AN ECONOMIC SURVIVAL GAME

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INTRODUCTION

The income of a firm will be created through efficient use of its assets. But the income of the firm will not necessarily be the same as the income of its owner. If the maximization of the income to the firm's owners (e.g. stockholders) can be taken as the purpose of the activity of the firm, it is convenient to consider the firm's accounts as divided into two parts. Following the terminology of [1] and [2], we will call them the enterprise or corporate account, and the dividend or withdrawal account. From this setting of the problem, the firm will try to maximize the withdrawal account over time.

Withdrawals will be made from the corporate account and the corporate account will increase or decrease according to the unforeseen positive or negative profit which may be determined by the activity of the firm in the complicated market situation. If the corporate account falls below a certain level because of a large loss or because of an excessive withdrawal, then the firm will be ruined. So, it may be dangerous to withdraw a large amount from the corporate account at a given time. On the other hand, if the firm hesitates to withdraw even a small amount although it possesses a relatively large corporate account, this too may be poor policy, since its owners (stockholders) depend on some withdrawals being made.

In this situation, what kind of withdrawal policy will be the firm's best strategy in the long run? To answer this question we will describe the model which reflects this situation in Section 1.

Before proceeding to this section we should like to make the following remarks. In our model of the economic survival game the firm's profit is interpreted as the realized value of the market random variable.

But we do not state explicitly what the distribution of this random variable depends on. In terms of usage in decision theory, it may be said that it is determined by the "world." So, in that sense, the economic survival game treated in this paper belongs to the class of games against nature.

In a real oligopolistic situation, it is more natural to assume that the profits of the firms will depend on the strategies taken by the firms in the market and the world. This is surely a situation with which game theory is concerned. For the moment let us restrict the number of the competitive firms to two and neglect the influence of the world. Then, taking into consideration the demand functions, cost functions and so on, we will be able to summarize the above duopolistic situation in the framework of a two-person non-zero-sum game even though this is a complicated process.

For this competitive situation, game theory tells us of the existence of an equilibrium point and a corresponding pair of equilibrium strategies, provided that we do not take into consideration the possibility of cooperation or negotiation between the firms. But in this case it would be very difficult both to calculate these equilibrium mixed strategies because of the tremendous number of possible strategies and to assume that each firm would use its equilibrium strategy in reality. To overcome these difficulties, it may be worthwhile to use the following approach for the equilibrium analysis of the duopolistic situation.

Each firm starts by taking some strategy which may be considered good in some sense. In the next period both firms will modify their strategies by using those which are optimal against the strategies that were taken by their opponents in the first period. In this way in each period each firm will take a strategy which is optimal against the strategies taken by his opponent in previous periods. If each firm proceeds in this way, then two questions arise: will the average of strategies taken by each firm converge to an equilibrium strategy and will their average gains converge to an equilibrium point? The existence of this equilibrium strategy and equilibrium point is proved in game theory as stated above.

In the theory of games this process is usually called either fictitious play — which is not really a suitable name for the situation — or a learning process. In the case of the zero-sum two-person game, the affirmative answer to the above question is proved in [3] and [4]. But the affirmative answer to the non-zero-sum case presented in this paper is conjectural and this is still one of the open problems in game theory. If the assertion could be proved, it would shed a new light on the analysis of the duopolistic situation.*

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1. THE SETTING OF THE PROBLEM

At the end of each time period $t = 0, 1, 2, \ldots$, (the present time point will be expressed as the end of the time period $t = 0$ which is at the same time the beginning of the time period $t = 1$), we assume that a corporation has two accounts, a corporate account $C(t)$ and a withdrawal account $W(t)$. The corporation has to choose a withdrawal strategy w among the set of all withdrawal strategies with a certain intention to be stated later. A withdrawal strategy is a rule which determines, at the end of each time period, the amount of withdrawal (or dividend) which should be transferred from the corporate account to the withdrawal account. In this way, a withdrawal strategy w is defined as a function on the corporate account $C(t)$ which takes a value $w(C(t))$ such that

$$(1.1) \qquad 0 \leq w(C(t)) \leq C(t)\ ,\ t = 0, 1, 2, \ldots .$$

If a strategy w is once chosen by the corporation, then, at the end of each time period $(t - 1)$, w determines the initial level $S(t)$ of the corporate account at the time period t according to the following equation:

$$(1.2) \qquad S(t) = C(t-1) - w(C(t-1)),\ t = 1, 2, \ldots .$$

During the time period t, the corporation will obtain a profit (or loss) $z(t)$ which is determined by many factors such as the market situation and the initial amount of the corporate account $S(t)$ at the time period t, and so on. Then the corporate account $C(t)$ at the end of the time period t is given by the equation

$$(1.3) \qquad C(t) = S(t) + z(t)\ ,\ t = 1, 2, \ldots .$$

(See Fig. 1.)

