SOLUTION AND ESTIMATION OF SIMULTANEOUS EQUATIONS UNDER RATIONAL EXPECTATIONS

Gregory C. Chow

Econometric Research Program
Research Memorandum No. 291

October 1981

Econometric Research Program
PRINCETON UNIVERSITY
207 Dickinson Hall
Princeton, New Jersey

SOLUTION AND ESTIMATION OF SIMULTANEOUS EQUATIONS UNDER RATIONAL EXPECTATIONS

Gregory C. Chow

1. INTRODUCTION

It is well recognized that, under the assumption of rational expectations, there are multiple solutions to a system of dynamic linear simultaneous equations if expectations of future endogenous variables appear. The existence of multiple solutions for such a model has been bothersome to econometricians wishing to study economic behavior under rational expectations. This paper provides a solution to the problem of multiple solutions by completing the model with additional parameters which can be estimated statistically. Thus the multiple-solutions problem is resolved by appealing to empirical evidence.

Once the proposed solution is employed, estimation and identification of the structural parameters in the original model will become simpler tasks than what are being suggested in the current literature, e.g. Wallis [1980]. In particular, it will not be necessary to formulate and estimate dynamic models for the exogenous variables. A desirable consequence of introducing exogenous variables in the simultaneous-equation model of the Cowles Commission is to allow division of labor in econometric model building. For the purpose of studying the behavior of a subset of economic variables, it is possible to treat some other variables as exogenous. Our proposed solution will simplify the problems of estimation and identification, bringing their treatment closer to the standard treatment of simultaneous equations without rational expectations, and preserve the desirable feature of modelling the endogenous and exogenous variables separately.

Section 2 gives the proposed solution to linear simultaneous equations with expectations of future endogenous variables formed by rational expectations. Section 3 discusses the rationale for the solution, and compares it with the solution suggested by Taylor [1977]. Section 4 comments on the solution proposed by Blanchard and Kahn [1980]. It should be acknowledged that I was led to working on the proposed solution of Section 2 after reading the papers of Taylor [1977] and Blanchard and Kahn [1980]. Section 5 treats the problems of estimation and identification of linear models employing the proposed solution. Section 6 provides an approximate treatment for estimating nonlinear simultaneous equations under rational expectations.

2. SOLUTION TO LINEAR EXPECTATIONS MODELS

Let a system of linear simultaneous equations be written as

$$(1)\qquad By_t + A_1 y_{t-1} + \cdots + A_p y_{t-p} + B_0 y_{t|t-1} + B_1 y_{t+1|t-1} + \cdots + B_q y_{t+q|t-1} + \Gamma z_t = u_t$$

where y_t is a vector of G endogenous variables; z_t is a vector of K exogenous variables; u_t is a vector of normal and serially uncorrelated random disturbances; $y_{t+i|t-1}$ is the expectations of y_{t+i} conditional on information up to the end of period t-1, such information including $y_{t-1}, y_{t-2}, \ldots, u_{t-1}, u_{t-2}, \ldots,$ and $z_t, z_{t-1}, \ldots$. Note that z_t is treated as given when the model is solved to explain y_t. This treatment of z_t appears to be in accord with econometric practice, and is accepted, for example, by Shiller [1978, p. 27] while reviewing the rational expectations literature. It also accords with the convention adopted in dynamic models in areas other than economics where z_{t-1} instead of z_t is used in a model explaining y_t.

Since some elements of $y_{t+q|t-1}$ may be absent from the model, the matrix B_q will often have columns of zeros corresponding to these elements. Let $y^a_{t+q|t-1}$ be a subvector consisting of q_1 elements actually appearing in the model, and write

$$B_q y_{t+q|t-1} = [B^a_q \quad 0] y_{t+q|t-1} = B^a_q y^a_{t+q|t-1} .$$

Consider the reduced-form equations for y^a_t, which we obtain by premultiplying (1) by the first g_1 rows of B^{-1}, denoted by B^{-1}_a

$$\text{(2)} \qquad B^{-1}_a [By_t + A_1 y_{t-1} + \dots + A_p y_{t-p} + B_0 y_{t|t-1} + \dots + B^a_q y^a_{t+q|t-1} + \Gamma z_t] = B^{-1}_a u_t \equiv v^a_t$$

where $B^{-1}_a By_t = y^a_t$. The proposed solution for y^a_t under the assumption of rational expectations is

$$\text{(3)} \qquad B^{-1}_a [By_t + A_1 y_{t-1} + \dots + A_p y_{t-p} + B_0 y_t + \dots + B^a_q y^a_{t+q} + \Gamma z_t] = (B^{-1}_a B^a_q) v^a_{t+q} + C_{q-1} v^a_{t+q-1} + \dots + C_0 v^a_t .$$

In other words, the solution for the reduced-form (2) for y^a_t is obtained by replacing all expectations variables by their actual values (dropping "$|t-1$" in the subscripts) and replacing the residual v^a_t by a linear combination of v^a_{t+q} , ... , v^a_t as indicated, with the coefficients C_{q-1}, ..., C_0 unspecified and to be determined empirically.

