A DEFINITION OF SUBJECTIVE PROBABILITY

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1. Introduction. It is widely recognized that the word "probability" has two very different main senses. In its original meaning, which is still the popular meaning, the word is roughly synonymous with plausibility. It has reference to reasonableness of belief or expectation. If "logic" is interpreted in a broad sense, then this kind of probability belongs to logic. In its other meaning, which is that usually attributed to it by statisticians, the word has reference to a type of physical phenomena, known as random or chance phenomena. If "physics" is interpreted in a broad sense, then this kind of probability belongs to physics. Physical probabilities can be determined empirically by noting the proportion of successes in some trials. (The determination is inexact and unsure, like all other physical determinations.)

In order to distinguish these two main senses, physical probabilities will be referred to as "chances," whereas "probability" unqualified will refer to logical probability.

Within the two main categories of logical probability (probability proper) and physical probability (chances), especially in the former, various lesser differences of meaning can be distinguished. In this paper we are concerned with the personal or subjective concept of probability, as considered by Ramsey [10] and Savage [11]. Probabilities and utilities are defined in terms of a person's preferences, in so far as these preferences satisfy certain consistency assumptions. The definition is constructive; that is, the probabilities and utilities can be calculated from observed preferences.

Some persons, especially those with scientific training, are acquainted with the mathematical theory of chances and consider it to be an adequate theory for some kinds of physical phenomena - the uncertain outcomes of the spin of a roulette wheel, the toss of a coin, the roll of a die, a random-number generator. They believe that equipment can be found whose output conforms well with the theory of chances, with stated values for the chances of the simple outcomes. It suffices for this purpose that they believe that some one such piece of equipment exists, for example, a fair coin, since any system of chances can be realized by multiple use of such equipment. (The relation of the theory of chances with chance phenomena has been well illustrated by Kerrich [3].)

For such a person, his utilities can be defined in terms of chances, as shown by von Neumann and Morgenstern [9]. The purpose of this note is to define the person's probabilities in terms of chances, by an extension of the von Neumann-Morgenstern theory. The addition of only two plausible assumptions to those of utility theory enables us to define probabilities with the appropriate properties, in a simple and natural way.

A discussion of our approach in relation to some of the literature will be found at the end of this note.

2. Lotteries.

All of the following considerations are based on the preferences of a single fixed individual, whom we call "you".

Let $\mathcal{A}$ be a set of prizes. A lottery on $\mathcal{A}$ is a device for deciding which prize in $\mathcal{A}$ you will receive, on the basis of a single observation that records which one of a set of mutually exclusive and exhaustive uncertain events took place. It is possible that with each of these uncertain events there is associated a known chance; for example, this would be so if we were observing a single spin of a well-made roulette wheel. On the other hand, it is possible that chances cannot be associated with the uncertain events in question, or that the values of such chances are unknown; for example, this would be so if we were observing a horse race. To distinguish between the two kinds of lotteries, we call the first a "roulette lottery," and the second a "horse lottery." To fix ideas, we shall discuss only horse lotteries that are based on a single, fixed horse race. Our object is to define the probabilities which you associate with each of the possible outcomes of this race.

In addition to the simple lotteries just described, we consider compound lotteries. These are constructed from simple lotteries by iteration; a lottery whose prizes are other lotteries is a compound lottery. A compound lottery may be compounded from roulette lotteries only, or from roulette lotteries and horse lotteries; but since we are considering only one horse race, a compound lottery cannot be compounded from horse lotteries only. Von Neumann-Morgenstern utilities are constructed on the basis of preference comparisons between compound roulette lotteries; in a similar manner, our subjective probabilities will be constructed on the basis of preference comparisons between lotteries that are compounded from horse and roulette lotteries.

3. Assumptions. Perhaps the most lucid published account of the von Neumann-Morgenstern utility theory is that of Luce and Raiffa [8]. As we do not want to take up room by reviewing utility theory here, we adopt their version and take it as known; however, a reader familiar with any version of utility theory will have no difficulty in following our discussion. We note that though Luce and Raiffa assume that the set $\mathcal{A}$ of basic prizes is finite, they make use of this assumption only in concluding that there is a most desired prize A_1 and a least desired prize A_r, such that A_1 is preferred to A_r $(A_1 \succ A_r)$ and for any

prize A_i the following two "preference or indifference" relations hold: $A_1 \gtrsim A_i$, $A_i \gtrsim A_r$. We shall assume this directly, and not assume that $\mathcal{A}$ is finite.