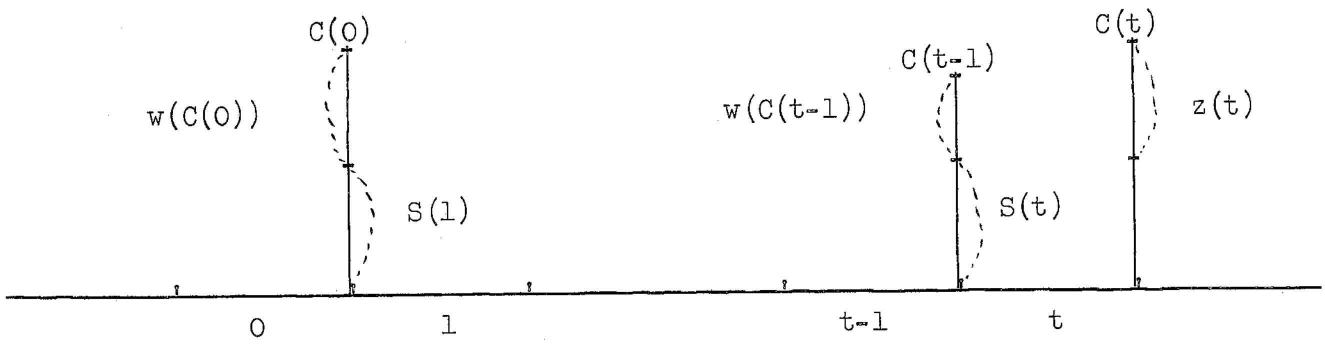

Fig. 1

Here let us assume that a profit $z(t)$ is a realized value of a random variable z whose distribution function F may depend on the value of the corporate account $S(t)$, because the market situation will be more favorable to the firm if it possesses a large corporate account. When it is necessary to make clear the dependency of F on $S(t) = s$, it will be expressed by $F[s]$. The random variable $z(t)$ will be called the market random variable.

Now we introduce the "ruin rule" of the corporation as follows: Once the corporate account $C(t)$ becomes zero or negative at some time point, then the corporation must disappear from the market at that point. Under the conditions stated above, the corporation's objective is to choose a withdrawal strategy w which maximizes the expected value of the sum of the discounted withdrawals, that is the expected value of

$$\sum_{t=0}^{\infty} \rho^t w(C(t)) , \tag{1.4}$$

where ρ is the discount rate, $0 < \rho < 1$. In this expression (1.4), once $C(t)$ takes a non-positive value at some time period $t = t'$, that is once it occurs $C(t') \leq 0$, then we assume $w(C(t')) = w(C(t'+1)) = \dots = 0$, since the corporation must be ruined at that point. The expected value of the amount (1.4) obtained by using a strategy w and starting with an initial corporate account

$$C(0) = x , \quad x > 0 ,$$

will be written $V(x, w)$. We will define the above situation an economic survival game Ω.

<u>Definition 1:</u> A strategy w^o which satisfies the condition

$$V(x, w^o) \geq V(x, w) \tag{1.5}$$

for all possible strategies w and for all initial corporate accounts $x \geq 0$, will be called an optimal strategy of the one-person economic survival game Ω.

The value of $V(x, w^o)$ associated with an optimal strategy w^o will be called the value of the game Ω starting with an initial corporate account x. This value will be written $V(x)$.

Our purpose is to characterize the optimal strategy of the economic survival game Ω.

Generally speaking, in this kind of decision problem, it is usually very difficult to give an explicit formulation of an optimal strategy, even if its existence may be proved. For example, see the very interesting article by M. Shubik and G. L. Thompson [1], in which they analyze the structure of the optimal strategy for an economic survival game. But the game Ω they treat is a very special one, in which the market random variable z can take only the two values -1 and 1 with certain fixed probabilities p and q respectively, independent of the level of the initial corporate account at each time period t.

The purpose of this paper is to determine the pattern of the optimal strategy in an economic survival game Ω in which the market random variable z can take any integral values greater than or equal to the least possible finite negative value.

2. SOME PRELIMINARIES

We will start by stating a very simple but fundamental theorem under a general condition.

Theorem 1: The value function $V(x)$ of the (one-person) economic survival game Ω satisfies the following functional equation:

$$(2.1) \qquad V(x) = \max_{0 \leq y \leq x} \{y + \rho \int_{-\infty}^{\infty} V(x-y+z)\, dF[x-y](z)\}, \text{ for } x > 0,$$

with the boundary condition

$$(2.2) \qquad V(x) = 0, \quad \text{for } x \leq 0.$$

Proof. If we withdraw an amount $0 \leq y \leq x$ immediately at $t = 0$, and then follow with an optimal strategy at all $t \geq 1$, we can obtain the discounted expected value

$$(2.3) \qquad y + \rho \int_{-\infty}^{\infty} V(x-y+z)\, dF[x-y](z).$$

Accordingly, the value $V(x)$ must be the maximum value of (2.3) with respect to y. That is, the value function $V(x)$ satisfies equation (2.1). Q.E.D.