To show that (3) is a solution to model (2) for explaining y^a_t, take expectations of both sides of (3) conditioned on information up to the end of t-1, yielding

$$B_a^{-1}[By_{t|t-1} + A_1 y_{t-1} + \dots + A_p y_{t-p} + B_0 y_{t|t-1} + \dots + B_q^a y^a_{t+q|t-1} + \Gamma z_t] = 0 \; . \tag{4}$$

Subtraction of (4) from (2) gives

$$B_a^{-1}[By_t - By_{t|t-1}] = y_t^a - y^a_{t|t-1} = v_t^a \; . \tag{5}$$

Hence, the proposed solution (3) is correct if it implies (5); for then, we can simply add (5) to (4) to obtain the original model (2). Subtracting q from the time subscripts of (3), we get

$$\begin{aligned} B_a^{-1}[By_{t-q} + A_1 y_{t-1-q} + \dots + A_p y_{t-p-q} + B_0 y_{t-q} + \dots + B_q^a y_t^a + \Gamma z_{t-q}] \\ = B_a^{-1} B_q^a v_t^a + C_{q-1} v^a_{t-1} + \dots + C_0 v^a_{t-q} \; . \end{aligned} \tag{3'}$$

(3') in effect is an equation explaining y_t^a, as an explicit solution for y_t^a can be obtained by premultiplying by $[B_a^{-1} B_q^a]^{-1}$, which is assumed to exist. By taking expectations of (3') conditioned on information up to t-1, and subtracting the result from (3'), we obtain (5) and complete the proof.

The solution (3) will be used to express $y^a_{t+i|t-1}$ $(0 \leq i \leq q)$ as functions of actual variables. To demonstrate the algebra involved, we let $p = 1$ and $q = 2$ in the following derivations. Denoting by y_t^b the vector of all other variables whose expectations appear in (1), we solve equation (3) for y^a_{t+2}, yielding

$$y^a_{t+2} = g^a(y^b_{t+1}, y^b_t, y_{t-1}, y^a_{t+1}, y^a_t, z_t) + v^a_{t+2} + C_1^a v^a_{t+1} + C_0^a v_t^a \tag{6}$$

where g^a stands for the linear function obtained by solving (3) for y^a_{t+2}. Equation (6) can be used to evaluate $y^a_{t|t-1}$, $y^a_{t+1|t-1}$ and $y^a_{t+2|t-1}$ successively:

$$y^a_{t|t-1} = g^a(y^b_{t-1}, y^b_{t-2}, y_{t-3}, y^a_{t-1}, y^a_{t-2}, z_{t-2}) + C_1^a v^a_{t-1} + C_0^a v^a_{t-2} \; ; \tag{7}$$

$$(8)\qquad y^a_{t+1|t-1} = g^a(y^b_{t|t-1}, y^b_{t-1}, y_{t-2}, y^a_{t|t-1}, y^a_{t-1}, z_{t-1}) + C^a_0 v^a_{t-1}$$

$$= f^a_1(y^b_{t|t-1}, y_{t-1}, y_{t-2}, y_{t-3}, z_{t-1}, z_{t-2}, v^a_{t-1}, v^a_{t-2})$$

where the function f^a_1 is obtained by substituting (7) for $y^a_{t|t-1}$ in the preceding line; and

$$(9)\qquad y^a_{t+2|t-1} = g^a(y^b_{t+1|t-1}, y^b_{t|t-1}, y_{t-1}, y^a_{t+1|t-1}, y^a_{t|t-1}, z_t)$$

$$= f^a_2(y^b_{t+1|t-1}, y^b_{t|t-1}, y_{t-1}, y_{t-2}, y_{t-3}, z_t, z_{t-1}, z_{t-2}, v^a_{t-1}, v^a_{t-2})$$

where the function f^a_2 is obtained by substituting (8) and (7) respectively for $y^a_{t+1|t-1}$ and $y^a_{t|t-1}$ in the preceding line. If y^b_t is null, our problem is solved, since (7), (8) and (9) will have converted all expectations variables into observables.