We use utility theory on two levels. First, we suppose that you have a preference ordering on $\mathcal{R}$ that satisfies the axioms of utility theory, where $\mathcal{R}$ is the set of all (simple or compound) roulette lotteries with prizes in $\mathcal{A}$. It follows that it is possible to define a utility function u on $\mathcal{R}$; we normalize this utility function so that $u(A_1) = 1$ and $u(A_r) = 0$. Second, let us assume that the horse race under consideration has exactly s mutually exclusive and exhaustive possible outcomes, denoted by $h_1, \ldots, h_s$. For any s roulette lotteries $R_1, \ldots, R_s$, the symbol $[R_1, \ldots, R_s]$ will denote the (compound) horse lottery that yields you the roulette lottery R_1 if the outcome of the race is h_1 , R_2 if the outcome is h_2 , and so on. The set of all such horse lotteries will be denoted by $\mathcal{H}$. We can now define roulette lotteries <u>whose prizes are such horse lotteries</u>, i.e., members of $\mathcal{H}$ rather than members of $\mathcal{A}$. We suppose that on this new set of roulette lotteries with prizes in $\mathcal{H}$ you also have a preference ordering that satisfies the axioms of utility theory (with $\mathcal{H}$ taking the place of the basic set $\mathcal{A}$ of prizes). Furthermore, we make the following two additional assumptions:

<u>ASSUMPTION 1.</u> (Monotonicity in the prizes) <u>If</u> $R_i \gtrsim R_i'$, <u>then</u>

$$[R_1, \ldots, R_i, \ldots, R_s] \gtrsim [R_1, \ldots, R_i', \ldots, R_s] .$$

<u>ASSUMPTION 2.</u> (Reversal of order in compound lotteries) <u>If</u> $(f_1 B_1, \ldots, f_k B_k)$ <u>denotes the roulette lottery that yields</u> B_1 <u>with chance</u> f_1 , B_2 <u>with chance</u> f_2 , <u>and so on, then</u>

$$(f_1[R_1^1, \ldots, R_s^1], \ldots, f_k[R_1^k, \ldots, R_s^k]) \sim$$
$$\sim [(f_1 R_1^1, \ldots, f_k R_1^k), \ldots, (f_1 R_s^1, \ldots, f_k R_s^k)] .$$

Assumption 1 says that if two horse lotteries are identical except for the prizes associated with one outcome, then your preference between the lotteries is governed by your preference between the prizes associated with that outcome. It is very much akin to Luce and Raiffa's "substitutability" assumption. Assumption 2 says that if the prize you receive is to be determined by both a horse race and the spin of a roulette wheel, then it is immaterial whether the wheel is spun before or after the race. This is akin in spirit to Luce and Raiffa's "reduction of compound lotteries" assumption (sometimes called the "algebra of combining" assumption), but is even more plausible than the latter. Here the "joy in gambling" is not abstracted away, and the chances f_i are not combined in any way.

4. <u>Existence of subjective probabilities.</u> From Assumption 1 it follows that $[A_1, \ldots, A_1]$ is the most desired, and $[A_r, \ldots, A_r]$ the least desired horse lottery. Since the axioms of utility theory hold, we can define a utility function u on $\mathcal{H}$ and on the roulette lotteries on $\mathcal{H}$; we normalize it so that $u[A_1, \ldots, A_1] = 1$ and $u[A_r, \ldots, A_r] = 0$.

<u>THEOREM.</u> <u>There are</u> s <u>non-negative numbers</u> $p_1, \ldots, p_s$ <u>summing to</u> 1, <u>such that for all</u> $[R_1, \ldots, R_s] \in \mathcal{H}$,

$$u[R_1, \ldots, R_s] = p_1 u(R_1) + \ldots + p_s u(R_s) .$$

The number p_i is called the <u>subjective probability</u> of the outcome h_i of the race.

The u on the left side of the displayed equation is the utility on $\mathcal{H}$ defined in the previous paragraph, while that on the right side is that defined on $\mathcal{R}$ in the previous section. From the theorem it follows that $u[R, \ldots, R] = u(R)$, so that we could identify $[R, \ldots, R]$

with R, and think of the preference order as being defined on all lotteries simultaneously; but this is not necessary for the proof of the theorem.

To prove the theorem, note first that by Assumption 1 any horse lottery is determined up to indifference by the utilities of its entries; so that by abusing our symbolism slightly, we can write our horse lotteries in the form $[r_1, \ldots, r_s]$, where $r_i = u(R_i)$. In particular, $u[1, \ldots, 1] = u[A_1, \ldots, A_1] = 1$ and $u[0, \ldots, 0] = u[A_r, \ldots, A_r] = 0$. Now define $p_i = u[0, \ldots, 1, \ldots, 0]$, where the 1 on the right side appears in spot i, and the other spots have 0.

<u>LEMMA.</u> <u>If for some</u> $k \geq 0$ <u>and for</u> $i = 1, \ldots, s$, <u>we have</u> $0 \leq r_i \leq 1$ <u>and</u> $0 \leq kr_i \leq 1$, <u>then</u>

$$u[kr_1, \ldots, kr_s] = ku[r_1, \ldots, r_s] .$$

<u>PROOF.</u> First assume $k \leq 1$. Then by Assumption 2 and the expected utility property for utilities on $\mathcal{R}$, we have

$$[kr_1, \ldots, kr_s] = [kr_1 + (1-k)0, \ldots, kr_s + (1-k)0] \sim$$

$$\sim (k[r_1, \ldots, r_s], (1-k)[0, \ldots, 0]) .$$

Hence

$$u[kr_1, \ldots, kr_s] = ku[r_1, \ldots, r_s] + (1-k)\, u[0, \ldots, 0] =$$

$$= ku[r_1, \ldots, r_s] .$$

Next, if $k > 1$, it follows from the first half that

$$u[r_1, \ldots, r_s] = u[kr_1/k, \ldots, kr_s/k]$$

$$= \frac{1}{k} u[kr_1, \ldots, kr_s] ;$$

and multiplying through by k, we complete the proof of the lemma.

