiom is motivated by the notion that if a subgame (with origin x_τ) of a game has only one plausible outcome H*, then player behavior must be consistent with the belief that if the node x_τ is reached, then H* will result.

IV. Inconsistency of the Axioms.

In this section we provide an example which shows that a strict subset of the four axioms is inconsistent. Consider the extensive form game Γ :



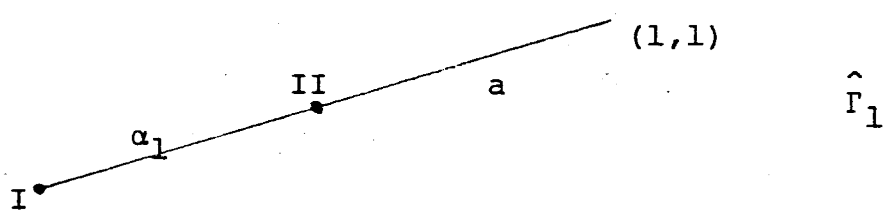
Γ

The pure strategies for player I are α_1 and α_2. Since player II has two choices at each of his two information sets, he has four pure strategies. These can be written

$$\beta_1 = (a,d)$$
$$\beta_2 = (a,c)$$
$$\beta_3 = (b,d)$$
$$\beta_4 = (b,c)$$

Notice that choice α_2 is dominated by choice α_1;
" b " " " " a ; and
" c " " " " d .

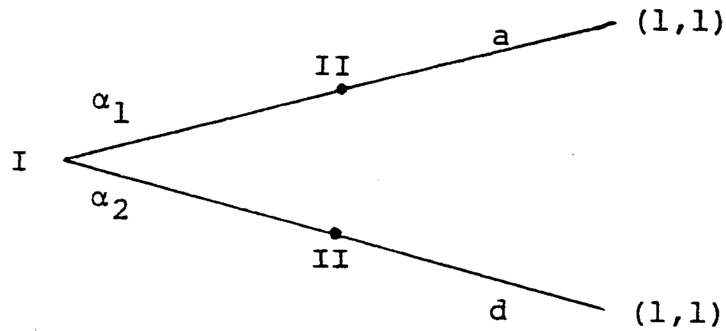
Removal of the dominated choices α_2, b, and c yields



Any strategy profile of Γ for which any of the choices α_2, b, or c is made with positive probability, is inadmissible with respect to $\hat{\Gamma}_1$. Hence (α_1,β_1) is the only strategy profile of Γ which is admissible with respect to $\hat{\Gamma}_1$. Thus $F(\Gamma) \neq \phi$ implies $F(\Gamma) = \{(\alpha_1,\beta_1)\}$.

But by (A2'), $F(\hat{\Gamma}_1) \neq \phi$ and by (A3)(ii), $F(\hat{\Gamma}_1) \neq \phi$ implies $F(\Gamma) \neq \phi$. Therefore, $F(\Gamma) = \{(\alpha_1, \beta_1)\}$ (1)

However, removal of the dominated choices b and c yields



$\hat{\Gamma}_2$

Applying (A3)(i) to Γ and $\hat{\Gamma}_2$, we have that $(\alpha_1, \beta_1) \in F(\Gamma)$ implies $(\alpha_1, \beta_1) \in F(\hat{\Gamma}_2)$. But $(\alpha_1, \beta_1) \in F(\hat{\Gamma}_2)$ implies $(\alpha_2, \beta_1) \in F(\hat{\Gamma}_2)$ (see Note 4). By (A3)(ii), $(\alpha_2, \beta_1) \in F(\hat{\Gamma}_2)$ implies $(\alpha_2, \beta_1) \in F(\Gamma)$. This contradicts (1). We have thus established the following proposition:

<u>Proposition 1.</u> There exists no F satisfying (A2') and (A3). Since (A2') implies (A2), we have also established:

<u>Corollary 1</u>. There exists no F satisfying (A2) and (A3).