Let all elements of y^b_t appear in $y_{t+q-1|t-1}$, i.e., in the first argument of g^a. We need to find a model to convert all $y^b_{t+i|t-1}$ into observables. To do so, we substitute (7), (8) and (9) for $y^a_{t|t-1}$, $y^a_{t+1|t-1}$ and $y^a_{t+2|t-1}$ in model (1) and obtain a model

$$(10)\qquad F(By_t, y_{t-1}, y_{t-2}, y_{t-3}, z_t, z_{t-1}, z_{t-2}, v^a_{t-1}, v^a_{t-2}, y^b_{t|t-1}, \tilde{B}^b_1 y^b_{t+1|t-1}) = u_t$$

which involves only the expectations $y^b_{t+i|t-1}$ $(0 \leq i \leq q-1)$. The reduced form for y^b_t is derived from premultiplying (10) by B_b^{-1}, with $B_b^{-1}By_t = y^b_t$. Applying the method of solution (3), we can write the solution to this reduced-form by replacing the expectations by the actual variables and adding a moving-average residual of order q-1,

$$(11)\qquad B_b^{-1}F(By_t, y_{t-1}, y_{t-2}, y_{t-3}, z_t, z_{t-1}, z_{t-2}, v^a_{t-1}, v^a_{t-2}, y^b_t, \tilde{B}^b_1 y^b_{t+1})$$

$$= (B_b^{-1}\tilde{B}^b_1) v^b_{t+1} + \tilde{C}^b_0 v^b_t$$

where $v_t^b = B_b^{-1}u_t$. Equation (11) can be solved for y_{t+1}^b, yielding

(12) $$y_{t+1}^b = g^b(y_t^a, y_t^b, y_{t-1}, y_{t-2}, y_{t-3}, z_t, z_{t-1}, z_{t-2}, v_{t-1}^a, v_{t-2}^a) + v_{t+1}^b + C_0^b v_t^b$$

where g^b stands for the linear function obtained by solving (11) for y_{t+1}^b. (12) can be used to evaluate $y_{t|t-1}^b$ and $y_{t+1|t-1}^b$ successively:

(13) $$y_{t|t-1}^b = g^b(y_{t-1}^a, y_{t-1}^b, y_{t-2}, y_{t-3}, y_{t-4}, z_{t-1}, z_{t-2}, z_{t-3}, v_{t-2}^a, v_{t-3}^a) + C_0^b v_{t-1}^b ;$$

(14) $$\begin{aligned} y_{t+1|t-1}^b &= g^b(y_{t|t-1}^a, y_{t|t-1}^b, y_{t-1}, y_{t-2}, y_{t-3}, z_t, z_{t-1}, z_{t-2}, v_{t-1}^a, v_{t-2}^a) \\ &= f_1^b(y_{t-1}, y_{t-2}, y_{t-3}, y_{t-4}, z_t, z_{t-1}, z_{t-2}, z_{t-3}, v_{t-1}^a, v_{t-2}^a, v_{t-3}^a, v_{t-1}^b) \end{aligned}$$

where the function f_1^b is obtained by substituting (7) for $y_{t|t-1}^a$ and (13) for $y_{t|t-1}^b$ in the preceding line.

If only a subset of the elements of y_t^b appears in $y_{t+q-1|t-1}$, or in the first argument of g^a, while the remaining elements appear in $y_{t+q-2|t-1}$, or in the second argument of g^a, the solution (11) would apply only to the former variables which form the last argument of the function F in (10) and (11). The second to the last argument of F in (10) and (11) would include both sets of variables, now redesignated y_t^b and y_t^c respectively, y_t^c being those variables appearing in $y_{t+q-2|t-1}$ but not in $y_{t+q-1|t-1}$ or $y_{t+q|t-1}$. Beginning with model (10), with its last two arguments so reinterpreted, we would provide a solution for y_t^b having a moving-average residual of order q-1 as in (11). This solution would be used in the same way that equation (6) was used, to replace the variables $y_{t+i|t-1}^b$ in (10) and to yield a solution for y_t^c having a moving-average residual of order q-2. In the exposition hereafter, we will assume that the introduction of y_t^c is not necessary for our model.

To recapitulate, we have constructed a model (12) for y_t^b involving only actual variables, the lagged variables v_{t-k}^a having been derived from the model

(3) for y_t^a. The model (12) in an ARMA model having other lagged dependent variables and exogenous variables present, the order of the MA process being q-1. By this model, we can compute $y^b_{t|t-1}$ and $y^b_{t+1|t-1}$ using (13) and (14). The results can be used to compute $y^a_{t+1|t-1}$ and $y^a_{t+2|t-1}$ using (8) and (9).

In the special case with $q = 0$, y_t^a will be a subvector consisting of all endogenous variables appearing in $y_{t|t-1}$, having a reduced form

$$B_a^{-1}[By_t + A_1y_{t-1} + \dots + A_py_{t-p} + B_0^a y^a_{t|t-1} + \Gamma z_t] = B_a^{-1}u_t = v_t^a .$$

The solution to this reduced-form equation is

$$(15) \qquad (I + B_a^{-1}B_0^a)y_t^a + B_a^{-1}[A_1y_{t-1}+\dots+A_py_{t-p}+\Gamma z_t] = (I + B_a^{-1}B_0^a)v_t^a$$

which can be used to compute $y^a_{t|t-1}$ in terms of the actual variables.