Returning to the proof of the theorem, we set $c = r_1 + \dots + r_s$. If $c = 0$, then all the $r_i = 0$, and the theorem is trivial. If $c > 0$, then the r_i/c are non-negative and sum to 1 , and hence from Assumption 2 we conclude that

$$[r_1/c ,\dots, r_s/c] \sim ((r_1/c)[1,0 ,\dots, 0] ,\dots, (r_s/c)[0 ,\dots, 0,1]) .$$

Hence by the lemma (with $k = 1/c$) and the expected utility property, we have

$$u[R_1 ,\dots, R_s] = u[r_1 ,\dots, r_s] = u[cr_1/c ,\dots, cr_s/c]$$
$$= cu[r_1/c ,\dots, r_s/c] = c((r_1/c)\, u[1,0 ,\dots, 0] + \dots + (r_s/c)\, u[0 ,\dots, 0,1])$$
$$= r_1p_1 + \dots + r_sp_s = p_1u(R_1) + \dots + p_su(R_s) ,$$

and the proof of the theorem is complete.

If our construction of subjective probabilities is applied to a set of exclusive and exhaustive outcomes h_i of some trial, such that each outcome has a known chance f_i , the "horse lotteries" degenerate into "roulette lotteries." (Formally this means that we are assuming $[R_1 ,\dots, R_s] \sim (f_1R_1 ,\dots, f_sR_s)$.) We now see at once from our definition of p_i that $p_i = f_i$ for all i . Thus in this case the subjective probability of any outcome is equal to the chance associated with that outcome. Since the two are equal, it does not matter much which word or symbol we use. The chance refers to the phenomenon, the probability refers to your attitude towards the phenomenon, and they are in perfect agreement.

To provide an adequate basis for the study of scientific inference, the above development of probability needs some extension. Horses must be translated into hypotheses, and the concept of observation must be introduced. We do not pursue these topics here.

5. Comparison with the literature. Ramsey [10], de Finetti [2], Koopman [4,5,6], Savage [11], Davidson and Suppes [1,13], Kraft, Pratt and Seidenberg [7], Smith [12] and others have given definitions of subjective probability. To our mind, the advantage of our approach lies in the simplicity and a priori plausibility of our assumptions, and the simplicity of our construction and proof. The disadvantage lies in the fact that we make use of chances; we do not consider it much of a disadvantage, but that is a matter of taste and viewpoint.

Anyone who wishes to avoid a concept of physical chance distinct from probability may reinterpret our construction as a method of defining more difficult probabilities in terms of easier ones. Such a person may consider that probabilities can be assigned directly to the outcomes of spins of a roulette wheel, flips of a coin, and suchlike, from considerations of symmetry. The probabilities may be so widely agreed on as to be termed impersonal or objective probabilities. Then our construction can be used to define subjective probabilities for other sorts of outcomes in terms of these objective probabilities.

To conclude, we make a comparison of our terminology and approach with Savage's [11]. Our "horse lottery" is his "act"; our "outcome of the race" is his "state of the world"; our "prize" is his "consequence." Of Savage's six postulates, which he numbers P1 through P6, we share with him explicitly P1 (ordering of the horse lotteries - it is among the assumptions of utility theory). We also share P5, which Suppes [13] terms a structure axiom rather than a rationality axiom; it is our assumption that $A_1 > A_r$. P2 and P3 (the "sure-thing" principle) are represented by our Assumption 1, which is close to P3. P4 can be translated to our context by saying that your guess about which horse will win is not affected by the size of the prize offered for each horse. Inherent in the intuitive background of P4 is the assumption that you explicitly guess which horse will win, and that you are able and willing to judge which of two horses has a higher probability of winning. Thus P4 implicitly assumes the existence of comparative probabilities; rather than constructing subjective probabilities from preferences, Savage has constructed numerical subjective probabilities from comparative subjective probabilities. There is nothing corresponding to P4 in our approach. P6 is a continuity postulate which plays a role in defining subjective probability similar to that of the continuity (or "Archimidean") assumption of utility theory in defining the utility function. Here again there is implicit use of the probability notion; but now we have not only comparative probabilities, but also the existence of probabilities that are "small." Although there is nothing directly corresponding to P6 in our approach, we may say, very roughly, that there is no need for such an assumption in our approach because we start out with chances, which have a continuous range.

The heart of Savage's approach is P4 , which asserts the existence of comparative probabilities. We feel that the preference relation is a step closer to experience than the comparative probability relation, and that it is desirable to base subjective probabilities on the preference relation only, without any a priori reference - implicit or explicit - to comparative probabilities.

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