From now on we will proceed under the following assumptions:

Assumption (i): At each time period $t = 1, 2, \ldots$, the market random variable z can take only integral values i, $i = 0, \pm 1, \pm 2, \ldots$ with probabilities $p_i(S(t))$, $i = 0, \pm 1, \pm 2, \ldots$, respectively, which may depend on the initial level of the corporate account $S(t)$ at the beginning of the time period t, where

$$p_i(S(t)) \geq 0, \quad \sum_{i=-\infty}^{\infty} p_i(S(t)) = 1.$$

Assumption (ii): The initial corporate account $C(0)$ is a positive integer.

Assumption (iii): The possible values of a withdrawal function w are restricted to non-negative integers.

Assumption (iv): The first time that the corporate account takes a non-positive value $C(t') \leq 0$ at $t = t'$, the game must be stopped at this moment.

Under these assumptions, the functional equation (2.1), which a value function $V(n)$ of the game Ω should satisfy can be written as follows:

$$V(n) = \max_{0 \leq k \leq n} \{k + \rho \sum_{i=-\infty}^{\infty} V(n-k+i)\, p_i(n-k)\} , \tag{2.4}$$

where all variables n, k, i are integers. If we define the function $G(m)$ by

$$G(m) = \rho \sum_{i=-\infty}^{\infty} V(m+i) p_i(m) , \tag{2.5}$$

then equation (2.4) can be rewritten as

$$V(n) = \max_{0 \leq k \leq n} \{k + G(n-k)\} . \tag{2.6}$$

At this point, to avoid the possible diversity of an optimal strategy which can be seen from (2.6), we restrict our considerations to the following.

Definition 2: An optimal strategy w^o of the game Ω which satisfies the condition

$$w^o(n) = 0 , \text{ if and only if} \tag{2.7}$$

$$G(n) > \max_{1 \leq k \leq n} \{k + \rho G(n-k)\} \tag{2.8}$$

is called a special optimal strategy of the game Ω .

From now on we assume that when we talk about an optimal strategy, it is a special optimal strategy. We also remark that the equation

$$V(n, w^{o}) = k + G(n-k) \tag{2.9}$$

implies

$$w^{o}(n) = k . \tag{2.10}$$

<u>Lemma 1:</u> For an optimal strategy w^{o} of the game Ω , there exists an integer $n_{o} (\geq 0)$ such that

$$w^{o}(1) = w^{o}(2) = \dots = w^{o}(n_{o}) = 0 \tag{2.11}$$

and

$$w^{o}(n_{o} + 1) > 0 . \tag{2.12}$$

<u>Proof:</u> If

$$w^{o}(m) = 0 , \quad \text{for all} \quad m = 1, 2, \dots ,$$

then it is clear that

$$V(n, w^{o}) = 0 , \quad \text{for all} \quad n = 1, 2, \dots .$$

But if $C(0) = n > 0$, then by withdrawing the whole corporate account n immediately, we can obtain $n > V(n, w^{o}) = 0$. This contradicts the optimality of w^{o} . So, there must exist an integer n_{o} which satisfies (2.11) and (2.12).

<u>Lemma 2:</u> If for a certain integer n we have

$$w^{o}(n) = 0 , \tag{2.13}$$

and for a certain integer $a \geq 1$

$$w^{o}(n + \ell) > 0 , \quad \text{for all} \quad 1 \leq \ell \leq a , \tag{2.14}$$

then we have

$$w^{o}(n + \ell) = \ell , \quad \text{for} \quad 1 \leq \ell \leq a , \tag{2.15}$$

and

$$w^o(n + a + 1) = 0 \quad \text{or} \quad a + 1 \; . \tag{2.16}$$

<u>Proof:</u> The fact $w^o(n) = 0$ means

$$\begin{aligned} V(n) &= \max_{0 \le k \le n} \{k + G(n-k)\} \\ &= G(n) \; . \end{aligned} \tag{2.17}$$

On the other hand we have

$$\begin{aligned} V(n+1) &= \max_{0 \le k \le n+1} \{k + G(n+1-k)\} \\ &= \max \left[G(n+1) , \max_{0 \le k \le n} \{1 + k + G(n-k)\}\right] \; . \end{aligned} \tag{2.18}$$

From (2.17) and (2.18), we have

$$V(n+1) = \max \left[G(n+1), 1 + G(n)\right] \; . \tag{2.19}$$

Accordingly, from (2.19), the assumption

$$w^o(n+1) > 0$$

means

$$V(n+1) = 1 + G(n) \; , \tag{2.20}$$

that is

$$w^o(n+1) = 1 \; .$$

From (2.18) and (2.20) we have

$$1 + G(n) = \max_{0 \le k \le n+1} \{k + G(n+1-k)\} \; . \tag{2.21}$$

Next we have

$$\begin{aligned} V(n+2) &= \max_{0 \le k \le n+2} \{k + G(n+2-k)\} \\ &= \max \left[G(n+2) , \max_{0 \le k \le n+1} \{1 + k + G(n+1-k))\right] \; . \end{aligned} \tag{2.22}$$

Then from (2.21) and (2.22) we have

(2.23) $$V(n+2) = \max\,[G(n+2),\ 2 + G(n)]\ .$$

Accordingly, from (2.23), the assumption

$$w^{o}(n+2) > 0$$

means that

$$V(n+2) = 2 + G(n)\ ,$$

that is

$$w^{o}(n+2) = 2\ .$$

In this way, from (2.13) and (2.14), we have (2.15).