3. RATIONALE FOR THE SOLUTION

To appreciate the rationale of the solution (3), consider a simple model for a scalar y_t:

$$(16) \qquad y_{t+1|t-1} = by_t + d + z_t + u_t .$$

Let the solution take the form:

$$(17) \qquad y_t = c + \alpha_0 u_t + \alpha_1 u_{t-1} + \dots + \beta_0 z_t + \beta_1 z_{t-1} + \dots$$

where the sums are infinite. (17) implies, with $z_{t|t-1} = z_t$,

$$(18) \qquad y_{t+1|t-1} = c + \alpha_2 u_{t-1} + \alpha_3 u_{t-2} + \dots + \beta_0 z_{t+1|t-1} + \beta_1 z_t + \beta_2 z_{t-1} + \dots$$

If the model (16) is to be valid, (18) must equal

$$(19) \qquad bc + b\alpha_0 u_t + b\alpha_1 u_{t-1} + \cdots + b\beta_0 z_t + b\beta_1 z_{t-1} + \cdots + d + z_t + u_t .$$

To find the coefficients β_i in (17) one could formulate a model for z_t and express $z_{t+1|t-1}$ as an infinite sum of z_t, z_{t-1}, z_{t-2}, ... Then the term $\beta_0 z_{t+1|t-1}$ in (18) can be replaced, and the coefficients of z_{t-i} $(i \geq 0)$ in (18) can be equated with the corresponding coefficients in (19), but this solution is complicated. We choose not to formulate a model for z_t, and obtain a solution by setting $\beta_0 = 0$. Then equating coefficients of (18) and (19) gives

$$(20) \qquad c = (1-b)^{-1}d \; ; \quad \alpha_0 = -b^{-1} , \quad \alpha_{i+1} = b\alpha_i \; (i \geq 1) \; ;$$

$$\beta_1 = 1 , \quad \beta_{i+1} = b\beta_i \; (i \geq 1) .$$

Using these coefficients for (17), we subtract by_{t-1} from y_t to give

$$(21) \qquad y_t = by_{t-1} + d + z_{t-1} - b^{-1}u_t + (\alpha_1+1)u_{t-1} .$$

We observe that the solution (21), with its time subscripts advanced by one, amounts to replacing the expectation variable in (16) by its actual value and forming a new residual by a linear combination of u_t and u_{t+1}, as we have done in writing (3).

There are three advantages in adopting the solution (3) for the simultaneous-equations model (1). First, it avoids the need to postulate and estimate a model for the exogenous variables as required by currently suggested approaches such as in Wallis [1980]. Second, the estimation and identification problems for the model can be handled by standard methods in econometrics. Third, the multiple-solutions problem arising in rational expectations models is solved by appealing to empirical data, rather than by imposing arbitrary conditions. As Taylor [1977] has pointed out, the model (16) has multiple solutions as represented by the free parameter α_1 in (21), which corresponds to the free parameters $C_{q-1}, \ldots, C_0$ in (3). We propose to let the data tell us what the values of these parameters are.

There is another way to resolve the multiple-solutions problem in rational expectations models. One may argue that, when multiple solutions exist, the econometrician has not completed his or her job in specifying a complete model. Take model (16) for example. Just to say that y_t depends on what people expect its value will be in t+1, and on other factors, is not a complete theory until one specifies how the expectation $y_{t+1|t-1}$ is formed. From this point of view, just to postulate rational expectations is not sufficient to complete the theory. The econometrician using model (16) should go back to the drawing board and specify something more about the expectation variable until a unique solution is obtained. If the econometrician does not wish to specify the model further, our proposal is to complete the model by introducing additional parameters $C_0, \ldots, C_{q-1}$ as indicated by (3). Following the suggestion of Taylor [1977] one would choose those parameter values which minimize the stationary covariance matrix of the system, according to some metric. On the other hand, we propose to estimate the values of these parameters empirically. Our method will also work when the system is explosive, for which a stationary covariance matrix does not exist.

4. THE SOLUTION OF BLANCHARD AND KAHN

A main assumption underlying the solution given by Blanchard and Kahn [1980] is that all explosive expectations should be ruled out. See [1980, p. 1310, equation (A4)]. For example, if they were confronted with model (6) with $b > 1$, and

$z_t = z_0$, their solution would be

$$y_t = -b^{-1}(d+z_0+u_t) + b^{-1}y_{t+1|t-1}$$

$$= -b^{-1}(d+z_0+u_t) - b^{-2}(d+z_0+u_{t+1|t-1}) - b^{-3}(d+z_0+u_{t+2|t-1}) - \cdots$$

$$= -b^{-1}(d+z_0)(1-b^{-1})^{-1} - b^{-1}u_t$$

which is obtained by repeated substitutions for $y_{t+i-1|t-1}$ using

$$y_{t+i|t-1} = by_{t+i-1|t-1} + d + z_0 + u_{t+i-1|t-1} \qquad (i \geq 2)$$

whereas our solution is (21), with its parameters to be estimated. We believe that it is arbitrary to impose the stationarity condition on otherwise explosive expectations variables in the system.