Note that (A1) and (A4) were not required in generating a contradiction.

V. Simultaneous Dominance.

In this section we replace the troublesome axiom of dominance by a related, but weaker axiom, which we call simultaneous dominance.

Simultaneous Dominance

If $\hat{\Gamma}$ is obtained from Γ by removing all dominated choices at all information sets, then

(i) $\{q_i\} \in F(\Gamma)$ implies

(a) q_i is admissible with respect to $\hat{\Gamma}$, for each i

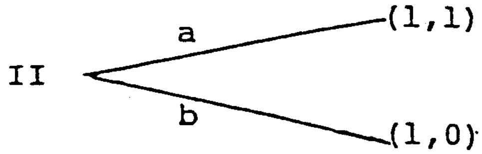
(b) $(\eta(q_1),\ldots,\eta(q_n)) \in F(\hat{\Gamma})$.

(ii) $\{\hat{q}_i\} \in F(\hat{\Gamma})$ implies that there exists $\{q_i\} \in F(\Gamma)$ such that q_i is admissible with respect to $\hat{\Gamma}$ for each i , and $\eta(q_i) = \hat{q}_i$, for each i .

The proof of the following proposition concerning simultaneous dominance is closely related to that of Proposition 1.

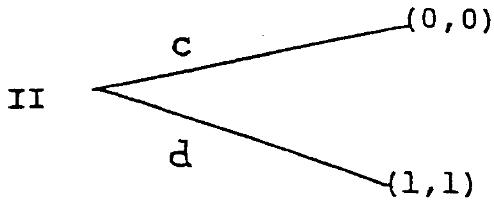
Proposition 2. There exists no F satisfying (A2'), simultaneous dominance, and (A4).

Proof: Consider the game Γ introduced in Section IV. As noted there, the set of dominated choices is $\{\alpha_2, b, c\}$. Removal of these yields $\hat{\Gamma}_2$, and by the arguments in the proof of Proposition 1, we have $F(\Gamma) = \{(\alpha_1,\beta_1)\}$. Removal of the subgames



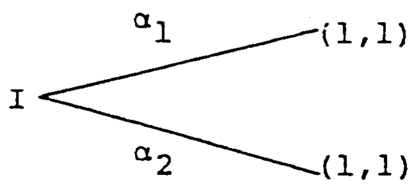
τ_1

and



τ_2

yields



$\tilde{\Gamma}$

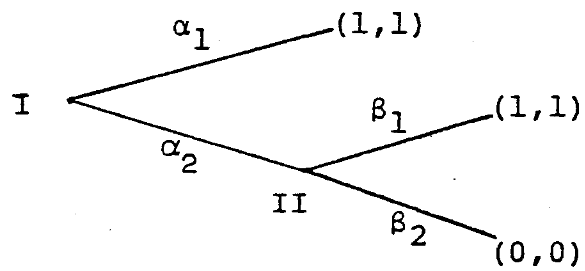
By (A4)(i)(b), $(\alpha_1,\beta_1) \in F(\Gamma)$ implies $(\alpha_1) \in F(\tilde{\Gamma})$. Also, $(\alpha_1) \in F(\tilde{\Gamma})$ implies $(\alpha_2) \in F(\tilde{\Gamma})$. (See Note 4). By (A4)(ii), $(\alpha_2) \in F(\tilde{\Gamma})$ implies $(\alpha_2,\beta_1) \in F(\Gamma)$. But this contradicts $\{(\alpha_1,\beta_1)\} = F(\Gamma)$, so the proof is complete.

VI. The Normal Form

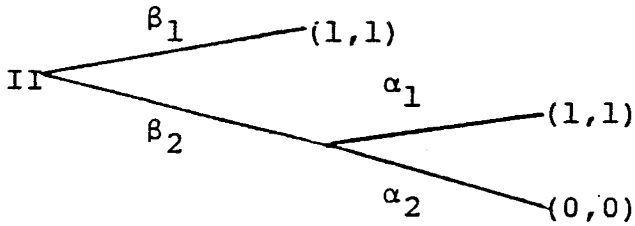
This section addresses the question of whether it is possible to require that subgame replacement be satisfied by a solution concept using only information from the normal form. The answer is perhaps the most revealing of the inconsistency results presented in this paper.