Then from (2.15), we know that

(2.24) $$\begin{aligned} V(n+a) &= a + G(n) \\ &= \max_{0 \le k \le n+a} \{k + G(n+a-k)\}\ . \end{aligned}$$

Now

(2.25) $$\begin{aligned} V(n+a+1) &= \max\,[G(n+a+1),\ \max_{1 \le k \le n+a+1} \{k + G(n+a+1-k)\}] \\ &= \max\,[G(n+a+1),\ \max_{0 \le k \le n+a} \{1 + k + G(n+a-k)\}]\ . \end{aligned}$$

Accordingly, from (2.24) and (2.25), we have

$$V(n+a+1) = \max\,[G(n+a+1),\ a+1+G(n)]\ .$$

This implies (2.16). Q.E.D.

3. THE MAIN THEOREM

In this section we will try to make clear the structure of the optimal strategy in a (one-person) economic survival game. But for that we are obliged to place some restrictions on the market random variable in the game.

Theorem 2. We assume the following: The market random variable z in the (one-person) economic survival game Ω is distributed independently over time and has the same discrete distribution — taking integral values — at each time period $t = 1, 2, \ldots$. Among the possible values of z , there exists the least finite negative integer K $(K > 0)$.

Then there exists the unique finite sequence of integers $n_o, n_1, \ldots, n_\ell$ and $m_o, m_1, \ldots, m_{\ell-1}$, such that

(3.1) $$0 \leq n_o < m_o \leq n_1 < m_1 \leq n_2 < m_2 \leq \cdots \leq n_{\ell-1} < m_{\ell-1} \leq n_\ell ,$$

where

$$0 < m_i - n_i \leq K , \; i = 0, 1, \ldots, \ell-1 .$$

And the optimal strategy w^o of the game Ω is given as follows:

(3.3) $$\begin{aligned} w^o(n) &= 0 \quad \text{for} \quad 0 \leq n \leq n_o \\ w^o(n) &= n - n_i , \text{ for } n_i < n < m_i , \; i = 0, 1, 2, \ldots, \ell-1 , \\ w^o(n) &= 0 , \text{ for } m_{j-1} \leq n \leq n_j , \; j = 1, \ldots, \ell , \end{aligned}$$

and

$$w^o(n) = n - n_\ell , \text{ for } n \geq n_\ell .$$

Remark 1: If $w^o(1) > 0$, that is $w^o(1) = 1$, then

$$w^o(n) = n , \text{ for all } n = 1, 2, \ldots .$$

(This will be proved in the proof of Theorem 2.) This situation will be expressed as a special pattern of Theorem 2 in which $n_o = n_\ell = 0$.

Definition 3: When there exist two different integers $n' < m'$ such that

$$w(n') = w(m') = 0$$

and

$$w(n) = n - n' , \text{ for } n' < n < m' ,$$

we call the set of integers $\{n', n'+1, \ldots, m'\}$ a wave of a strategy w, and denote it by $W(n', m')$. We will define the length of a wave $W(n', m')$ by

$$m' - n' + 1 . \tag{3.5}$$

When there exists an integer $n*$ such that

$$w(n*) = 0$$

and

$$w(n) = n - n* , \text{ for all } n > n* ,$$

a strategy w enters in what we will call a stable state from $n*$.

Remark 2: Using the terminology defined above, Theorem 2 may be stated as follows: The optimal strategy w^o of the game Ω has a finite number of waves each of which has a length less than or equal to $K + 1$, and then which enters in a stable state from a certain point.

Corollary: If the least possible negative value of the market random variable z is -1, then the optimal strategy w^o of the game Ω is given as follows: There exists a unique integer N, and

$$\begin{aligned} & w^o(n) = 0 , \text{ for } n \leq N , \\ & \text{and} \\ & w^o(n) = n - N , \text{ for } n > N . \end{aligned} \tag{3.6}$$

(Compare with the results in [1].)

Proof of Corollary: In the case of $-K = -1$, Theorem 2 states the following: Even if an optimal strategy were to have a wave, its length cannot exceed $K + 1 = 2$. On the other hand, from our definition, any wave has a length greater than or equal to 3. This proves that in the case of $-K = -1$, an optimal strategy w^o cannot have any wave; that is, w^o is given as in (3.6). Q.E.D.

<u>Proof of Theorem 2:</u> Let w^o be an optimal strategy of the game Ω . $w^o(1)$ can have two possible values:

(3.7) $$w^o(1) = 0 ,$$

and

(3.8) $$w^o(1) = 1 .$$

At first let us consider the case (3.7). In this case, from Lemma 1, we know the existence of a certain integer n_o, $n_o \geq 1$, such that

(3.9) $$w^o(1) = w^o(2) = \dots = w^o(n_o) = 0 ,$$

and

(3.10) $$w^o(n_o + 1) = 1 .$$

If it were

$$w^o(n) > 0 , \text{ for all } n > n_o ,$$

that is, by Lemma 2

(3.11) $$w^o(n) = n - n_o , \text{ for all } n > n_o ,$$

then Theorem 2 holds with $\ell = 0$.