However, if Blanchard and Kahn were confronted with the model (16) with $b < 1$, they would not be able to offer a unique solution. Essentially, when dealing with a multivariate linear system, Blanchard and Kahn transform the system into one explaining the canonical variables. Let the system, written in first-order form, consist of n variables, not counting the expectations variables, and let the entire system consist of $n + m$ variables. If exactly n of the n+m characteristic roots of the system are stable and m are explosive, Blanchard and Kahn would provide a unique solution since all canonical variables associated with the explosive roots are determined by the stationarity assumption, using their equation (A4). If there are more than n stable roots, multiple solutions exist.

While disagreeing with Blanchard and Kahn's stationarity assumption which is used to obtain a unique solution in the special case of exactly n stable roots, we wish to point out that their approach, if properly interpreted, can be adopted to solve the general linear model (1) and not just some special cases as they claim, and, second, that the proof of their main result as applied to the general case can be made much simpler. To conform to their notation, let the model explaining y_{t+1} be written as

$$(22)\qquad y_{t+1} = A_0 y_t + A_1 y_{t-1} + A_2 y_{t-2} + B_1 y_{t+1|t} + B_2 y_{t+2|t} + B_3 y_{t+3|t} + \Gamma_0 z_t$$

which can be rewritten as

$$(23)\qquad \begin{bmatrix} y_{t-1} \\ y_t \\ y_{t+1} \\ \hline y_{t+2|t} \\ y_{t+2|t} \\ y_{t+3|t} \end{bmatrix} = \begin{bmatrix} 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 & 0 & I \\ -B_3^{-1}A_2 & -B_3^{-1}A_1 & -B_3^{-1}A_0 & B_3^{-1} & -B_3^{-1}B_1 & -B_3^{-1}B_2 \end{bmatrix} \begin{bmatrix} y_{t-2} \\ y_{t-1} \\ y_t \\ \hline y_{t+1} \\ y_{t+1|t} \\ y_{t+2|t} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ \Gamma_0 \end{bmatrix} z_t$$

The assumption that B_3^{-1} exists is used only for expositional convenience, and can easily be dropped.

Defining

$$x_t = \begin{bmatrix} y_{t-2} \\ y_{t-1} \\ y_t \end{bmatrix} \qquad \text{and} \qquad p_t = \begin{bmatrix} y_{t+1} \\ y_{t+1|t} \\ y_{t+2|t} \end{bmatrix}$$

we can rewrite (23) as

$$(24)\qquad \begin{bmatrix} x_{t+1} \\ p_{t+1|t} \end{bmatrix} = A \begin{bmatrix} x_t \\ p_t \end{bmatrix} + \Gamma z_t \; .$$

This model does not fit into the Blanchard-Kahn framework because x_t does not satisfy their definition of predetermined variables, i.e., $x_{t+1|t} = x_{t+1}$. In the above model $y_{t+1|t} \neq y_{t+1}$. To satisfy the Blanchard-Kahn requirement, we need to drop y_{t+1} in (22), and deal with a linear model involving only y_t, $y_{t-1}, y_{t-2}, \ldots, y_{t+1|t}, y_{t+2|t}, y_{t+3|t}, \ldots$, without y_{t+1} itself.

To solve model (22), we redefine a vector of predetermined variables to be $x_{t|t} = x_t$, i.e., x_t is given information at time t. (Blanchard and Kahn defined x_{t+1} to be given information at time t.) Introduce canonical variables Y_t and Q_t which satisfy

(25) $$\begin{bmatrix} x_t \\ p_t \end{bmatrix} = B \begin{bmatrix} Y_t \\ Q_t \end{bmatrix} \quad ; \qquad A = BJB^{-1} = C^{-1} \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix} C$$

where the diagonal matrix J_1 consists of all stable roots of A and J_2 consists of all roots larger than one in absolute value. Assume the number of predetermined variables to equal the number of stable roots of A. Q_t is found by Blanchard-Kahn's équation (A4), obtained by repeated substitutions for future $Q_{t+i|t}$ $(i \geq 1)$.

