

LECTURE NOTES ON THE THEORY OF VOTING

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Exercises are proposed at the end of Chapters 1 to 8. No precise allocation of the papers listed in the bibliography to the proposed results is given. Also, I have neglected to mention the very numerous distinct terminologies that are attached to each concept.

Chapters 1 to 5 present the social choice à la Arrow, where profiles of individual preferences must be converted into a collective preference or, more

modestly, into a binary relation expressing society's opinion over pairs of outcomes but lacking transitivity. The former leads to impossibility results (Chapter 2) while the latter generates practical ways (algorithms) to deal with the majority relation. Finally, Chapter 5 explores the single-peaked restriction of the domain where Condorcet winners (in a slightly more general acceptance of the word) are the uniquely best Arrowian aggregators.

Chapters 6 to 9 compare voting rules (for which Arrow's Independence of Irrelevant Alternatives is no longer an issue) by their strategic features. Scoring methods (generalizing the familiar Borda rule) behave nicely if we are ready to accept a random outcome (and do not insist upon ex-ante efficiency); moreover they choose consistently across societies of variable sizes (Chapter 7). On the other hand, voting by veto methods displays a remarkable robustness to non-cooperative and cooperative behaviors of the agents (Chapters 8, 9).

Chapter 1 : SOCIAL CHOICE CORRESPONDENCES AND

SOCIAL WELFARE ORDERINGS.

Basic definitions.

A collective decision problem arises when several individual agents together choose among several outcomes about which their opinions conflict. We denote by N with current element i the set of agents (society) and by A with current element a , the set of outcomes (or alternatives).

Social choice correspondences idealize systematical methods for achieving collective choice (voting rules) that do not preclude any pattern of individual opinions nor any restriction on the set of available outcomes. They are universalizable choice methods (institutions) that ignore, as a matter of principle, where the actual outcomes come from and why the living agents are endowed with such particular preference pattern.

Denote by $L(A)$ the set of linear orders (complete, transitive, antisymmetric binary relations : any two outcomes are comparable and indifferences are ruled out) and by $u_i \in L(A)$ the opinion of agent i (his preference ordering). Then a preference profile u associates to each agent a particular ordering :

$$u = (u_i)_{i \in N} \quad \text{where } u_i \text{ belongs to } L(A)$$

A social choice correspondence (in short s.c.c.) is a mapping S associating to any preference profile $u \in L(A)^N$ and any (nonempty) subset B of A , a (nonempty) subset $S(u;B)$ of B , called the choice set of B at profile u . When the choice set is a singleton for all u, B , we say that the s.c.c. is decisive ; equivalently we call it a social choice function.

Note that individual preferences are restricted only inasmuch as indifferences are forbidden ; this restriction, moreover, is mainly pedagogical : most of the results presented below can be rewritten (sometimes in a less transparent form) to allow for preference preorderings (complete, transitive, reflexive relations), (see e.g. Exercise 1, Chap. 1). Typically our statement of Arrow's theorem on $L(A)$ is stronger than in the set $R(A)$ of preference preorderings.

2 . Binary choices :

This is virtually the only context where one family of social choice correspondences emerge unambiguously from axiomatic arguments.

Let $A = \{a,b\}$ be the outcome set. A preference profile is now an element u in $\{a,b\}^N$ ($u_i = a$ stands for $u_i(a) > u_i(b)$: remember indifferences are ruled out for simplicity).

The s.c.c. S is said to be anonymous if it is a symmetrical mapping of its $|N|$ variables (equal influence of each and every opinion). Next we say that S is :

monotonic if a new supporter of outcome x can not ruin x 's election : for all profile u and agent i :

$$\{x \in S(u) , u_i \neq x\} \Rightarrow \{x \in S(v_i, u_i) \text{ where } v_i = x\}$$

strictly monotonic if a new supporter of an outcome already in the choice set, makes it the unique winner : for all profile u and agent i :

$$\{x \in S(u) , u_i \neq x\} \Rightarrow \{S(v_i, u_i) = \{x\} \text{ where } v_i = x\}$$

For binary choices monotonicity is equivalent to strategyproofness (see Chap. 6 below).

For any two nonnegative integers p, q such that $p+q \leq n+1$ (where $n = |N|$), define the following s.c.c. :

$$\begin{cases} S_{p,q}(u) \ni a & \text{iff at least } p \text{ agents vote } a \\ S_{p,q}(u) \ni b & \text{iff at least } q \text{ agents vote } b \end{cases}$$

Notice that $S_{p,q}$ is a s.c. function if and only if $p+q = n+1$.

Lemma 1

The s.c.c. S is anonymous and monotonic if and only if $S = S_{p,q}$ for some integers p, q as above. Further, it is strictly monotonic if $p+q = n$ or $n+1$.

Corollary May [1952]

There is exactly one s.c.c. which is together anonymous, strictly monotonic and neutral (non discriminating among outcomes : permuting a, b in every agent's opinion permutes correspondingly the choice set) : it is the (binary) majority rule S^* :

$$\text{i.e. : } \begin{cases} S^* = S_{p,p} & \text{if } n = 2p \text{ or } n = 2p-1, \quad p > 1 \\ x \in S^*(u) & \text{iff at least as many agents vote for } x \text{ as for } y. \end{cases}$$

3. Social welfare orderings and Arrow's independence of irrelevant alternatives.

From now on the set A of outcomes has at least three distinct elements.

Take a social choice correspondence S and fix a preference profile u . Consider the choice function $B \rightarrow S(u, B)$ selecting a choice set

$S(u, B)$ within any conceivable subset B of A . If society was made of a single agent, the choice set from B would be simply the best preferred outcome (or outcomes if indifferences are allowed) of this agent over B . Thus the natural, anthropomorphic, inclination is to require the same level of rationality from a collective choice function, whatever the number of agents: "il faut que les procédés d'une assemblée délibérante se rapprochent autant qu'il est possible de ceux que suit l'esprit d'un seul individu dans l'examen d'une question" (Condorcet). Formally we need a preordering $R(u)$ over A (complete and transitive binary relation) such that for all $B \subset A$ and $a \in B$:

$$a \in S(u, B) \iff \{a R(u) b \text{ for all } b \in B\}$$

We shall write $S(u, B) = \underset{B}{\operatorname{argmax}} R(u)$ and say that $R(u)$ rationalizes the choice function $S(u, \cdot)$. In this case the s.c.c. S is completely described by the mapping $u \rightarrow R(u)$ (from preference profiles into preorderings over A). The latter is called a social welfare ordering and was originally defined by Arrow [1963] in his seminal formalization of collective choice. In the s.w.o. approach, individual opinions are aggregated into a "social" preordering; although in general no actual agent is endowed with that representative opinion, some ideal leader exists, who would internalize exactly this opinion: s.w.o. are convincingly simple choice methods. Their shortcomings stem from Arrow's IIA axiom and in turn from Arrow's impossibility theorem: Chapter 3. Our illustrative example uses Borda's s.w.o., namely $B_0(u)$ is derived from the vector of Borda scoring $\beta(u)$:

$$\beta(u; a) = \sum_{i \in N} u_i(a)$$

where u_i is the conventional fixed-scale utility with range $\{0, \dots, p-1\}$ (thus $u_i(a) = p-1$ means that a is top of u_i while $u_i(a) = 0$ means that it is bottom).

An argument in the restaurant.

In this fancy restaurant a Soufflé for dessert is a must ; the smallest piece is for three persons and monoflavoured. Today our party - John, Kenneth and Truman - must choose among Chocolate, Vanilla, Strawberry or Peach. You order before the meal but quite often by the end of it, some flavours are out. "We don't want to bother about choices with full stomach, says John. So let's rank these flavours by their Borda scores, and we will have the first on our list of whatever is available". This gives :

V	P	C	3
C	V	V	2
S	C	S	1
P	S	P	0
John	Kenneth	Truman	Borda scores

hence, from the Borda scores $V(7) > C(6) > P(3) > S(2)$. So when the waiter announces that only Vanilla and Chocolate are left, they order Vanilla at once. One minute later, Truman strikes his forehead : I am very sorry, gentleman : I have put Strawberry and Peach at bottom because for years I was allergic to them ; I just remember that in my last test a week ago, these items proved safe to me, which induces me to change my ranking as :

$$C > S > P > V$$

As Truman is known to be absent-minded but very honest, John willy nilly performs the Borda count again and this time :

$$C(6) > V(5) > P(4) > S(3)$$

Angry Kenneth to John : "your method is poor because our actual choice (between Chocolate and Vanilla) depends so much on our tastes for soufflés

that don't even exist for us and that we would not choose anyhow. What if I ask that we think about Lemon soufflés (they were on the list yesterday) or Camembert soufflés (they could be on the list tomorrow) ? True, they don't show up on the list today, but what's the difference with unavailable Strawberrys ? We ought not to hurry and choose only among actually available soufflés. Since only Chocolate and Vanilla are, majority vote in the unavoidable voting rule, so we shall have Vanilla".

At this point the waiter comes back and tells that Strawberry and Peach are available afterall, which makes John laugh : "see how your method is puzzling : among the four flavours we choose Chocolate (by Borda again) so you should think Chocolate is our "optimum optimorum" ; yet between Chocolate and Vanilla you want us to choose Vanilla so Chocolate is not our first best afterall ; no rational person would be so inconsistent". "I don't see why a collective body should behave according to the rational that we feel is natural for individuals". "Condorcet himself,..." "Gentleman, here comes our Soufflé ; what flavour again ?".