So, we consider the case where an integer $m_o > n_o$ exists such that

(3.12) $$w^o(n) = n - n_o , \text{ for } n_o < n < m_o ,$$

and

(3.13) $$w^o(m_o) = 0 .$$

Then we will prove that

(3.14) $$m_o - n_o \leq K .$$

Now let us tentatively assume that

(3.15) $$m_o - n_o \geq K + 1 .$$

We will then consider the following two games $\Omega(m_o, w^o)$ and $\Omega(m_o-1, w^*)$, where the notation $\Omega(n, w)$ expresses an economic survival game Ω which starts with an initial corporate account $C(0) = n$, using a strategy w^*. The strategy w^* will be defined below. Let $S^o(t)$, $S^*(t)$, and $C^o(t)$, $C^*(t)$ be the starting corporate accounts and the corporate accounts at a time period t of the games $\Omega(m_o, w^o)$ and $\Omega(m_o-1, w^*)$ respectively. Then we have

(3.16) $$C^o(0) = m_o \text{ , and } C^*(0) = m_o-1 .$$

Now let us define the strategy w^* as follows: As long as $C^o(t)$ continues to take on the values n such that

$$w^o(C^o(t)) = 0 ,$$

we define w^* by

$$w^*(C^*(t)) = 0 .$$

But, once at some time period $t = t' > 0$, $C^o(t)$ reaches a value $C^o(t') = n$ for which

$$w^o(C^o(t')) > 0 ,$$

then after that point, that is for all $t \geq t'$, we define w^* following just the same rule as for w^o.

Then we will show that under assumption (3.15), we have

(3.17) $$V(m_o, w^o) < V(m_o-1, w^*) + 1 ,$$

where $V(m_o, w^o)$ and $V(m_o-1, w^*)$ express the values obtained in the games $\Omega(m_o, w^o)$ and $\Omega(m_o-1, w^*)$ respectively.

Let us divide the set of all positive and negative integers and zero into three disjoint subsets Q_1, R_1 and S_1 defined as follows:

(3.18) $$Q_1 = \{ n \mid n \geq m_o \ , \ w^o(n) = 0 \} \ ,$$

(3.19) $$R_1 = \{ n \mid w^o(n) > 0 \} \ ,$$

(3.20) $$S_1 = \{ n \mid n \leq n_o \} \ .$$

And let us consider the movement of $C^o(t)$ over time. From the definition (3.13) of m_o, we know that $C^o(t)$ starts, at $t = 0$, from the point $C^o(0) = m_o$ in Q_1. Accordingly, concerning the movement of $C^o(t)$, the following three cases are conceivable:

Case A_1: $C^o(t)$ continues to stay in Q_1, that is

(3.21) $$C^o(t) \in Q_1 \ , \text{ for all } t = 0, 1, 2, \ldots \ .$$

Case B_1: After $C^o(t)$ varies in Q_1, at some point $t = t' + 1$, $C^o(t)$ enters for the first time in R_1. That is for some $t = t' \geq 0$,

(3.22) $$C^o(t) \in Q_1 \ , \text{ for } 0 \leq t \leq t' \ ,$$

and

(3.23) $$C^o(t' + 1) \in R_1 \ .$$

Case C_1: After $C^o(t)$ varies in Q_1, at some point $t = t'' + 1$, $C^o(t)$ enters for the first time in S_1. That is, for some $t = t'' \geq 0$,

(3.24) $$C^o(t) \in Q_1 \ , \text{ for } 0 \leq t \leq t'' \ ,$$

and

(3.25) $$C^o(t'' + 1) \in S_1 \ .$$

Let us consider these three cases separately.

Case A_1: In this case, we always have

$$w^o(C^o(t)) = 0 \ , \quad \text{for } t = 0, 1, 2, \ldots \ .$$

Accordingly, from the definition of w^*, we also have

$$w^*(C^*(t)) = 0 \ , \quad \text{for } t = 0, 1, 2, \ldots \ .$$

That is

$$(3.26) \qquad w^{o}(C^{o}(t)) - w^{*}(C^{*}(t)) = 0 \text{ , for } t = 0, 1, 2, \ldots .$$

And we have

$$C^{*}(t) = C^{o}(t) - 1 \text{ , for } t = 0, 1, 2, \ldots .$$

So it is conceivable that $C^{*}(t)$ may reach zero (that is the game $\Omega(m_o-1, w^{*})$ must be stopped) before $C^{o}(t)$ reaches zero. But in this case, from (3.21), we always have

$$C^{o}(t) \geq m_o \geq 3 .$$

That is

$$C^{*}(t) \geq 2 \text{ , for all } t = 0, 1, 2, \ldots .$$

Consequently, the above situation can never occur, and the equality (3.26) surely holds.