Proposition 3: There exists no F satisfying (A1), (A2), and (A4).

Proof: Consider the games Γ_1 and Γ_2 :



Γ_1



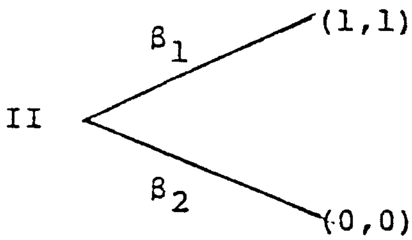
Γ_2

Γ_1 and Γ_2 have the same normal form, which may be written as G below:

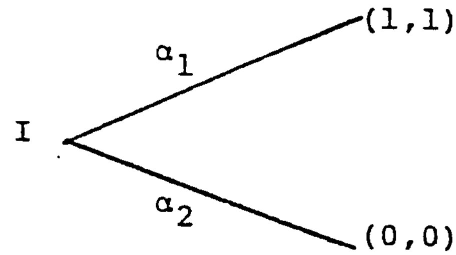
		II: β_1	II: β_2
I	α_1	(1,1)	(1,1)
	α_2	(1,1)	(0,0)

G

(A1) implies that $F(\Gamma_1) = F(\Gamma_2)$. Replacing the subgames τ_1 and τ_2

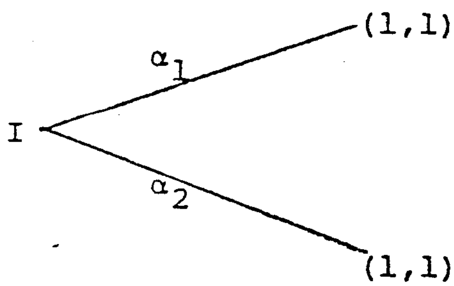


τ_1

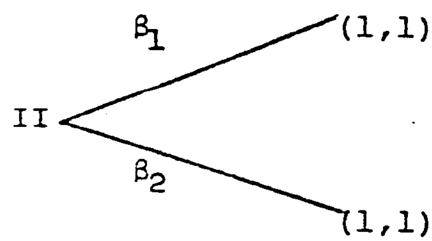


τ_2

of Γ_1 and Γ_2 respectively, we obtain $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$:



$\tilde{\Gamma}_1$



$\tilde{\Gamma}_2$.

(A4)(i)(a) applied to Γ_1 and $\tilde{\Gamma}_1$ gives:

$(\alpha,\beta) \in F(\Gamma_1)$ implies $\beta = \beta_1$, where α and β denote mixed strategies for players I and II respectively, and β_1 may be regarded as the mixed strategy that assigns probability 1 to the pure strategy β_1.

Applying (A4)(i)(a) to Γ_2 and $\tilde{\Gamma}_2$, we have $(\alpha,\beta) \in F(\Gamma_2)$ implies $\alpha = \alpha_1$. Thus, $(\alpha,\beta) \in F(\Gamma_1) \cap F(\Gamma_2)$ implies $(\alpha,\beta) = (\alpha_1,\beta_1)$. But by (A1), $F(\Gamma_1) = F(\Gamma_2)$, and hence $(\alpha,\beta) \in F(\Gamma_i)$, $i=1,2$, implies $(\alpha,\beta) = (\alpha_1,\beta_1)$. Combining this with nonemptiness (A2), we have

$$(\alpha_1,\beta_1) \in F(\Gamma_1) .$$

Then by (A4)(i)(b), $\alpha_1 \in F(\tilde{\Gamma}_1)$, which implies $\alpha_2 \in F(\tilde{\Gamma}_1)$ (see Note 4). (A4)(ii) implies $(\alpha_2,\beta_1) \in F(\Gamma_1)$, a contradiction.