Definition

Given A , the set of outcomes, and society N , a social choice correspondence S is Arrow's independent of irrelevant alternatives (in short AIIA) if for any two profiles u, v and any subset B of A we have :

$$\left\{ \begin{array}{l} \text{the restriction of } u \text{ and } v \text{ to } B \\ \text{coincide} \end{array} \right\} \Rightarrow \{S(u, B) = S(v, B)\}$$

(where the premises mean : $u_i(a) < u_i(b) \Leftrightarrow v_i(a) < v_i(b)$ for all a, b in B).

This says that choice among a subset of outcomes depends only upon preferences over that subset. Thus the choice of the maximal universe A

encompassing all possible outcomes is not conflictual ; a posteriori constraints that narrow down the scope of choice also achieve informational decentralization. In other words, the cost of finding out individual preferences can be kept at their minimum since only opinions about outcomes ultimately feasible, matter.

Typically the s.c.c. associated with the Borda s.w.o. violates AIIA axiom. On the other hand, consider for any subset B of A the restriction of profile u to B and pick the outcomes with best corresponding Borda score. This defines a Borda-like s.c.c. that i) satisfies the AIIA' axiom, ii) is no-longer derived from a s.w.o..

As Arrow's theorem (and the related results of Chap. 2) will show there is no way to reconcile the s.w.o. approach with the AIIA axiom as long as individual preferences over A are not restricted and A contains at least three different outcomes. Hence three routes : the first (Chap. 7) studies social welfare orderings inspired by and more general than the Borda scoring ; the second, more explored route (Chap. 2 to 4) insists on Arrow's IIA axiom and seeks to preserve as much of rationality as possible in their choice function : this yields several extensions of the binary majority rule ; the last route (Chap. 5) seeks to sufficiently restrict the domain of individual preferences so that s.w.o.'s exist that satisfy the AIIA axiom. Of course, when the set A of outcomes is fixed once and for all, the contradiction vanishes because the AIIA axiom disappears: the correspondence $u \rightarrow S(u;A)$ only matters. This is the point of view of the strategic analysis of voting rules (Chapters 6, 8, 9).

Exercices on Chapter 1Exercize 1. Generalization of May's result when indifferences are allowed.

Individual preferences take now the form :

$$u_i = x \quad \text{if agent } i \text{ strictly prefers } x \text{ to } y, \\ \text{where } \{x, y\} = \{a, b\} . \\ = \{a, b\} \quad \text{if he is indifferent.}$$

Say that S is monotonic if for all u and i :

$$\{x \in S(u) ; u_i = y \neq x\} \Rightarrow \{x \in S(v_i, u_{-i}) \text{ where } v_i = \{a, b\}\}$$

and

$$\{x \in S(u) ; u_i = \{a, b\}\} \Rightarrow \{x \in S(v_i, u_{-i}) \text{ where } v_i = \{x\}\}$$

Generalize similarly strict monotonicity.

a) Show that there is exactly one anonymous strictly monotonic and neutral s.c.c. given by :

$$x \in S^*(u) \quad \text{iff at least as many agents strictly prefer } x \text{ to } \\ y \text{ as } y \text{ to } x .$$

b) How does the characterization of anonymous + efficient + monotonic (resp. strictly monotonic) s.c.c. S generalize ?

Exercize 2. Generalization of Lemma 1 : Monotonic social choice functions.

A strong simple game (see Chap. 2.2) is a subset W of $2^N \setminus \emptyset$ of which the elements are called the winning coalitions and such that :

$$T \in W \iff N \setminus T \notin W \quad \text{all } T \subset N$$

$$\{T \in W \text{ and } T \subset T'\} \Rightarrow \{T' \in W\}$$

1) To any strong simple game, associate the following binary s.c.f. :

$$S_W(u) = x \quad \text{iff} \quad \{i \in N / u_i(x) > u_i(y)\} \in W$$

where $\{x, y\} = \{a, b\}$

show that S_W is a monotonic s.c.f. and that conversely any monotonic s.c.f. can be written as $S = S_W$ for some strong simple game W .

2) Given a strong simple game W , define M to be the set of inclusion minimal elements of W . Prove that M satisfies

$$\{W, W' \in M \text{ and } W \subset W'\} \Rightarrow \{W = W'\} \quad \text{all } W, W'$$

and

$$\text{for all } W \subset N, \{\exists W_0 \in M : W_0 \subset W\} \Leftrightarrow \{\forall W_0 \in M : W_0 \cap W \neq \emptyset\}$$

Conversely any subset M of coalitions satisfying these two properties is the set of inclusion minimal elements of some strong simple game W .

3) For $|N| = 4$ there is (up to a permutation of agents) exactly three strong simple game with associated M

$$M = \{\{1\}\}, \quad M' = \{\{12\}\{23\}\{31\}\}, \quad M'' = \{\{12\}\{13\}\{14\}\{234\}\}$$

For $|N| = 5$ determine the seven types of strong simple games.

4) To any vector $(\alpha_1, \dots, \alpha_n)$ of convex weights such that for all coalition W we have :

$$\sum_{i \in W} \alpha_i \neq \frac{1}{2}$$

one can associate the weighted strong simple game :

$$W \in \mathcal{W} \iff \sum_{i \in W} \alpha_i > \frac{1}{2}$$

Prove that all strong simple games are of this type for $|N| \leq 5$, but not for $|N| \geq 6$.

Exercise 3. Unavoidable ties in social choice correspondences.

(Moulin [1983]).

Let A and N both finite with respective cardinality p and n .

Clearly if $n = p = 2$, anonymity and neutrality together are not compatible with single-valuedness: one cannot solve the tie:

$$u_1(a) > u_1(b)$$

$$u_2(b) > u_2(a)$$

without breaking either anonymity (let agent 1 arbitrate) or neutrality (let outcome a prevail). To pursue this argument we define formally the neutrality axiom (no discrimination among outcomes) :

Neutrality : (no discrimination among outcomes) : if σ is a permutation of A into itself (a relabeling of the outcomes) through which profile u is permuted as u^σ ($u_i^\sigma(a) = u_i(\sigma^{-1}(a))$) then $S(u^\sigma, \sigma(B)) = \sigma(S(u, B))$.

a) Prove that the following two statements are equivalent :

- i) n has no prime factor less than or equal to p .
- ii) there exists on A, N an anonymous and neutral s.c. function

Hint for i) \Rightarrow ii). Use a repeated version of the plurality s.c.c..

Under assumption i) one can construct an anonymous and neutral s.c. function which in addition, is efficient (namely selects only Pareto optimal outcomes) : for instance by repeating the plurality s.c.c.. Suppose now that the set A of feasible outcomes is fixed and we seek an anonymous, neutral and single-valued function: $u \rightarrow S(u, A)$. Then the efficiency axiom becomes "costly" in the following sense:

b) There exists an anonymous, neutral, efficient and single-valued function $u \rightarrow S(u; A)$ if and only if property i) holds.

c) There exists an anonymous, neutral and single-valued function $u \rightarrow S(u, A)$ if and only if iii) holds :

iii) p can not be written as the sum of non trivial divisors of n .

Hint : Take $n = 2$, $p = 3$ (hence iii) holds but i) fails) and set $A = \{a, b, c\}$. If both u_1, u_2 agree on their top outcome, take it as $S(u_1, u_2)$; otherwise take the remaining outcome. In general, fix n and set D_n to be the set of integers q that can be written as the sum of non trivial divisors of n with the convention $0 \in D_n$; the proof goes by induction on p . Assume $p \notin D_n$ and fix a profile u . For all t , $0 \leq t \leq n$, denote by A_t the (possibly empty) subset of outcomes that are ranked first by exactly t agents. Since p is not in D_n , and $p = \sum_{0 \leq t \leq n} |A_t|$, at least one t exists such that $|A_t| \notin D_n$ and $|A_t| < p$. Take the largest such t and apply the induction assumption.

This proves the "if" statement. To prove "only if" suppose $p \in D_n$; it can be written as $p = q_1 p_1 + \dots + q_k p_k$ where $q_1 \dots q_k$ are non zero integers and p_1, \dots, p_k are pairwise distinct prime divisors of n ; hence $(p_1 \dots p_k)$ is a divisor of n . Construct a profile among $p_1 \dots p_k$ agents and $p_1 + \dots + p_k$ outcomes that cannot be decisively arbitrated upon under anonymity and neutrality; next replicate agents and outcomes.

Chapter 2 : RATIONALIZABLE SOCIAL CHOICE CORRESPONDENCES :IMPOSSIBILITY RESULTS.1 . Arrow's theorem.Definition 1

Given A and N , a s.c.c. S is said to be rationalizable if for all profile u there is a binary relation $R(u)$ on A such that

$$S(u, B) = \underset{B}{\operatorname{argmax}} R(u) = \{a \in B / a R(u) b \text{ all } b \in B\} \text{ for all } B$$

Whenever $R(u)$ is a preordering of A for all u , the mapping R is just a social welfare ordering (see Chap. 1.3 above) and S is the associated s.c.c. : we call it transitively rationalizable. Yet transitivity of $R(u)$ is not necessary for rationalizability : we can still interpret $a R(u) b$ as "society does not strictly prefer b to a " as long as relation $R(u)$ is acyclic namely its asymmetric component $P(u)$ ($a P(u) b$ iff $a R(u) b$ and no $b R(u) a$) has no cycles.

Lemma 1

If S is rationalizable, for all profile u , the binary relation $R(u)$ is called the base relation of the choice function $S(u, \cdot)$. It is given by :

$$a R(u) b \quad \text{iff} \quad a \in S(u, \{a, b\})$$

Moreover $R(u)$ is complete and acyclic.

We say that S is transitively rationalizable if it is rationalized by a transitive relation. Equivalently S is deduced from a social welfare ordering.