Case B_1: In this case we have

$$C^{o}(t) \in Q_1 \text{ , for } 0 \leq t \leq t' .$$

Accordingly, because of the definition of Q_1 , we have

$$(3.27) \qquad w^{o}(C^{o}(t)) = 0 \text{ , for } 0 \leq t \leq t' .$$

Then, from the definition of w^{*} , we also have

$$w^{*}(C^{*}(t)) = 0 \text{ , for } 0 \leq t \leq t' ,$$

that is

$$w^{o}(C^{o}(t)) - w^{*}(C^{*}(t)) = 0 \text{ , for } 0 \leq t \leq t' .$$

Now let

$$C^{o}(t'+1) = n' \in R_1 .$$

Then, from the definition (3.19) of R_1 , we have

$$w^{o}(C^{o}(t'+1)) > 0 .$$

And from the reasoning in Lemmas 1 and 2, there must exist an integer n^* such that

(3.29) $$w^o(n^*) = 0 ,$$

and

(3.30) $$w^o(n) = n - n^* , \text{ for } n^* < n \leq n' .$$

Here the value of n^* must be such that

(3.31) $$m_o \leq n^* < n' ,$$

or

(3.32) $$n^* = n^o < n' .$$

In any case we have

(3.33) $$w^o(C^o(t'+1)) = n' - n^* \quad (> 0) .$$

Then because of the definition of w^*, (3.27) and (3.33) imply the following:

$$w^*(C^*(t)) = 0 , \quad \text{for } 0 \leq t \leq t' ,$$

and

(3.34) $$S^*(t+1) = C^*(t) = C^o(t) - 1 = S^o(t+1) - 1 , \quad \text{for } 0 \leq t \leq t' .$$

Accordingly from (3.28) and (3.34) we have

(3.35) $$C^*(t+1) = n' - 1 .$$

Furthermore, from the definition of w^*, in this case, w^* follows the rule of w^o at all $t \geq t'$. Accordingly, from (3.35) and (3.30), we have

(3.36) $$w^*(C^*(t'+1)) = w^o(n'-1) = n' - 1 - n^* .$$

Then, from (3.33) and (3.36), we have

(3.37) $$w^o(C^o(t'+1)) - w^*(C^*(t'+1)) = 1 .$$

At the same time, we have

$$S^o(t'+2) = n' - (n'-n^*) = n^* ,$$

$$S^*(t'+2) = (n'-1) - (n'-1-n^*) = n^* ,$$

that is

$$S^{o}(t'+2) = S^{*}(t'+2) \; . \tag{3.38}$$

Therefore, from (3.38), after $t = t' + 2$, the two strategies w^{o} and w^{*} give completely the same results.

From the above reasoning, in Case B_1, we may conclude as follows: In the whole process of the two games $\Omega(m_o, w^{o})$ and $\Omega(m_o-1, w^{*})$, the difference between the two withdrawals accruing from w^{o} and w^{*} is just one unit which occurs at a certain time period $t = t' + 1$. The present value of this difference is $\rho^{t'+1} < 1$.

Case C_1: In this case, as was shown in Case B_1, we have

$$w^{o}(C^{o}(t)) = w^{*}(C^{*}(t)) = 0 \; , \quad \text{for} \quad 0 \leq t \leq t'' \; ,$$

and

$$S^{o}(t''+1) = S^{*}(t''+1) + 1 \geq m_o \; . \tag{3.39}$$

Accordingly, in order that $C^{o}(t''+1)$ takes a value in S_1, a realized value z'' of the market random variable z at $t = t'' + 1$ must be such that

$$z'' \leq - (m_o - n_o) \; . \tag{3.40}$$

But under assumption (3.15), (3.40) implies

$$z'' \leq -(K+1) \; . \tag{3.41}$$

But this is impossible, because $-K$ is the least possible negative value of z. Accordingly, under assumption (3.15), Case C_1 can never occur.

From the above considerations we can synthetically state the following conclusion: In the two games $\Omega(m_o, w^{o})$ and $\Omega(m_o-1, w^{*})$, under the assumption (3.15), a positive difference between withdrawals accruing from w^{o} and w^{*} can only occur in Case B_1, and the present value of this difference is $1 \cdot \rho^{t}$ with some $t \geq 1$, that is

$1 \cdot \rho^t < 1$. Therefore, the inequality (3.17) holds. But this inequality implies the following: In the game Ω starting with corporate account m_o , if we withdraw one unit immediately, and then follow the strategy w^* , then we can obtain more than $v(m_o, w^o)$. This result contradicts the optimality of the strategy w^o . This contradiction comes from the assumption (3.15). Accordingly, the inequality (3.14) must hold. That is, the length of the possible first wave $W(n_o, m_o)$ cannot exceed $K + 1$.