Given Q_t the solution for x_t and p_t is very simple, using the definition

(26) $$C_{21}x_t + C_{22}p_t = Q_t$$

and the first part of the model (24)

(27) $$x_{t+1} = A_{11}x_t + A_{12}p_t + \gamma_1 z_t \ .$$

Solving (26) for p_t, we have

(28) $$p_t = - C_{22}^{-1}C_{21}x_t + C_{22}^{-1}Q_t \ .$$

Substituting (28) into (27), we have

$$(29) \qquad x_{t+1} = (A_{11}-A_{12}C_{22}^{-1}C_{21})x_t + A_{12}C_{22}^{-1}Q_t + \gamma_1 z_t$$

$$= B_{11}J_1B_{11}^{-1}x_t + A_{12}C_{22}^{-1}Q_t + \gamma_1 z_t$$

which is identical with the Blanchard-Kahn solution, noting

$$A_{11} - A_{12}C_{22}^{-1}C_{21} = B_{11}J_1B_{11}^{-1}$$

$$A_{12} = B_{11}J_1C_{12} + B_{12}J_2C_{22}$$

by the definitions of A, B, C, J_1 and J_2.

We observe that Blanchard and Kahn have introduced a condition $x_{t+1|t} = x_{t+1}$ which is unnecessary and restrictive. Furthermore, they use this condition in their equation

$$(A6) \qquad 0 = x_{t+1|t} - x_{t+1} = B_{11}(Y_{t+1}-Y_{t+1|t}) - B_{12}(Q_{t+1}-Q_{t+1|t})$$

to obtain their solution. This step is unnecessary.

5. ESTIMATION OF LINEAR MODELS

We first consider maximum likelihood estimation of the parameters of (1). The solution method of Section 2 is used to convert model (1) into three submodels involving no expectations variables, namely (3) for y_t^a, (12) for y_t^b (assuming only two sets of expectations variables in model (1)), and a third submodel for the remaining variables, say y_t^c, obtained by using (7), (8), (9), (13) and (14) to replace the expectations variables in the reduced-form equations for y_t^c. A likelihood function for these submodels can be constructed, under the assumptions that the residuals v_t^a, v_t^b and v_t^c are jointly normal. Since the coefficients of these submodels are known functions of the structural parameters in (1), the likelihood function can be evaluated numerically given the parameters of (1). A gradient or conjugate gradient algorithm can be applied to maximize the likelihood function numerically with respect to the parameters of (1) and of the moving-average processes for the residuals of (3) and (12). If the total number of parameters is not too large, say below 100, the above method of maximum-likelihood estimation using brute force can be recommended. The practitioner may reduce the number of parameters by the assumption that the coefficient matrices of the vector moving-average processes in (3) and (12) are diagonal.

For simultaneous-equations models under rational expectations, the solution submodels for y_t^a, y_t^b and y_t^c correspond to the reduced-form equations of the traditional models. The parameters of these submodels, like the parameters of the reduced-form equations, are functions of the structural parameters. To obtain maximum-likelihood estimates of the structural parameters, one can set up the likelihood function for the reduced form, or the solution submodels, and maximize it with respect to the parameters of the structure using a numerical method.

When the number of structural parameters is very large, it may be computationally more convenient to set up the likelihood function for the structural equations (1) and maximize it iteratively in two steps as follows. The likelihood function of (1) is as given in standard texts, except that the expectations variables themselves should be interpreted as functions of the structural parameters as well as the parameters of the MA residuals of (3) and (12), via equations (7), (8), (9), (13) and (14). In principle, one can substitute the right-hand side of (7), (8), (9), etc. for these expectational variables in the likelihood function and maximize it accordingly. However, since efficient algorithms have already been devised for the maximum-likelihood estimation of (1) with the expectations variables treated as given, for example by Chow and Fair [1973], it may be desirable to divide our task into two steps. First, treating the values of the functions $y^a_{t+i|t-1}$ and $y^b_{t+i|t-1}$ as fixed tentatively, use an existing algorithm for traditional simultaneous-equations models to estimate the parameters of (1). Second, recompute the values of the functions $y^a_{t+i|t-1}$ and $y^b_{t+i|t-1}$ using revised estimates of the parameters of (1) and of the MA residuals in (3) and (12).

The second step is executed in the following manner. Form the coefficients of the autoregressive processes in (3) and (12) using the revised structural parameters and maximize the likelihood function for (3) and (12) with respect only to the moving-average parameters. Given the parameters of (3) and (12), we evaluate $y^a_{t+i|t-1}$ and $y^b_{t+i|t-1}$ using (7), (8), (9), (13) and (14). This two-step procedure can be iterated. If it converges, the result is a set of maximum-likelihood estimates at least in the sense of satisfying the first-order conditions for maximizing the likelihood function which are imposed in each of the two steps.