Theorem 1 (Arrow's theorem)

Suppose A contains at least three outcomes. Then S is a transitively rationalizable s.c.c. satisfying the AIIA axiom as well as the following unanimity condition :

$$\{u_i(a) > u_i(b) \text{ all } i \in N\} \Rightarrow \{S(u; \{ab\}) = \{a\}\}$$

(collective choice is compatible with the Pareto dominance).

if and only if S is a dictatorial s.c.c. :

$$S = S_i^* \text{ for some } i \in N, \text{ where } S_i^*(u, B) = \underset{B}{\operatorname{argmax}} u_i \text{ all } u, B$$

Of course the result could be stated for social welfare orderings : a s.w.o. R satisfies AIIA and unanimity (the Pareto dominance relation at u is contained in $R(u)$) if and only if R is dictatorial (for some i , $R(u) = u_i$ all u). Yet the above presentation is more akin to the next two theorems.

Remark

The above statement is slightly stronger than in Arrow [1963] (where unanimity is replaced by a monotonicity axiom - Condition 2 - and non imposition - Condition 5 - : these two together are stronger than unanimity).

When the collective preference relation $R(u)$ is required only to be quasi-transitive (namely : its antisymmetric component $P(u)$ is transitive) then the family of oligarchic s.c.c.s. emerge. Given a coalition of agents T (a nonempty subset of N) we define the T -oligarchic s.c.c. S_T^* as follows :

$$S_T^*(u, B) = \{a \in B / \text{for no } b \in B : u_i(a) < u_i(b) \text{ all } i \in T\},$$

Thus $S_T(u, B)$ is just the B-Pareto set w.r.t. the oligarchy T . It is rationalized by the quasi-transitive relation R_T :

$$a R_T(u) b \text{ iff } \{\text{for some } i \in T, u_i(b) < u_i(a)\}$$

Theorem 2 Brown [1975], Mas Colell and Sonnenschein [1972] (Review of Economic Studies, 39, 185-192).

Suppose A contains at least three outcomes. Then S is a quasi-transitively rationalizable s.c.c. satisfying the AIIA axiom and the unanimity condition, if and only if S is an oligarchic s.c.c. ($S = S_T^*$ for some coalition T).

In particular if we insist on anonymity (one man, one vote) we are left with an impossibility result (for transitive rationalization) and a depressing possibility result (among quasi-transitively rationalizable s.c.c., only the Pareto s.c.c. survives the AIIA axiom and the unanimity condition).

2. Nakamura's theorem.

Let us call arrovian the rationalizable s.c.c. satisfying the AIIA axiom. Arrovian s.c.c.s. contain more than oligarchic s.c.c.s.. In the latter there is a sharp trade-off between decisiveness and symmetry among agents. In fact, there are arrovian s.c.c.s. distributing the decision power in a much more flexible way. Nakamura's theorem fully characterizes the arrovian s.c.c.s. which are, in addition, neutral.

Definition 2

Let \mathcal{W} be a simple game over society N namely a subset of $2^N \setminus \emptyset$, whose members are called the winning coalitions and such that :

$$W \in \mathcal{W} \Rightarrow N \setminus W \notin \mathcal{W}$$

$$\{W \in \mathcal{W}, W \subset W'\} \Rightarrow W' \in \mathcal{W}$$

To any profile u in $L(A)^N$ we associate the dominance relation $R_{\mathcal{W}}(u)$ on A

$$a R_{\mathcal{W}}(u) b \Leftrightarrow \{i \in N / u_i(a) < u_i(b)\} = N(u; b, a) \notin \mathcal{W}$$

Suppose relation $R_{\mathcal{W}}(u)$ is acyclic for all profiles u : it rationalizes a s.c.c. called the core of \mathcal{W} , which clearly satisfies the AIIA axiom.

$$C_{\mathcal{W}}(u; B) = \operatorname{argmax}_B R(u) = \{a \in B / \text{for no } b \in B : N(u; b, a) \in \mathcal{W}\}$$

This s.c.c. is also neutral and monotonic in the following sense : for all $u, v \in L(A)^N$ and all $a \in A$ if the only change from u to v is that a has improved w.r.t. any other outcome (i.e. the restriction of u and v on $A \setminus \{a\}$ coincide and $u_i(a) > u_i(b) \Rightarrow v_i(a) > v_i(b)$ all $i \in N, b \in A$) then election of a is not threatened : if $a \in S(u, B)$ then $a \in S(v, B)$. This generalizes the definition of Chap. I.2.

Lemma 2

Conversely let S be a neutral, monotonic, arrobian s.c.c..

Then S can be uniquely written as the core correspondence of a simple game ω :

$$S(u, B) = C_{\omega}(u, B) \quad \text{for all } u, B .$$

It remains to characterize the simple games ω of which the dominance relation is acyclic. We observe first that these games can not be decisive over each pair of alternatives, i.e. many ties are necessary.

Fix A and a simple game ω on N . The dominance relation $R_{\omega}(u)$ is anti-symmetric (among 2 outcomes, one strictly dominates the other) for all u if and only if the game ω is strong :

$$W \in \omega \Leftrightarrow N \setminus W \in \omega$$

Lemma 3 (Condorcet paradox)

Suppose A contains at least 3 outcomes and ω is a strong simple game over N . Then its dominance relation is acyclic if and only if ω is dictatorial, namely $\omega = \{W \subset N / i^* \in W\}$ for some dictator $i^* \in N$.

Proof

If ω is i -dictatorial, the relation $R_{\omega}(u)$ is just agent i 's ordering, hence an acyclic relation. Suppose now ω is non dictatorial ; then pick an inclusion minimal winning coalition N_1 (hence $N_1 \neq N$). Pick an agent i in N_1 . Since i is not a dictator, coalition $N \setminus i$ is winning. Since N_1 is inclusion minimal, $N_1 \setminus i$ is not winning, therefore $\{i\} \cup (N \setminus N_1)$ is winning. Now we construct a profile u such that

$$\begin{array}{ll} \text{for } i & : \quad a > b > c \\ \text{for } j \in N_1 \setminus i & : \quad c > a > b \\ \text{for } j \in N \setminus N_1 & : \quad b > c > a \end{array}$$

One computes easily :

$$\begin{aligned} N(u, a, b) &= N_1 & N(u, b, c) &= \{i\} \cup (N \setminus N_1) \\ N(u, c, a) &= N \setminus i \end{aligned}$$

Henceforth

$$\begin{aligned} N(u, a, b) &\in W \\ N(u, b, c) &\in W \\ N(u, c, a) &\in W \end{aligned}$$

So that $R_W(u)$ has a cycle.

The ordinary Condorcet paradox corresponds to the strong majority game (with a particular agent breaking possible ties).

Theorem 3 Nakamura [1975]

Given a simple game W on N , its Nakamura number $v(W)$ is the minimal number of winning coalitions with empty intersection

$$v = +\infty, \quad \text{if } \bigcap_{T \in W} T \neq \emptyset$$

$$v = \inf \left\{ |T| / T \subset W \text{ and } \bigcap_{T \in T} T = \emptyset \right\}$$

Given A , the two following statements on (A, W) are equivalent :

- i) for all u in $R(A)^N$, the relation $R_W(u)$ is acyclic on A .
- ii) $|A| < v(W)$.

Proof

Suppose $|A| < v(W)$ and that i fails : for some profile u , the strict component of relation $R_W(u)$ has a cycle $a_1, \dots, a_K, a_{K+1} = a_1$,

namely $N(u; a_1, a_2) \in W, \dots, N(u; a_k, a_{k+1}) \in W, \dots, N(u; a_K, a_1) \in W$.

Since $K < v$ there exists an agent i such that

$$i \in \bigcap_{k=1, \dots, K} N(a_k, a_{k+1}) .$$

This implies a contradiction :

$$u_i(a_1) < u_i(a_2) < u_i(a_3) < \dots < u_i(a_K) < u_i(a_1) .$$

Conversely, suppose that $v \leq |A|$. We construct a profile u such that $R_W(u)$ is cyclic.

Denoting $p = |A|$, we can find a sequence T_1, \dots, T_p of winning coalitions with an empty intersection :

$$T_k \in W, \text{ all } k=1, \dots, p \text{ and } \bigcap_{k=1, \dots, p} T_k \neq \emptyset .$$

Since $\bigcup_{k=1, \dots, p} T_k^c = N$ we can find a sequence R_1, \dots, R_p of pairwise disjoint (possibly empty) coalitions such that

$$R_k \subset T_k^c, \text{ } k=1, \dots, p \text{ and } \bigcup_{k=1, \dots, p} R_k = N .$$

Next, order arbitrarily the outcomes

$$A = \{b_1, \dots, b_p\} .$$

Since T_k is winning $R_k^c = \bigcup_{k' \neq k} R_{k'}$, is winning as well.

We now construct a profile u such that :

$$\begin{aligned}
\text{on } R_1 & : b_1 < b_2 < \dots < b_p \\
\text{on } R_2 & : b_2 < b_3 < \dots < b_p < b_1 , \\
& \vdots \\
\text{on } R_k & : b_k < b_{k+1} < \dots < b_p < b_1 < \dots < b_{k-1} , \\
\text{on } R_p & : b_p < b_1 < \dots < b_{p-1}
\end{aligned}$$

We claim that relation $R_{\omega}(u)$ is cyclic. Namely we have :

$$\begin{aligned}
N(b_2, b_1) & = R_2^c \Rightarrow b_2 P_{\omega}(u) b_1 \\
& \vdots \\
N(b_k, b_{k+1}) & = R_{k+1}^c \Rightarrow b_{k+1} P_{\omega}(u) b_k \\
& \vdots \\
N(b_p, b_1) & = R_1^c \Rightarrow b_1 P_{\omega}(u) b_p
\end{aligned}$$

For instance the Nakamura number of the T-oligarchy game $\omega_T = \{W / T \subset W\}$ is $+\infty$.