Next, from the optimality of w^o , it is clear that it is impossible to have

$$w^o(n) = 0 \text{ , for all } n \geq m_o \text{ .}$$

So, we must have some integer n_1 ($\geq m_o$) such that

(3.42) $$w^o(n) = 0 \text{ , for } m_o \leq n \leq n_1 \text{ ,}$$

and

(3.43) $$w^o(n_1 + 1) = 1 \text{ .}$$

If we have

$$w^o(n) = n - n_1 \text{ , for all } n \geq n_1 \text{ ,}$$

then Theorem 2 holds with $\ell = 1$. So, we assume the existence of the second wave $W(n_1, m_1)$, that is the existence of an integer $m_1 (> n_1)$ such that

(3.44) $$w^o(n) = n - n_1 \text{ , for } n_1 < n < m_1 \text{ ,}$$

and

(3.45) $$w^o(m_1) = 0 \text{ .}$$

Then we will prove that

(3.46) $$m_1 - n_1 \leq K \text{ .}$$

First, as before, let us tentatively assume, on the contrary, that

(3.47) $$m_1 - n_1 \geq K + 1 \text{ .}$$

Then we compare the two games $\Omega(m_1, w^o)$ and $\Omega(m_1-1, w^*)$, where the strategy w^* is defined just the same way as in the case of the first wave $W(n_o, m_o)$, but starting with

$$C^o(0) = m_1 \ , \text{ and } \ C^*(0) = m_1 - 1$$

respectively.

Now let us divide the set of whole integers and zero into the following three disjoint subsets Q_2, R_2 and S_2 :

$$Q_2 = \{ n \mid n \geq m_1 \ , \ w^o(n) = 0 \} \ , \tag{3.48}$$

$$R_2 = \{ n \mid n \geq m_o \ , \ w^o(n) > 0 \} \ , \tag{3.49}$$

$$S_2 = \{ n \mid n \leq n_1 \} \ . \tag{3.50}$$

Then, from the fact that $C^o(0) = m_1 \in Q_2$, concerning the movement of $C^o(t)$ over time, the following three possibilities are conceivable:

<u>Case A_2:</u> $C^o(t) \in Q_2$, for all $t = 0, 1, 2, \ldots$.

<u>Case B_2:</u> There exists a certain time period $t = t' \geq 0$ such that

$$C^o(t) \in Q_2 \ , \text{ for } \ 0 \leq t \leq t'$$

and

$$C^o(t'+ 1) \in R_2 \ .$$

<u>Case C_2:</u> There exists a certain time period $t = t'' \geq 0$ such that

$$C^o(t) \in Q_2 \ , \text{ for } \ 0 \leq t \leq t''$$

and

$$C^o(t''+ 1) \in S_2 \ .$$

With the same reasoning as in the case of the first wave $W(n_o, m_o)$, it is clear that we can make the following statements:

In Case A_2, we always have

$$w^o(C^o(t)) = w^*(C^*(t)) = 0 \ ,$$

and

$$w^{o}(C^{o}(t)) - w^{*}(C^{*}(t)) = 0 \text{ , for } t = 0, 1, 2, \ldots .$$

In Case B_2, just as in Case B_1, we have

$$w^{o}(C^{o}(t)) - w^{*}(C^{*}(t)) = 0 \text{ , for } 0 \leq t \leq t' ,$$

and

$$w^{o}(C^{o}(t'+1)) - w^{*}(C^{*}(t'+1)) = 1 .$$

And at all $t \geq t' + 2$, the two strategies w^{o} and w^{*} give completely the same results. As in the case of the first wave $W(n_o, m_o)$, it can be easily shown that Case C_2 can never occur under assumption (3.47). In this way, under assumption (3.47), we arrive at the contradiction

$$V(m_1, w^{o}) < V(m_1-1, w^{*}) .$$

This contradiction proves the inequality (3.46).

Proceeding as above, we can conclude that the length of any wave $W(n_i, m_i)$ cannot exceed $K + 1$, that is

$$m_i - n_i \leq K .$$

Now we will prove that the number of possible waves $W(n_i, m_i)$ associated with an optimal strategy w^{o} is finite. First, we will tentatively assume that infinitely many waves $W(n_i, m_i)$, $i = 0, 1, 2, \ldots$, were associated with an optimal strategy w^{o}. Then, in the game $\Omega(N, w^{o})$, even if we assume that at each time period $t = 0, 1, 2, \ldots$ we had positive withdrawals $w^{o}(C^{o}(t))$, the amount of the withdrawals $w^{o}(C^{o}(t))$ cannot exceed $K - 1$, since the length of each wave $W(n_i, m_i)$ cannot exceed $K + 1$. Accordingly, the sum of the present values of withdrawals can never exceed

$$(K-1)(1+\rho+\rho^2+\ldots) = \frac{1}{1-\rho} K ,$$

that is

$$V(N, w^{o}) \leq \frac{1}{1-\rho} K , \tag{3.51}$$

for any initial corporate account $C^{o}(0) = N$.

On the other hand, it is clear that

(3.52) $$V(N, w^{o}) \geq N \; .$$

So, if we start with a sufficiently large $C^{o}(0) = N$, then from (3.51) and (3.52), we have the following contradiction

$$V(N, w^{o}) \leq \frac{1}{1-\rho} K < N \leq V(N, w^{o}) \; .$$

Accordingly, the number of possible waves associated with an optimal strategy w^{o} must be finite. That is, if $w^{o}(1) = 0$, then the optimal strategy w^{o} must have the structure stated in Theorem 2.