VII. Conclusion

One way of responding to the questions raised by Propositions 1-3 is to explore the possibility that the various sets of axioms can be satisfied "almost everywhere." If, however, one is looking for axioms that must be satisfied universally, the results here rule out dominance as a satisfactory criterion: it conflicts with weak nonemptiness, a truly innocuous axiom. The status of the remaining axioms is unclear. Our feeling is that subgame replacement, (A4), is a very attractive and reasonable requirement. Consequently, its incompatibility with (A1), dependence on normal form, (given nonemptiness), is a serious matter. Perhaps the information that is lost in the transition from the extensive form to the normal form of a game is less inconsequential than is sometimes supposed.

Princeton University
July 1981

Note 1. In the text we provided an informal description of the removal of a choice. This note gives a formal definition.

We say that game $\hat{\Gamma} = (\hat{K}, \hat{P}, \hat{U}, \hat{C}, \hat{h})$ is obtained from game $\Gamma = (K, P, U, C, h)$ by removing some choice $c^* \in C_{u^*}$ at some information set $u^* \in U$, if the elements of $\hat{\Gamma}$ satisfy:

$$\hat{X} = \{x \in X \mid x \text{ does not come after } c^*\}$$
$$\hat{Z} = Z \cap \hat{X}$$
$$\hat{A} = \{a \in A \mid a \notin c^* \text{ and } a \text{ does not come after } c^*\}$$
$$\hat{P} = (\hat{P}_1, \ldots, \hat{P}_n) \text{ where } \hat{P}_i = P_i \cap \hat{X} \text{, for each } i .$$
$$\hat{U} = (\hat{U}_1, \ldots, \hat{U}_n) \text{ where } \hat{U}_i = \{u \cap \hat{X} \mid u \in U_i\} \text{, for each } i .$$
$$\hat{C} = \{ c \cap \hat{A} \mid c \in C\}$$

Also, if u is defined by $\hat{u} = u \cap \hat{X}$, $u \in U$, then

$$\hat{A}_{\hat{u}} = A_u \cap \hat{A} \text{, for each } \hat{u} \in \hat{U} .$$
$$\hat{C}_{\hat{u}} = \{c \cap \hat{A}_{\hat{u}} \mid c \in C_u\} \text{ for each } \hat{u} .$$
$$\hat{h}(z) = h(z) \text{ for each } z \in \hat{Z} .$$

Note 2. We provide here a formal definition of subgame replacement. The game $\tilde{\Gamma} = (\tilde{K}, \tilde{P}, \tilde{U}, \tilde{C}, \tilde{h})$ is obtained from the game $\Gamma = (K, P, U, C, h)$ by the replacement of a subgame τ (with origin x_τ) having a unique Nash equilibrium $r = (r_1, \ldots, r_n)$, if the elements of $\tilde{\Gamma}$ satisfy:

$\tilde{X} = \{x \in X \mid x$ does not come after $x_\tau\}$

$\tilde{Z} = \{x_\tau\} \cup (Z \cap \tilde{X})$

$\tilde{Z}^c = \{x \in \tilde{X} \mid x \notin \tilde{Z}\}$

$\tilde{A} = \{a \in A \mid a$ does not come after $x_\tau\}$

$\tilde{P} = \{\tilde{P}_i\}$ where $\tilde{P}_i = P_i \cap \tilde{Z}^c$, for each i.

$\tilde{U} = \{\tilde{U}_i\}$ where $\tilde{U}_i = \{u \cap \tilde{Z}^c / u \in U_i\}$, for each i.

$\tilde{A}_{\tilde{u}} = A_{\tilde{u}}$ for each $\tilde{u} \in \tilde{U}$.