As another example consider the q-quota game W_q (where q is an integer $\frac{|N|}{2} < q \leq |N|$) of which the winning coalitions are those with at least q agents (typically it is not strong if $q > \frac{|N|+1}{2}$). Its Nakamura number is worth

$$\nu(W_q) = \left\lceil \frac{n}{n-q} \right\rceil \quad \text{where } \lceil x \rceil \text{ is the smallest integer not inferior to } x.$$

Hence for a 5-outcomes problem among n agents a quota q strictly above $\cdot 8n$ (e.g. 81 out of 100) is necessary to guarantee non emptiness of the core $C_q(u)$

$$a \in C_q(u) \quad \text{iff no other outcome is preferred to } a \text{ by as many as } \cdot 8n \text{ agents.}$$

Although Arrowian s.c.c.s. are rather numerous, they are very undecisive (they declare socially indifferent a huge number of outcomes) as Nakamura's result (see also Exercise 2) imply. Thus in any practical sense we will keep in mind that the rationalizability property and the Arrow's IIA axiom are incompatible.

Proof of Arrow's theorem.

Step 1

Because S is transitively rationalizable, it satisfies the following property, called Chernoff's condition (see Chap. 4.1 below) :

$$S(u, B_1 \cup B_2) \subset S(u, B_1) \cup B_2 \quad \text{all } u, B_1, B_2$$

Therefore S is efficient, for if a is Pareto superior to b ($u_i(a) > u_i(b)$ all i) then $S(u, \{a, b\}) = a$ by unanimity hence

$$a, b \in B \Rightarrow S(u, B) \subset S(u, \{a, b\}) \cup B \setminus \{a, b\} = B \setminus b$$

Step 2

For all a, b define $N(a, b)$ to be the following subset of $2^N \setminus \emptyset$:
 a subset T of agents is in $N(a, b)$ if and only if

$$(1) \quad \text{for all profile } u \quad [\{i \in N / u_i(a) > u_i(b)\} = T] \Rightarrow [S(u, \{a, b\}) = a]$$

which by AIIA is equivalent to

$$(2) \quad \text{there is a profile } u \text{ such that } \{i / u_i(a) > u_i(b)\} = T \\ \text{and } S(u, \{a, b\}) = a$$

Observe that $N \in N(a, b)$ for all a, b (a reformulation of unanimity).

Next we claim :

$$(3) \quad T \in N(a,b) \Leftrightarrow N \setminus T \notin N(b,a) \quad \text{all } T, a, b$$

Take a profile u where $\{i / u_i(a) > u_i(b)\} = T$ i.e. $\{i / u_i(b) > u_i(a)\} = N \setminus T$. If $T \in N(a,b)$ and $N \setminus T \in N(b,a)$ then we get a contradiction of (1), hence \Rightarrow . To prove \Leftarrow , suppose, per absurdum

$$T \notin N(a,b) \quad \text{and} \quad N \setminus T \notin N(b,a)$$

By (2) this means that for all profile u :

$$(4) \quad \{i / u_i(a) > u_i(b)\} = T \Rightarrow S(u; \{a,b\}) = \{a,b\}$$

Pick a third outcome c (remember $|A| \geq 3$) and consider a profile where

$$\text{for } i \in T \quad u_i(a) > u_i(c) > u_i(b)$$

$$\text{for } i \in N \setminus T \quad u_i(b) > u_i(a) > u_i(c)$$

By (4) and unanimity we get successively $S(\{a,b\}) = \{a,b\}$ and $S(\{a,c\}) = a$. Since S is transitively rationalizable this implies $S(\{b,c\}) = b$ henceforth $N \setminus T \in N(b,c)$. Similarly consider a profile such as

$$\text{for } i \in T \quad u_i(c) > u_i(a) > u_i(b)$$

$$\text{for } i \in N \setminus T \quad u_i(b) > u_i(c) > u_i(a)$$

We get $S(\{a,b\}) = \{a,b\}$, $S(\{a,c\}) = c$, hence $S(\{b,c\}) = c$ so that $T \in N(c,b)$.

Step 3

$N(a,b) = N^*$ is independent of a and b . Pick 3 distinct outcomes a, b, c and $T \in N(a,b)$. Next construct a profile such as

$$u_i(c) > u_i(a) > u_i(b) \quad i \in T$$

$$u_i(b) > u_i(c) > u_i(a) \quad i \in N \setminus T$$

Then $S(a,b) = a$ by (2) and $S(c,a) = c$ by unanimity. Hence $S(b,c) = c$ (by transitive rationalization) so that $T \in N(c,b)$. Therefore $N(a,b)$ does not depend on a . A symmetrical argument shows that it does not depend on b either.

Step 4 N^* is a filter.

For T, T' in N^* we must prove $T \cap T' \in N^*$ so suppose per absurdum $T \cap T' \notin N^*$ i.e. $N \setminus (T \cap T') \in N^*$. Let u be a profile such that :

$$u_i(a) > u_i(b) > u_i(c) \quad i \in T \cap T'$$

$$u_i(c) > u_i(a) > u_i(b) \quad i \in T \setminus T'$$

$$u_i(b) > u_i(c) > u_i(a) \quad i \in T' \setminus T$$

$$u_i(c) > u_i(b) > u_i(a) \quad i \in N \setminus (T \cup T')$$

Then, omitting u for simplicity, we have :

$$N \setminus (T \cap T') \in N^* \Rightarrow S(\{a,c\}) = c$$

$$T \in N^* \Rightarrow S(\{a,b\}) = a$$

$$T' \in N^* \Rightarrow S(\{b,c\}) = b$$

From Chernoff's condition it follows that $S(\{a,b,c\})$ contains neither a, b or c , contradiction.

Step 5

From step 2 the filter N^* is maximal

$$(T \in N^* \Leftrightarrow N \setminus T \notin N^*) \quad \text{and contains } N .$$

These three properties together imply (this is well-known and easy to check) that N^* is an ultrafilter, which amounts (since N is finite) to the existence of an agent i^* such that

$$T \in N^* \Leftrightarrow i^* \in T$$

Let us prove that i^* is a dictator. Fix a profile u and a subset B and denote by a the top outcome of u_{i^*} in B . For all b in $B \setminus a$ we have $\{i / u_i(a) > u_i(b)\} \in N^*$ hence $S(u, \{a, b\}) = a$. By Chernoff's condition again this implies $b \notin S(u, B)$ so $S(u, B) = \{a\}$ after all, which was to be proved.

Exercizes on Chapter 2Exercise 1

Prove the theorem on quasi-transitively rationalizable s.c.c. by copying the proof of Arrow's theorem. Step 1,2,3 are unaffected except for property (3). Then N^* is just a prefilter : it is stable by intersection.

Exercise 2

Prove that the only arroviaian social choice functions are dictatorial.

Exercise 3 Anonymous arroviaian s.c.c.s.

Let S be anonymous and monotonic arroviaian s.c.c. with associated binary relation R .

a) Show the existence for all pair (a,b) of distinct outcomes of an integer $n_{a,b}$, $0 < n_{a,b} < n$ ($n = |N|$) such that

$$(5) \quad aR(u) b \Leftrightarrow |N(u,a,b)| > n_{a,b}$$

b) Show that along a cycle of length K , the sum of $n_{a,b}$ is bounded above by $n+K-1$:

for all K -sequence a_1, \dots, a_K of distinct outcomes:

$$\sum_{k=1}^K n_{a_k, a_{k+1}} < n+K-1 \quad \text{where we set } a_{K+1} = a_1$$

c) Conversely, let $(a,b) \rightarrow n_{a,b}$ be an integer valued mapping satisfying the above inequalities for all $K \leq |A|$ and let $R(u)$ be the relation associated by (5). Show that it defines an anonymous and monotonic arrowian s.c.c..

Chapter 3 : VOTING BY BINARY CHOICES.

1 . Binary social choice correspondences.

To avoid the depressing undecisiveness of arrovian s.c.c.s., we will weaken the rationalizability requirement into the property of binariness. To any s.c.c. S and any profile u we associate the base relation $R(u)$ ($a R(u) b$ iff $a \in S(u, \{a, b\})$ see Chap. 2.1) that describes collective choice among doubletons. A binary s.c.c. is one that aggregates only this information.

Definition 1

The s.c.c. S is binary if we have :

$$\text{for all } u, v \quad \{R(u) = R(v)\} \Rightarrow \{S(u, B) = S(v, B) \text{ all } B\}$$

In other words the s.c.c. S can be written as

$$S(u) = \rho [R(u)] \quad \text{where } S(u) \text{ is the choice function} \\ B \rightarrow S(u, B)$$

Here ρ is a mapping from complete relations into choice functions ; we call it a relation aggregator.

For instance a rationalizable s.c.c. is a binary s.c.c. where, in addition, the aggregator ρ is just the argmax operator. In general, however, binary s.c.c.s. will produce cyclic base relations $R(u)$ (e.g. when $R(u)$ is the ordinary majority relation) and aggregate them into decisive or not too indecisive choice functions.

Lemma 1

We shall say that S is a regular binary s.c.c. if it is monotonic, efficient, binary and satisfies the AIIA axiom. In this case :

i) The base relation $R(u)$ is described by a family of simple games $w_{a,b}$, one for each pair a,b of distinct outcomes :

$$a R(u) b \Leftrightarrow N(u,b,a) = \{i \in N / u_i(a) < u_i(b)\} \notin w_{b,a}$$

ii) The relation agregator ρ is AIIA in the following sense : $\rho(R)(B)$ depends only upon the restriction of R to B , and monotonic in the following sense : fix a, R, R' ; if the only change from R to R' is that some b such that $b P a$ are now $b I' a$, and some b such that $a I b$ are now $a P' b$, then the election of a is not threatened ; $a \in \rho(R)(B) \Rightarrow a \in \rho(R')(B)$.

iii) The agregator ρ selects uniquely the winner of R whenever there is one :

$$\{a P b \text{ for all } b \in A\} \Rightarrow \{\rho(R)(A) = \{a\}\} \quad \text{all } a$$

(where P is the asymmetric component of R).