To economize on computations and still obtain consistent estimates of the structural parameters of (1), one may revise step two by finding consistent estimates of the parameters of (3) and (12), by maximum likelihood for example, without imposing the nonlinear restrictions on their AR parameters as derived from the structure. These estimates are used to evaluate $y^a_{t+i|t-1}$ and $y^b_{t+i|t-1}$. Given the values of the expectations variables, the method of two-stage least squares, for example, can be applied to estimate the parameters of (1). To show that the method of 2SLS provides consistent estimates of the structural parameters in the present case, we only need to show that the reduced-form parameters in the first stage of 2SLS are consistently estimated. If the "composite" predetermined variables $y^a_{t+i|t-1}$ and $y^b_{t+i|t-1}$ in the reduced form are formed by the true parameters of (3) and (12), they are uncorrelated with the residuals of the reduced form and the method of least squares in the first stage is consistent. But consistent estimates of the matrix coefficients of (3) and (12) are also uncorrelated with the residuals v_t^a and v_t^b in the limit. Hence, the predetermined variables $y^a_{t+i|t-1}$ and $y^b_{t+i|t-1}$ formed by using these estimated coefficients are also uncorrelated with the reduced-form residuals in the limit, which implies consistency of the least-squares estimates of the reduced-form coefficients. Therefore, when one applies an existing method such as 2SLS to estimate the parameters of (1) under rational expectations, the additional complication lies in having to estimate (3) and (12) consistently in order to form the predetermined variables $y^a_{t+i|t-1}$ and $y^b_{t+i|t-1}$.

Identification conditions for the structural parameters in (1) are identical with those given in standard texts when we treat all $y_{t+i|t-1}$ as predetermined variables. By the use of models (3) and (12), all expectations variables are converted into predetermined variables and the standard treatment on identification applies.

6. ESTIMATING NONLINEAR MODELS

Following the two-step iterative procedure suggested for linear models, one can estimate the parameters of nonlinear simultaneous equations using any existing method once the expectations variables can be estimated. To estimate the expectations variables, we propose to divide them into y_t^a and y_t^b as in Section 2 and follow the nine steps taken for linear models: (a) construct a set of reduced-form equations for y_t^a involving expectations variables; (b) form a solution to these reduced-form equations by replacing the appropriate expectations by the actual variables and using a moving-average residual of order q; (c) estimate the moving-average parameters in the above solution; (d) using the estimated model in step (c), evaluate $y^a_{t+i|t-1}$ by (7), (8) and (9); (e) transform model (1) into model (10) by substituting for $y^a_{t+i|t-1}$; (f) construct a reduced form for y^b_t from (10); (g) form a solution (12) of this reduced form by replacing all expectations and using a moving-average residual of order q-1; (h) estimate the moving-average parameters in the above solution; and (i) using the estimated model in step (h), evaluate $y^b_{t+i|t-1}$ and $y^a_{t+i|t-1}$.

To follow these steps for a nonlinear system, let the reduced form for y^a_t be the solution for y^a_t from the model

$$(30) \qquad \Phi(y_t, y_{t-1}, \ldots, y_{t-p}, y_{t|t-1}, \ldots, y^a_{t+q|t-1}, z_t) = u_t .$$

There is no need to write down the solution in explicit form. In step (b), we form a model by replacing all expectations variables in (30) by the actual values, conceive y^a_{t+q} as a function g^a of $y_{t+q-1}, \ldots, y_{t-p}$ by solving the model for y^a_{t+q} ignoring the residual u_t (an approximation), and propose the following solution to the reduced form for y^a_t:

$$(31) \qquad y^a_{t+q} = g^a(y_{t+q-1}, \ldots, y_{t-p}, z_t) + v^a_{t+q} + C^a_{q-1} v^a_{t+q-1} + \ldots + C^a_0 v^a_t .$$

Since, given Φ, the value of g^a can be computed numerically, we can estimate the MA parameters in (31) as required in step (c).

To evaluate $y^a_{t+i|t-1}$ in step (d), using (31), we construct functions (7), (8) and (9) as in the linear case. When we write the function $g^a(\cdot)$ in (31), we only mean a computer program which generates a numerical value for this function using the values of its arguments as inputs. Similarly, the functions g^a, f^a_1 and f^a_2 in (7), (8) and (9) for the nonlinear case refer to computer programs which generate values for these functions given their arguments. In the computations of these conditional expectations, we employ the approximate, though incorrect, rule that the expectation of a nonlinear function g^a is the nonlinear function of the expectations. That is, (8) and (9) are equivalent to (31) after conditional expectations are taken of the arguments of g^a. Similarly, the function F in (10) as required in step (e) is derived from the function Φ in (30) where all the expectations $y^a_{t+i|t-1}$ are evaluated by the above functions (programs) specified in (7), (8) and (9).