Next, we consider the case

(3.53) $$w^{o}(1) = 1 \; .$$

In this case let us tentatively assume that a certain wave $W(0, m)$ is associated with an optimal strategy w^{o} . Then by comparing the two games $\Omega(m, w^{o})$ and $\Omega(m-1, w*)$ as above, it can easily be shown that

$$V(m, w^{o}) < V(m-1, w*) + 1 \; .$$

This contradiction proves the following: In case (3.53), a wave $W(0, m)$ cannot be associated with an optimal strategy w^{o} . That is, in this case the optimal strategy w^{o} must have the form stated in Remark 1. In order to complete the proof of Theorem 2, only the proof of the uniqueness of the optimal strategy w^{o} remains. Let us assume that the strategy w^{o} has the structure stated in Theorem 2. Let $w*$ be another optimal strategy of the game Ω . We define the functions V^{o}, $V*$, G^{o} and $G*$ by the following:

(3.54) $$V^{o}(n) = V(n, w^{o}), \; V*(n) = V(n, w*) \; ,$$

(3.55) $$G^{o}(m) = \rho \sum_{i} V^{o}(m + z_i) p_i \; ,$$

(3.56) $$G^*(m) = \rho \sum_i V^*(m + z_i) p_i$$

where z_i are possible values of the market random variable and p_i are their probabilities.

First, we remark that

(3.57) $$V^o(n) = V^*(n) , \quad \text{for all} \quad n = 1, 2, \ldots ,$$

because of the optimality of both w^o and w^*. Now

(3.58) $$V^o(1) = \max \{ 1, G^o(1) \} ,$$

(3.59) $$V^*(1) = \max \{ 1, G^*(1) \} .$$

In the case

$$V^o(1) = 1 = V^*(1) ,$$

that is

$$w^o(1) = 1 = w^*(1) ,$$

the uniqueness of the optimal strategy can be proved in the same way as for the case

(3.60) $$w^o(1) = w^*(1) = 0 .$$

Therefore we show the proof only for the latter case. Let us assume that there exists an integer $\ell (\leq n_o)$ such that

(3.61) $$w^o(n) = w^*(n) = 0 , \quad \text{for} \quad 1 \leq n \leq \ell - 1 ,$$

and

(3.62) $$w^*(\ell) = 1 .$$

Of course we have

(3.63) $$w^o(\ell) = 0 ,$$

since $\ell \leq n_o$.

Then (3.61), (3.62) and (3.63) imply the following:

(3.64) $$V^{o}(n) = G^{o}(n) \; , \; V^{*}(n) = G^{*}(n) \; , \text{ for } 1 \leq n \leq \ell - 1 \; ,$$

(3.65) $$V^{o}(\ell) = G^{o}(\ell) > 1 + G^{o}(\ell - 1) \; ,$$

and

(3.66) $$V^{*}(\ell) = 1 + G^{*}(\ell - 1) \; .$$

From (3.57) and (3.64), we have

(3.67) $$G^{o}(\ell - 1) = G^{*}(\ell - 1) \; .$$

Then from (3.65), (3.66), and (3.67), we have

(3.68) $$V^{o}(\ell) > V^{*}(\ell) \; .$$

This contradicts (3.57). Accordingly we must have

(3.69) $$w^{o}(n) = w^{*}(n) = 0 \; , \quad \text{for } 1 \leq n \leq n_{o} \; .$$

Next let us assume that

(3.70) $$w^{*}(n_{o} + 1) = 0 \; .$$

Then from (3.70) and from the fact that

(3.71) $$w^{o}(n_{o} + 1) = 1 \; ,$$

we have the following:

(3.72) $$V^{o}(n_{o} + 1) = 1 + G^{o}(n_{o}) \; ,$$

and

(3.73) $$V^{*}(n_{o} + 1) = G^{*}(n_{o} + 1) > 1 + G^{*}(n_{o}) \; .$$

On the other hand, from (3.57) and (3.69), we have

(3.74) $$G^{o}(n_{o}) = G^{*}(n_{o}) \; .$$

Accordingly, from (3.72), (3.73) and (3.74), we have the contradiction

(3.75) $$V^{*}(n_{o} + 1) > V^{o}(n_{o} + 1) \; .$$

This proves

$$w^*(n_o+1) = w^o(n_o+1) = 1 . \tag{3.76}$$

Now let us assume that

$$w^o(n_o+2) = 2 , \tag{3.77}$$

but

$$w^*(n_o+2) = 0 . \tag{3.78}$$

(Here we remark, by Lemma 2, that the possible values of $w^o(n_o+2)$ are zero and 2.) These imply that

$$V^o(n_o+2) = 2 + G^o(n_o) , \tag{3.79}$$

and

$$V^*(n_o+2) = G^*(n_o+2) > 2 + G^*(n_o) . \tag{3.80}$$

Then from (3.74), (3.79) and (3.80), we have

$$V^o(n_o+2) < V^*(n_o+2) .$$

This contradiction proves

$$w^*(n_o+2) = w^o(n_o+2) = 2 .$$

Proceeding in this way, we know that w^* coincides with w^o . This proves the uniqueness of the optimal strategy in the one-person economic survival game Ω .

Q.E.D.

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