A regular binary s.c.c. is made up of two quite independent pieces : first a family $w_{a,b}$ of simple games describing collective binary choices, and restricted only so as to ensure completeness of relation $R(u)$:

$$(1) \quad T \in w_{a,b} \Rightarrow N \setminus T \notin w_{b,a} \quad \text{all } T, a, b.$$

To fix ideas we shall assume from now on that $w_{a,b}$ is for all a,b the majority game namely :

$$T \in W^* \Leftrightarrow |T| > \frac{|N|}{2}$$

In other words we impose anonymity and neutrality of binary choices. In fact the anonymity assumption could be dropped without affecting Theorem 1 and 2 below ; on the other hand neutrality is needed in these results.

Second a mapping ρ that selects the winner of R whenever there is one, i.e. meets the majority principle due to Condorcet : whenever there is an outcome that beats any other outcome in pairwise comparisons (a Condorcet winner) it should be uniquely elected.

From the AIIA property of ρ follows that we will fix the feasible outcome space A once for all and concentrate on the mapping $\rho^* : R \rightarrow \rho^*(R) = \rho(R)(A)$ associating to each (possibly cyclic) relation R of social preferences a subsets $\rho^*(R)$ of socially good outcomes. Implicit in this simplification is that $\rho(R)(B)$ is determined similarly by means of the restriction of R to B .

Our task now is to propose reasonable relation agregators $R \rightarrow \rho^*(R) \subset A$. This is essentially a one person decision problem : this person ("social body") is supposedly able to formulate binary comparisons to be interpreted as preferences yet it can not guarantee transitivity of these binary preferences. How are we to decide upon an outcome or subset of outcomes amidst cycles (i.e. whenever no Condorcet winner exists) ?

Definition 2

For any complete relation R and A , we denote by $\overset{\vee}{R}$ its transitive closure and we define the top cycle of R as the set of maximal elements of $\overset{\vee}{R}$:

$$tc(R) = \{a / a \overset{\star}{R} b \text{ all } b\} = \left\{ \begin{array}{l} a / \text{for all } b \text{ there is a} \\ \text{R-path from } a \text{ to } b : \\ a = a_1 R a_2 R \dots R a_k = b. \end{array} \right\}$$

Lemma 2

The relation agregator ρ^\star of any regular binary s.c.c. satisfies :

$$\rho^\star(R) \subset tc(R) \quad \text{all } R$$

This result strenghtens property iii) of Lemma 1. The topcycle tc is a monotonic relation agregator ; when combined with the majority game it yields a binary s.c.c. satisfying AIIA as well, yet not a regular binary s.c.c. because efficiency is violated.

Here is an example : we consider a 3-agent society deciding upon a 4-alternatives issue with the following profile

r

c	d	b
d	a	c
a	b	d
b	c	a
u_1	u_2	u_3

The majority game yields the following asymmetric relation.



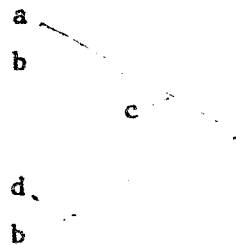
Since A itself is a R-cycle, it coincides with its top cycle. Yet outcome a is Pareto dominated by d .

The trouble with the top cycle is that it is "too big" : if R coincides with the linear ordering $a_1 > a_2 > \dots > a_p$ except for $a_p \perp a_1$, then A itself is a R -cycle and coincides with its top cycle. Yet a_1 is never defeated in binary contests and wins almost all the time !

We construct now some regular binary s.c.c.s.. This will be done in two ways : in Section 2 non neutral but decisive examples, in Section 3 neutral and undecisive examples.

2 . Voting on a binary tree ; the provisional winner algorithm.

A tree is a connected graph with no cycle, where a particular node is singled out as the origin. Thus each node has exactly one predecessor. A binary tree is a tree where each node has zero or two successors. Consider a finite binary tree and attach to each terminal node an outcome from A in such a way that each outcome appears at least once. Finally at each non terminal node, mark up one branch as a tie breaking device. This gives a binary game tree on A . Here is a typical example :



$A = \{a, b, c, d\}$

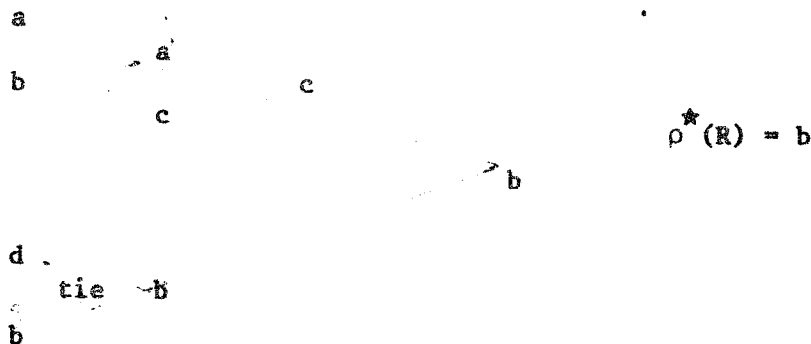
ties are broken in favour
of the bottom branch.

Any binary game tree defines a single-valued relation aggregator $\rho^*(R) \in A$ that selects from any complete relation R a single outcome : pick first a node with two terminal nodes, say x, y , for successors

and make it a terminal node with the R-winner of x,y attached to it ; if x and y are R-indifferent use the tie breaking mark up. Then repeat the operation until a single node is left with a single outcome attached to it : this outcome is $\rho^*(R)$. In the example above, suppose R is given by



This gives :



Lemma 3

The relation agregator ρ^* derived from a binary game tree is a selection from the topcycle :

$$\rho^*(R) \in tc(R) \quad \text{all } R$$

To derive a regular binary s.c. function from a binary game tree we must i) overcome the possible lack of efficiency noticed above, ii) guarantee the monotonicity of the agregator ρ^* ; indeed not every binary game tree yields a monotonic ρ^* : see Exercize 2 below.

These two difficulties are solved by the following algorithm.

Definition 3

Let σ^* be a linear ordering of A according to which we write $A = 1, 2, \dots, p$. Given a complete relation R on A , the algorithm of provisional winners is defined by :

$$\alpha_1 = 1 ; \text{ for all } k=1, \dots, p ; \alpha_k = \begin{cases} k & k R \alpha_1, \dots, k R \alpha_{k-1} \\ \alpha_{k-1} & \text{otherwise} \end{cases}$$

We set $\rho_{\sigma^*}^*(R) = \alpha_p$, to be called the σ^* -winner of R .

Theorem 1 Moulin [1979]

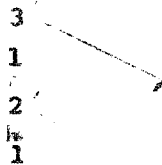
The above relation agregator is derived from a binary game tree. When combined with the majority game it yields a regular binary social choice function.

Proof :

To fix ideas, suppose that $A = \{1, 2, 3\}$ so that the algorithm of provisional winners is :

$$\alpha_1 = 1 ; \alpha_2 = \begin{cases} 1 & \text{if } 1 P 2 \\ 2 & \text{if } 2 R 1 \end{cases} ; \rho_{\sigma^*}^*(R) = \alpha_3 = \begin{cases} 1 & \text{if } 1 R 2, 1 R 3, \\ 3 & \text{if } 1 R 2, 3 P 1, \\ 2 & \text{if } 2 P 1, 2 R 3, \\ 3 & \text{if } 2 P 1, 3 P 2, 3 P 1 \end{cases}$$

which derives from the binary game tree



ties are broken in favour of the bottom branch.

The general proof goes by fixing $A = \{1, \dots, p\}$ and uses the notation :

$$R(x, y) = \begin{cases} x & \text{if } x P y \\ y & \text{if } y R x \end{cases}$$

Then we define $(p-1)$ mapping $\varphi^1, \dots, \varphi^{p-1}$ by the following induction :

$$\begin{aligned} \varphi^1(x) &= R(x, 1) && \text{all } x = 2, \dots, p \\ \varphi^2(x) &= R(\varphi^1(x), \varphi^1(2)) && \text{all } x = 3, \dots, p \\ &\vdots && \\ \varphi^k(x) &= R(\varphi^{k-1}(x), \varphi^{k-1}(k)) && \text{all } x = k+1, \dots, p \\ \varphi^{p-1}(p) &= R(\varphi^{p-2}(p), \varphi^{p-2}(p-1)) \end{aligned}$$

One checks by induction that for all $k=1, \dots, p-1$

$$\varphi^k(x) = \begin{cases} x & \text{if } x P \alpha_1, \dots, x P \alpha_k \\ \alpha_k & \text{otherwise} \end{cases} \quad \text{all } x = k+1, \dots, p$$

$$\varphi^k(k+1) = \alpha_{k+1}$$

On the other hand the induction formulas for φ^k allow to construct a binary game tree guaranteing the election of $\varphi^{p-1}(p)$ for all R .

To prove the second statement of the theorem, we compound the mapping u (profile) $\rightarrow R^*(u)$ (majority relation associated with u) with the aggregator ρ_G^* and must prove that $u \rightarrow \rho_G^*(R(u))$ is both monotonic and efficient. Monotonicity is immediately checked on the algorithm of provisional winners. For efficiency, suppose per absurdum $\rho_G^*(R(u)) = a$ and b Pareto dominates a . By construction we have $a R^*(u) \alpha_k$ for all provisional winners $\alpha_1, \dots, \alpha_p$ so that b is not

among them. Thus, by definition of the algorithm again, there is a provisional winner α_k such that $\alpha_k P^*(u) b$. From $|N(u, a, \alpha_k)| \geq \frac{|N|}{2}$ $|N(u, \alpha_k, b)| > \frac{|N|}{2}$ results that $N(u, a, \alpha_k)$ intersects $N(u, \alpha_k, b)$ hence at least one agent prefers a to b , a contradiction.