$\tilde{C} = \{c \cap \tilde{A} \mid c \in C\}$

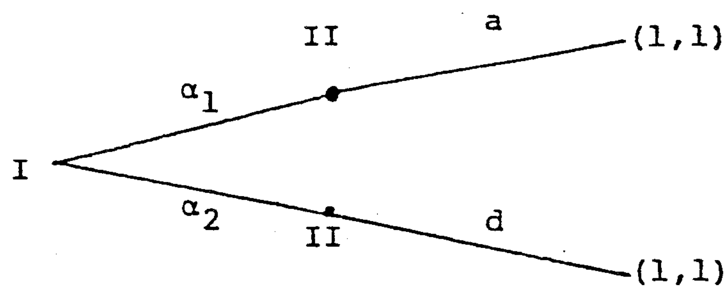
$\tilde{C}_{\tilde{u}} = C_{\tilde{u}}$ for each $\tilde{u} \in \tilde{U}$.

$\tilde{h}(z) = h(z)$, for each $z \in Z \cap \tilde{X}$; $\tilde{h}(x_\tau) = H^\tau(r)$.

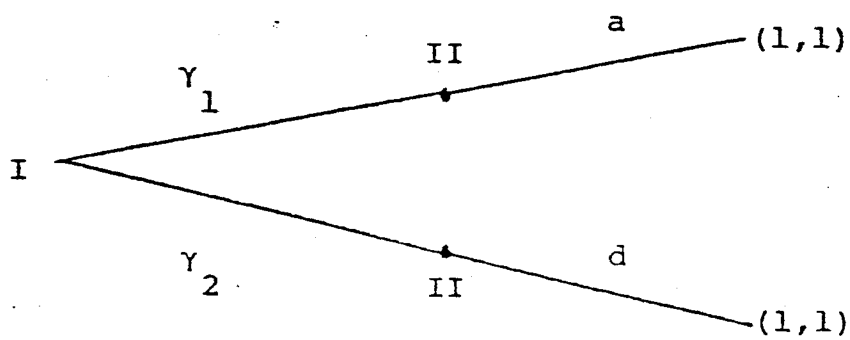
Note 3. We refer to (A4) as weak subgame replacement, since more generally one might want to permit the replacement of subgames with unique Nash equilibria in mixed strategies, or even subgames having multiple equilibria that are equivalent and interchangeable (see Luce and Raiffa (1957)). For convenience, in the remainder of the paper we call (A4) subgame replacement.

Note 4. It is clear that a given extensive form game has many geometric representations, each of which may be labelled according to taste. We take it to be understood that the solution set must be independent of the particular geometric orientation employed, or the labels one chooses to use.

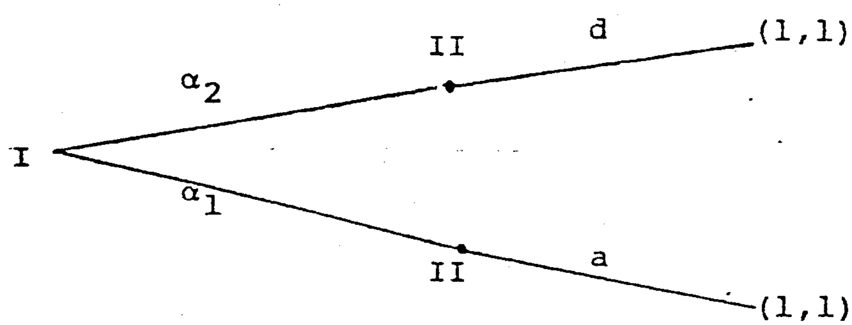
The game $\hat{\Gamma}_2$ (of section IV) has the following equivalent representations:



(i)