In steps (f) and (g), it is required to construct a function corresponding to g^b in (12):

$$(32) \qquad y^b_{t+q-1} = g^b(y^b_{t+q-2},\ldots,y_t,\ldots,y_{t-p-q};z_t,\ldots,z_{t-q},v^a_{t-1},\ldots,v^a_{t-q}) + v^b_{t+q-1} + C^b_{q-2}v^b_{t+q-2} + C^b_0 v^b_t .$$

The function g^b in (32) is a computer program to solve for y^b_{t+q-1} using the function F, or the revised function Φ, in step (e); that is, given the arguments of g^b we evaluate all $y^a_{t+i|t-1}$ $(0 \leq i \leq q)$ in the function Φ of (30) by the programs for (7), (8) and (9), and solve the nonlinear equations for y^b_{t+q-1} (or y^b_{t+1} in our exposition), the result being denoted by the vector g^b. In step (h), the MA parameters in (32) are estimated by maximum likelihood numerically. Finally, in step (i), we evaluate approximately the conditional expectations $y^b_{t+i|t-1}$ by taking conditional expectations of the arguments of g^b, and complete the evaluations of $y^a_{t+i|t-1}$ $(0 \leq i \leq q)$ and $y^b_{t+i|t-1}$ $(0 \leq i \leq q-1)$. These expectations variables will be used for estimating the parameters of the nonlinear system (20) by any existing method. The process can be iterated.

The above solution is approximate because we have repeatedly used the approximation that the expectation of a nonlinear function equals the nonlinear function of the expectations. To begin with, (31) is only an approximate solution to the reduced-form equation for y^a_t. Our proof for the solution (3) for linear models does not carry over to nonlinear models because we cannot equate the conditional expectations of $g^a(y_{t+q-1},\ldots,y_{t-p},z_t)$ to $g^a(y_{t+q-1|t-1},\ldots,$ $y_{t-p|t-1},z_{t|t-1})$, but if the model is not hghhly nonlinear, such operations may not be in gross error. Again, to evaluate $y^a_{t+i|t-1}$ approximately using (31), we replace the arguments y^a_{t+q-1}, etc. by $y^a_{t+q-1|t-1}$, etc. to form (8) and (9).

Our method for estimating nonlinear simultaneous equations under rational expectations may be considered computationally expensive. As compared with estimating nonlinear systems without rational expectations, we need to use a number of programs to evaluate and solve nonlinear functions in order to compute g^a in (31), g^b in (32), and the right-hand sides of (7), (8), (9), (13) and (14); we also need to maximize the likelihood functions for two MA processes of orders q and q-1 respectively (or only one MA process of first order if q = 1). All these additional computations are for the purpose of constructing the required expectations variables approximately. Given the expectations variables, the computational burden is the same as for traditional systems without rational expectations. It will be several times as much if several iterations are required. Unlike the linear case where only one iteration will suffice when the unconstrained parameters of (3) and (12) are consistently estimated to evaluate the expectations variables, the functions g^a in (31) and g^b in (32) in the nonlinear case have to be evaluated by using the structural parameters of (30). However, computations of this nature are not considered the most difficult in the current practice of econometrics.

REFERENCES

[1] Blanchard, O. J. and C. M. Kahn, "The Solution of Linear Difference Models under Rational Expectations," Econometrica, 38 (1980), 1305-1311.

[2] Chow, G. C., and R. Fair, "Maximum Likelihood Estimation of Linear Equation Systems with Auto-Regressive Residuals," Annals of Economic and Social Measurement, 2 (1973), 17-28.

[3] Shiller, R., "Rational Expectations and the Dynamic Structure of Macroeconomic Models: A Critical Review," Journal of Monetary Economics, 4 (1978), 1-44.

[4] Taylor, J. B., "Conditions for Unique Solutions in Stochastic Macroeconomic Models with Rational Expectations," Econometrica, 45 (1977), 1377-1385.

[5] Wallis, K. F., "Econometric Implications of the Rational Expectations Hypothesis," Econometrica, 48 (1980), 49-73.

FOOTNOTE

*I would like to thank Avinash Dixit, Cheng Hsiao, and especially John B. Taylor for helpful comments and acknowledge financial support from the National Science Foundation through Grant #SES80-12582.

After this paper was written, I have come across an unpublished paper by Michael K. Salemi of the University of North Carolina at Chapel Hill entitled "The Solution of Linear Rational Expectations Models." The differences between our approaches are:

(i) My solution allows for the existence of exogenous variables in the model for which no stochastic processes are assumed. On the other hand, Salemi does not allow for the existence of exogenous variables and, if his solution were to be modified to include exogenous variables, it would become extremely complicated and stationary stochastic processes for the exogenous variables would have to be postulated.

(ii) The solution of this paper does not require that the endogenous variables be covariance stationary, whereas Salemi's solution does.

(iii) The method of undetermined coefficients was used by Salemi to obtain a solution, and this is not the case for our solution (3). The method of undetermined coefficients works only for covariance-stationary models, which we do not assume.

(iv) I have expressed in Section 4 disagreement with the solution proposed by Blanchard and Kahn to which Salemi subscribes. In contrast with an assertion of Salemi, the method of this paper yields a solution even when more than m roots lie outside the unit circle, as pointed out in Section 4.