3 . Uncovered set and Copeland set.

We construct now two neutral regular binary s.c.c., at the cost of decisiveness.

Definition 4

Given a complete relation R we say that outcome b covers outcome a if we have :

$$\left. \begin{array}{l} a R z \Rightarrow b R z \\ z R b \Rightarrow z R a \end{array} \right\} \text{all } z \in A$$

with at least one of these two implications being not an equivalence.

The uncovered set $un(R)$ is the set of those outcomes that are not covered by any other outcome.

Definition 5

Given a complete relation R , the Copeland score of outcome a is defined as

$$E(a) = 2 \{ | \{ b / a P b \} | + | \{ b / a I b \} | \}$$

where P and I are the asymmetric and symmetric component of R (strict preference and indifference respectively).

The Copeland set $C(R)$ is the set of outcomes with maximal Copeland score.

Theorem 2 (Miller [1977])

The Copeland set and uncovered set, when combined with the majority relation $R^*(u)$ both define a neutral regular binary s.c.c.. Moreover the Copeland set is a subset of the uncovered set.

Proof

The monotonicity of both relation agregators un and C are clear. Efficiency of $un(R^*(u))$ follows by the same argument as in Theorem 1 once we observe the following : if a is uncovered in R and b is any other outcome, at least one of the three following statements holds :

- i) $a R b$
- ii) $a R z$ and $z P b$ for some z
- iii) $a P z$ and $z R b$ for some z

To prove efficiency of $C(R^*(u))$ it suffices to check the inclusion $C(R) \subset un(R)$ which is our last claim. Incidentally, this inclusion proves non emptiness of $un(R)$. Let a in $C(R)$ and suppose a is covered by outcome b . Then

$$\{z / a R z\} \subset \{z / b R z\}$$

and

$$\{z / a P z\} \subset \{z / b P z\}$$

Observe that $C(a) = |\{z / a P z\}| + |\{z / a R z\}|$. Thus because $C(b) \leq C(a)$ we conclude that both inclusions are equalities so that $a R z \Leftrightarrow b R z$ and $z R a \Leftrightarrow z R b$, all z , contradiction.

The Copeland set defines a more appealing binary s.c.c. than the uncovered set, inasmuch as it is always more decisive. Yet a point in the uncovered set is easier to obtain than one in the Copeland set, for the fairly simple algorithm of provisional winners always ends up within $un(R)$ (see the proof of Theorem 1 ; also Chap. 4.3, where it is shown that not every element of $un(R)$ can be obtained in this way). The following question is an open problem : does it exist a binary game tree of which the associated s.c. function always ends up within the Copeland set ?

Notice finally that Copeland scores also define a social welfare ordering on A , which is a particular member of the supporting size methods studied in Chapter 7 .

Exercizes on Chapter 3

Exercise 1

Prove that Theorem 1 and 2 hold when the majority game w^* is replaced by any neutral binary comparisons, namely $w_{a,b} = w$ for some fixed simple game w .

Exercise 2 Non monotonic binary game trees

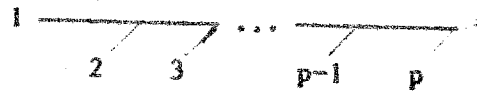
Consider the following binary game tree on $A = \{a,b,c,d\}$:



Prove that it yields a non monotonic relation agregator.

Exercise 3 Voting by sequential elimination

It is the following binary game tree, where $A = \{1, \dots, p\}$ is linearly ordered :



ties broken in favour
of the bottom branch

Thus the associated relation aggregator is defined by the algorithm

$$\beta_1 = 1, \beta_2 = R(2,1), \dots, \beta_k = R(k, \beta_{k-1}), \dots, \beta_p = \rho_\sigma(R)$$

where $R(x,y)$ is defined as in the proof of Theorem 1.

- a) Prove that it is a monotonic relation aggregator.
- b) Prove that when σ varies, $\rho_\sigma(R)$ describes the whole top cycle $tc(R)$.

Hint for b) : $tc(R)$ is a R -cycle and $a P b$ for all a in $tc(R)$ and b outside it : see Chapter 4.4.

Exercise 4

In the sequential elimination algorithm above as well as in the provisional winners algorithm, the outcome ranked first is at an obvious disadvantage : it wins only if it is a Condorcet winner (strictly preferred to any other outcome) in which case it would be elected for any other ordering of A .

Prove more : it is always an advantage in both aggregator to be ranked later :

Let $\sigma = \{1, 2, \dots, p\}$ and σ' equals to σ , except that k and $k+1$ have been permuted. Then we have

$$\rho_\sigma^*(R) = k \Rightarrow \rho_{\sigma'}^*(R) = k$$

where ρ_σ^* is the aggregator attached to either algorithm.

Chapter 4 : CHOICE FUNCTIONS AND BINARY RELATIONS.

This chapter is a technical appendix to Chapters 3,4. We explore the (one person) problem of converting a complete binary relation (presumably expressing society's binary preferences) into a choice function (prescribing society's decision) through a relation aggregator. Conversely, starting with an arbitrary choice function, we give axioms ensuring that it derives from an acyclic relation through the argmax operator (Section 1) or from its base relation through a satisfactory relation aggregator (Section 2).

1 . Acyclically rationalizable choice functions.

We recall first some definitions about complete relations R on a set A (for all a,b , $a R b$ and/or $b R a$). We denote by P the asymmetric component of R ($a P b$ iff $\text{No } b R a$) and by I its symmetric component ($a I b$ iff $a R b$ and $b R a$). We say that R is acyclic if P has no cycles such as $a_1 P a_2 P a_3 \dots P a_k P a_1$. Next R is quasitransitive if P is transitive ($a P b$ and $b P c$ imply $a P c$). If, in addition, relation I is transitive, then R itself is transitive and we call it a preordering. Finally if R is transitive and asymmetric ($R = P$) we call it an ordering.

Representing an ordering is easy : if A is finite, its elements are just linearly ordered as in $A = \{a,b,c,d,e,f\}$. When R is a preordering relation I is an equivalence relation, and P induces an ordering of equivalence classes : whence the conventional representation :

				g
		c		f
A =	a	b	d	e

where a column is an equivalence class and relation P goes left to right.

Suppose near R is quasitransitive. Then its asymmetric component can be written as the intersection of at most $\frac{|A|}{2}$ linear orderings R_1, \dots, R_K :

$$a P b \text{ iff } a R_k b \quad \text{all } k=1, \dots, K$$

Thus $\underset{B}{\operatorname{argmax}} R$ is just the Pareto set over B for the K -criteria R_1, \dots, R_K . Equivalently R can be pictured as a Hasse diagram :



$$R = R_1 \cap R_2 \quad \text{where} \quad \begin{aligned} R_1 &= a b c d e \\ R_2 &= a d e b c \end{aligned}$$



$$R = R_1 \cap R_2 \cap R_3 \quad \text{where} \quad \begin{aligned} R_1 &= a e b f d c \\ R_2 &= e f a d c b \\ R_3 &= f a c e b d \end{aligned}$$

A downward line from x to y means $x P y$. If no line connects x and y then $x I y$.

A good reference on these results is Roberts, F.J. Measurement Theory, 1979, Addison Wesley.

We turn now to choice functions.

Given the finite set A of outcomes, a choice function S on A is a mapping from $2^A \setminus \emptyset$ into itself such that :

for all $B \subset A$ $S(B) \subset B$

Definition 1

A choice function S is acyclically (resp. quasitransitively, resp. transitively, resp. linearly) rationalizable if there exists an acyclic relation R on A (resp. a quasitransitive relation, resp. a preordering, resp. an ordering) such that

$$S(B) = \underset{B}{\operatorname{argmax}} R = \{a \in B / a R b, \text{ all } b \in B\} \quad \text{all } B \subset A$$

As we noticed earlier (Chap. 3), if S is rationalizable at all, relation R must be the base relation of S namely :

$$a R b \quad \text{iff} \quad a \in S(ab)$$

Theorem 1

Given a choice function S the three following statements are equivalent :

- i) S is acyclically rationalizable.
- ii) for all a, B with $a \in B$: $\{a \in S(B)\} \Leftrightarrow \{a \in S(ab) \text{ all } b \in B\}$
- iii) S satisfies Chernoff's condition and the reinforcement property :

Chernoff :

$$B \subset B' \Rightarrow S(B') \cap B \subset S(B) \quad \text{all } B, B'$$

reinforcement :

$$S(B) \cap S(B') \subset S(B \cup B') \quad \text{all } B, B'$$

Chernoff's condition can be given a number of equivalent formulations (see Exercise 1) some of them akin to the reinforcement property, e.g. :

$$S(B \cup B') \subset S(B) \cup S(B') \quad \text{all } B, B'$$

Notice also that Chernoff's condition implies $S^2 = S$.