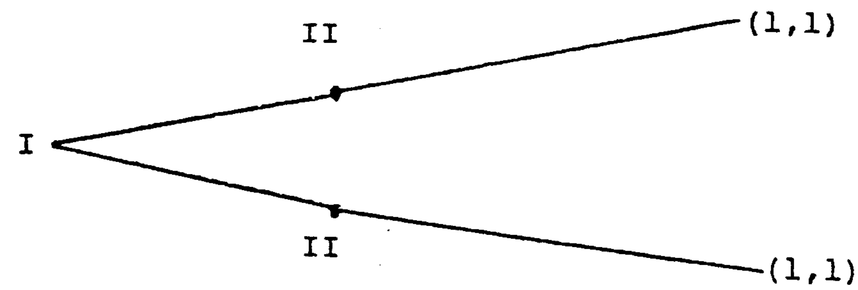


(ii)



(iii)

each of which orients or labels in a different way the underlying skeletal form



Since player II has, in each case, only one choice at each of his information sets, and hence only one strategy, we may without ambiguity refer to this strategy as β_1 in each of the three representations above.

(ii) may be obtained from (i) via the labelling rule

$$\alpha_1 \to \gamma_1$$
$$\alpha_2 \to \gamma_2$$
$$a \to a$$
$$d \to d$$

It follows that the strategy profile (α_1, β_1) of (i) is equivalent to the profile (γ_1, β_1) of representation (ii). We express this symbolically as

$$(\alpha_1, \beta_1)_{(i)} \leftrightarrow (\gamma_1, \beta_1)_{(ii)}$$

Now (ii) may also be obtained from (iii), via the rule

$$\alpha_2 \to \gamma_1$$
$$\alpha_1 \to \gamma_2$$
$$a \to d$$
$$d \to a .$$

Consequently we write

$$(\gamma_1,\beta_1)_{(ii)} \leftrightarrow (\alpha_2,\beta_1)_{(iii)} .$$

Furthermore, we may regard (iii) as game (i) oriented differently. This implies

$$(\alpha_2,\beta_1)_{(iii)} \leftrightarrow (\alpha_2,\beta_1)_{(i)}$$

Collecting these observations, we have

$$(\alpha_1,\beta_1)_{(i)} \leftrightarrow (\gamma_1,\beta_1)_{(ii)} \leftrightarrow (\alpha_2,\beta_1)_{(iii)} \leftrightarrow (\alpha_2,\beta_1)_{(i)}$$

Thus, $(\alpha_1,\beta_1)_{(i)} \leftrightarrow (\alpha_2,\beta_1)_{(i)}$, so that $(\alpha_1,\beta_1) \in F(\hat{\Gamma}_2)$ implies that $(\alpha_2,\beta_1) \in F(\hat{\Gamma}_2)$, as asserted in the proof of Proposition 1. (A similar, but simpler argument establishes that, in game $\tilde{\Gamma}$ of Section V, $(\alpha_1) \in F(\tilde{\Gamma})$ implies $(\alpha_2) \in F(\tilde{\Gamma})$.)

It is possible to restate the heuristic discussion above rigorously and in general terms, using the formal definition of an extensive form game, and by introducing bijections between pairs of these games. This, however, proves to be very clumsy, without providing further illumination. Since the point being made in this note is essentially very simple, we deemed the less formal discussion given above more appropriate here.

BIBLIOGRAPHY

Harsanyi, J. C., "A Solution Concept for n-Person Noncooperative Games," Int. Journal of Game Theory, Vol. 5, Issue 4, 1976, pp. 211-225.

Luce, R. D., and H. Raiffa, Games and Decisions, John Wiley and Sons, New York, 1957.

Myerson, R. B., "Refinements of the Nash Equilibrium Concept," Int. Journal of Game Theory, Vol. 7, Issue 2, 1978, pp. 73-80.

Nash, J. F., "Non-cooperative Games," Annals of Mathematics 54, 1951, pp. 286-295.

Selten, R., "Spieltheoretische Behandlung eines Oligopolmodells mit Nachfrageträgheit," Zeitschrift für die Gesamte Staatswissenschaft 121, 1965, pp. 301-324 and 667-689.

Selten, R., "Reexamination of the Perfectness Concept for Equilibrium Points in Extensive Games," Int. Journal of Game Theory, Vol. 4, 1975, pp. 25-55.