Theorem 2

Given a choice function S the four following statements are equivalent :

- i) S is transitively rationalizable
- ii) S satisfies Chernoff's condition and the strong reinforcement property :

strong reinforcement :

$$S(B) \cap S(B') \neq \emptyset \Rightarrow S(B) \cup S(B') \subset S(B \cup B') \quad \text{all } B, B'$$

- iii) S satisfies Nash's independence of irrelevant alternatives :

Nash's IIA :

$$\{B \subset B' \text{ and } S(B') \cap B \neq \emptyset\} \Rightarrow \{S(B) = S(B') \cap B\} \quad \text{all } B, B'$$

- iv) S satisfies the weak axiom of revealed preferences :

WARP

$$\{a \in B \setminus S(B), b \in S(B)\} \Rightarrow \text{No}\{a \in S(B'), b \in B'\} \quad \text{all } a, b, B, B'$$

Corollary

The choice function S is linearly rationalizable if and only if it is binary decisive ($S(ab)$ is a singleton, all a, b) and satisfies Chernoff's condition.

Theorem 3

Given a choice function S the three following statements are equivalent :

- i) S is quasi-transitively rationalizable
- ii) S is acyclically rationalizable and satisfies the condition (\star) condition (\star) :

$$\{B \subset B' \text{ and } S(B') \subset B\} \Rightarrow \{S(B) \subset S(B')\}$$

- iii) S satisfies Plott's Path Independence and the reinforcement property :

Plott's P.I.:

$$S(B \cup B') = S(S(B) \cup S(B')) \quad \text{all } B, B'$$

One common features in the above three results is that rationalizability is equivalent to a combination of two properties :

- i) one "expansion" property of the form : if we know the choices over small sets B, B' , we know something about the choice over the bigger set $B \cup B'$. See the reinforcement or strong reinforcement property.

- ii) One "contraction" property : if we know the choice over a big subset B , we know something about the choice over subsets of B . See Chernoff's condition or Plott's Path Independence.

2 . Relation agregators

Definition 2

Given A , a relation agregator is a mapping ρ associating to any complete relation R on A a choice function $\rho(R) = S$.

Three basic examples of relation agregators are the topcycle tc (Ch. 3.1) the uncovered set un and the Copeland set CO (Ch. 3.3).
We set :

$tc(R)(B) = tc(R/B)$ where R/B denotes the restriction of R to B .

(and a similar definition holds for un and CO).

So that these agregators satisfy the following version of Arrow's IIA axiom :

AIIA : $\{R \text{ and } R' \text{ coincide on } B\} \Rightarrow \{\rho(R)(B) = \rho(R')(B)\}$
all R, R', B .

They also satisfy neutrality and consistency :

Neutrality :

if σ is a permutation of A then :

$$\rho(R^\sigma)(\sigma(B)) = \sigma[\rho(R)(B)]$$

where we set $a R^\sigma b$ if $\sigma^{-1}(a) R \sigma^{-1}(b)$

Consistency :

$\{a P b \text{ for all } a \in B, \text{ and all } b \in B'\} \Rightarrow \{\rho(R)(B \cup B') = \rho(B)\}$
all R, B, B' .

Lemma 1

If a relation agregator ρ satisfies the consistency property, then

$$\rho(R) \subset tc(R) \quad \text{all } R$$

To get characterization results for our three agregators tc , un and CO , we restrict the range of R to tournaments namely asymmetric and complete relations. Then we speak of tournament agregators.

Theorem 4 Moulin [1983]

Whenever R is a tournament, the choice function $un(R)$ (resp. $tc(R)$) satisfies the reinforcement property (resp. strong reinforcement).

Conversely, any tournament agregator ρ satisfying AIIA, neutrality and reinforcement (resp. strong reinforcement) must contain the agregator un (resp. the agregator tc) :

$$un(R)(B) \subset \rho(R)(B) \quad \text{all } R, B$$

Since the inclusion $CO \subset un \subset tc$ always hold (see Ch. 3) we view the Copeland agregator as more satisfactory from a normative standpoint. Its axiomatic characterizations, however are harder to obtain.

Observe that the Copeland tournament agregator amounts to select from B the subset of outcomes b that beat the greatest number of opponents in B . Then it is easily checked that CO satisfies the two following properties :

Strict monotonicity :

for all R, R', B, a and B , if the only change from R to R' is that for some a, b , $b R a$ while $a R' b$ and if $a \in \rho(R)(B)$ then $\{a\} = \rho(R')(B)$.

Independence of preferences on Indifferent subsets :

for all R, R' , and B if R and R' differ only on $B \times B$ and $\rho(R)(B) = \rho(R')(B) = B$, then $\rho(R)$ and $\rho(R')$ coincide

Theorem 5 Rubinstein [1980], Henriot [1983].

The Copeland agregator CO is the unique tournament agregator satisfying AIIA, neutrality, strict monotonicity and Independence of preferences on Indifferent subsets.

The above result can also be adapted to characterize the social welfare ordering induced by the Copeland scores (where to each outcome is attached the number of outcomes that it beata).

Exercizes on Chapter 4Exercize 1 Equivalent formulations of Chernoff's condition.

Prove that Chernoff's condition is equivalent to any of the following properties :

- | | | | |
|---|---|---|-------------|
| a) | $S(B \cup B') \subset S(B) \cup B'$ | } | all B, B' |
| b) | $S(B \cup B') \subset S(B) \cup S(B')$ | | |
| c) | $S(B \cup B') \subset S(S(B) \cup B')$ | | |
| d) | $S(B \cup B') \subset S(S(B) \cup S(B'))$ | | |
| e) to h) same as above, stated for all pairs B, B' of | | | |
| <u>disjoint</u> subsets of A . | | | |

Exercize 2 One more characterization of acyclic rationalizability.

Prove that S is acyclically rationalizable iff :

$$S(B) \cap S(B') = S(B \cup B') \cap B \cap B' \quad \text{all } B, B'$$

Exercize 3 One more characterization of transitive rationalizability.

Prove that S is transitively rationalizable iff it satisfies Chernoff's condition and the following Sen's condition :

Sen : $B \subset B'$ and $S(B') \cap S(B) \neq \emptyset \Rightarrow S(B) \subset S(B')$ all B, B'

Exercize 4 Plott's Path Independence condition.

1) Prove that Plott's P.I. condition can be rewritten equivalently as a) or b) :

- | | | |
|----|---|--------------------------|
| a) | $S(B \cup B') = S(S(B) \cup B')$ | all B, B' |
| b) | $S(B_1 \cup \dots \cup B_K) = S(S(B_1) \cup \dots \cup S(B_K))$ | all K, B_1, \dots, B_K |

2) Prove that Plott's I.I. implies Chernoff's condition but the converse does not hold (give a counterexample with $|A| = 3$).

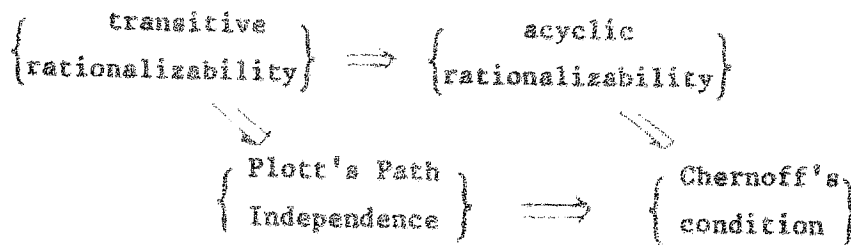
3) Prove that Plott's P.I. is equivalent to Chernoff's condition and Aizerman's condition :

Aizerman's :

$$B \subset B' \setminus S(B') \Rightarrow S(B' \setminus B) = S(B') \quad , \quad \text{all } B, B' .$$

Exercise 5

Give an example (with $|A| = 3$) showing that the converse of the implications below do not hold :



Show that no implication hold between Plott's P.I. and acyclic rationalizability.

Exercise 6 Weak rationalizability

Say that a choice function S is weakly rationalizable if it contains an acyclically rationalizable choice function.

1) Prove that S is weakly rationalizable if it contains a linearly rationalizable s.c.c..

2) Prove that S is weakly rationalizable iff it satisfies Deb's condition :

Deb : for all B there exists $a \in S(B)$: $\{B' \subset B \text{ and } a \in B'\} \Rightarrow \{a \in S(B')\}$, all B' .

Exercise 7 More on the top cycle agregator.

1) For any complete relation R , the top cycle $tc(R)(A)$ is the largest undominated R -cycle and the smallest head of A (we say that $B \subset A$ is dominated if for some $a \in A$, we have $a P b$, all $b \in B$). We say that B is a head of A if $b P a$ for all $b \in B$ and $a \in A \setminus B$.

2) For any complete relation R the choice function $S = tc(R)$ satisfies Bordes' condition :

Bordes : $\{B \subset B' \text{ and } S(B') \cap B \neq \emptyset\} \Rightarrow \{S(B) \subset S(B')\}$, all B, B' .

3) Conversely any tournament agregator satisfying AIIA, neutrality, consistency and Bordes' condition must contain the top cycle agregator.

Exercise 8 Uncovered set, Maxtransitive set and Copeland set.

We fix a tournament R and denote $U = un(R)(A)$, $C = CO(R)(A)$.

1) Show that U contains a iff for all $b \neq a$, $a R b$ and/or there exists c such that $a R c R b$.

Show that U contains a iff there exists an undominated subset B such that $a = \underset{B}{\operatorname{argmax}} R$.

2) Say that a subset B is transitive if the restriction of R to B is transitive. Denote by T the maxtransitive set of R namely the set of outcomes which are on top of some undominated transitive subset of A :

$$a \in T \Leftrightarrow \exists B \left\{ \begin{array}{l} B \text{ transitive, } B \text{ undominated} \\ a = \underset{B}{\operatorname{argmax}} R \end{array} \right.$$

Show that the algorithm of provisional winners associated with an ordering σ of A (Ch. 3.2) always ends up in T , and that when σ varies, all elements of T are reached.

3) The uncovered set U contains both C and T . Yet C and T are not related by any systematical inclusion.

CHAPTER 5: SINGLE PEAKED PREFERENCES AND CONDORCET WINNERS

When individual preferences are restricted to vary within a strict subset of $R(A)$, the set of preorderings over A , we speak of a restricted domain. On such a domain the concept of neutrality vanishes so that even May's justification of majority voting over doubletons does not hold any more. Here we explore the much studied restriction of single-peaked preferences where A is linearly ordered as

$$A = \{0 = a_0, a_1, \dots, a_p = 1\} \quad (\text{discrete model})$$

or alternatively A is the real interval $[0,1]$ (continuous model).

We say that a preference v on A is single peaked if there is an outcome p , called the peak of v , such that v is strictly increasing before p and strictly decreasing after it:

$$\begin{aligned} a < b \leq p &\Rightarrow u(a) < u(b) && \text{all } a, b \\ p \leq a < b &\Rightarrow u(a) > u(b) \end{aligned}$$

Note that indifferences across the peak are allowed. We denote by $SP(A)$ the set of single peaked preferences.

1. Ordinary Majority Relation

Given a profile u for society N , we denote by $M(u)$ the majority relation $aM(u)b$ iff $|N(u,a,b)| \geq |N(u,b,a)|$ where $N(u,a,b) = \{i \in N \mid u_i(a) > u_i(b)\}$ and by $\tilde{M}(u)$ its asymmetric component: $a\tilde{M}(u)b$ if $|N(u,a,b)| > |N(u,b,a)|$.

We call outcome a Condorcet winner at profile u on A if it is a maximal element of $M(a)$. Let $CW(u)$ denote the set of Condorcet winners at u .

Lemma 1.

Let $u \in SP(A)^N$ be a single-peaked profile. Then $CW(u)$ is nonempty. It contains outcome a iff at least half of the individual peaks are before a and at least half are after it.

$a \in CW(u) \Leftrightarrow |\{i \in N/p_i \leq a\}| \geq \frac{n}{2}$ and $|\{i \in N/a \leq p_i\}| \geq \frac{n}{2}$

(where n is the cardinality of N).

Moreover,

- if n is odd, $CW(u)$ is a singleton, namely the median peak (with rank $\frac{n+1}{2}$).
- if n is even, $CW(u)$ is the interval from the peak with rank $\frac{n}{2}$ to peak with rank $\frac{n}{2} + 1$.

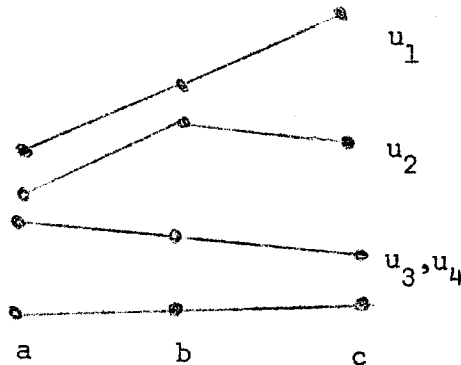
Lemma 2.

The strict majority relation $\tilde{M}(u)$ is transitive.

For n odd the majority relation $M(u)$ is single peaked, and its peak is the Condorcet winner.

For n even, relation $M(u)$ is not necessarily transitive.

Thus a sharp contract: if n is odd, the majority relation is a beautiful social welfare ordering satisfying Arrow's IIA, anonymity, monotony and efficiency (henceforth it is strategy proof: see Chapter 6). For n even, it is not even transitive (notice that \tilde{M} is not complete) hence no Arrowian aggregator:



An example of nontransitive $M(u)$: a against b and a against c yield a tie.

Yet b defeats c three to one.

2. Generalized Majority Relations and Condorcet Winners

A simple trick to overcome the difficulty raised when n is even is to add to society $\{1, \dots, n\}$ a number, with opposite parity, of phantom voters namely fixed single peaked preferences. For instance if $n = 4$, $M(u_1, u_2, u_3, u_4, v^*)$ (where v^* is fixed and single peaked) yields a social welfare ordering sharing all the properties listed above. In fact we could add any number of phantom voters without jeopardizing any of them except for efficiency:

Definition 1:

A generalized majority relation is a relation M_V taking the form:

$M_V(u) = M(u_1, \dots, u_n; v_1^*, \dots, v_{n-1}^*)$ all $u_1, \dots, u_n \in SP(A)$ where $v_1^*, \dots, v_{n-1}^* \in SP(A)$ are fixed single peaked preferences, called the phantom voters. The generalized Condorcet winner CW_q associated with M_V is given by

$$CW_q(u) = CW(p_1, \dots, p_n; q_1^*, \dots, q_{n-1}^*) \text{ where } q_i^* \text{ is the peak of } v_i^* .$$

It is easy to check that any generalized Condorcet winner defines a social welfare ordering ($u \rightarrow M_V(u)$) aggregating single peaked profiles into a single peaked collective preference and satisfying moreover, AIIA, anonymity, monotony and efficiency. The associated generalized Condorcet winner defines a fairly simple voting rule: each agent casts his peak, the $(n-1)$ fixed ballots q_1^*, \dots, q_{n-1}^* are thrown into the urn and finally the overall median is elected. We give some examples:

Positional Dictatorships

Take $q_1^* = \dots = q_{n-1}^* = 0$; the associated preference v_0 is called the leftist preference ($a > b$ iff $v_0(a) > v_0(b)$). Then $CW_q(u)$ is just the minimum of p_1, \dots, p_n hence the maximally left-biased Pareto optimal outcome (observe that at profile u_1, \dots, u_n the Pareto optimal outcomes cover the interval $[\min_{i=1, \dots, n} p_i, \sup_{i=1, \dots, n} p_i]$).

Take $q_1^* = \dots = q_{k-1}^* = 0$, $q_k^* = \dots = q_n^* = 1$ (thus $(k-1)$ phantom voters are leftist and $n-k$ are rightist)., then $CW_k(u)$ is k -positional dictator namely the k -th ranked peak with respect to the fixed ordering of A . For instance if n is even, we obtain the leftist-Condorcet winner for $k = \frac{n}{2} + 1$ and the rightist Condorcet winner for $k = \frac{n}{2}$.

Uniform Condorcet Winner

Take F to be a strictly increasing function from A into itself, to be interpreted as the inverse of the cumulative distribution of a nonatomic probability distribution on A (assume $F(0) = 0$, $F(1) = 1$) in the continuous model. In the discrete model F is associated with a probability P weighing all outcomes. Then set for a society with size n :

$$q_k^n = F\left(\frac{k}{n}\right) \quad 1 \leq k \leq n-1$$

Here the fixed ballots coincide with the n -quantiles of P . For n large, we can think of $(q_k^n)_{k=1, n-1}$ as uniformly spread over A according to P .

An important feature of generalized Condorcet winners CW_q is that their parameters q_1, \dots, q_{n-1} are determined by the outcomes chosen (over A) at those simple profiles where each agent is either rightist ($p_i = 1$) or leftist ($p_i = 0$). Indeed, if r agents are rightist and $(n-r)$ are leftist, $1 \leq r \leq n-1$, the collective peak $CW_q(p)$ is just q_r . Thus if the social ruler is convinced by the arguments developed in Section 3 below, that collective choice should be made by some generalized Condorcet winner mechanism, but he cannot make up his mind about parameters q_1, \dots, q_{n-1} , he must only find out the answer to $(n-1)$ simple questions, namely: which outcome is best if society splits into two homogenous antagonistic coalitions, with r rightist against $(n-r)$ leftist?

Now suppose he gets the following abrupt answer: if r rightist oppose $(n-r)$ leftist, comply with the rightist whenever $r \geq k$, otherwise comply with the leftist. This amounts to use k as the quota necessary and sufficient to enforce

any rightist move (while $(n-k+1)$ is the quota for leftist moves). In this case the k -positional dictatorship is the necessary voting rule. Thus positional dictatorships essentially enforce a quota-majority principle that is but a variant of Condorcet's majority principle.

Suppose, on the contrary, that the compromise that he views as equitable for an antagonistic society (r rightist and $(n-r)$ leftist) is outcome $\frac{n-r}{n}$ (continuous model): in other words he feels that the linear nature of the outcome space leaves room for flexible decisions where a marginal switch from rightist to leftist moves the elected outcome just a bit to the left. In that case we have a uniform Condorcet winner (here associated with the uniform distribution on $[0,1]$). See Exercises 1, 2 for more intuition on these voting rules.

3. Characterization Results

We prove now that the family of generalized Condorcet winner S.C.C. are uniquely characterized as transitively rationalizable S.C.C. satisfying the AIIA axiom. This justifies the majority relation itself in the single peaked context.

Definition 2

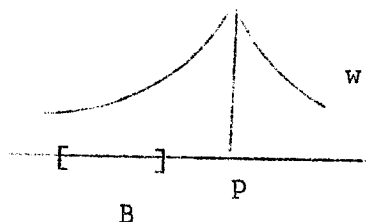
A single peaked social choice function is a mapping S , associating to any single peaked profile $u \in SP(A)^n$ and any subinterval B of A a unique outcome $S(u,B) \in B$.

For a single peaked profile u , any generalized majority relation $M_V(u)$ is single peaked hence it yields a decisive choice function over any subinterval B of A (not over any subset of A).

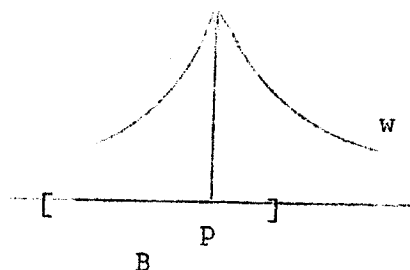
The point is the following fact:

if $w \in SP(A)$ is single peaked with its peak at p , and B is a subinterval of A then w reaches uniquely its maximum over B at the projection of p on B :

$$\operatorname{argmax}_B w = \operatorname{proj}_B^*(p)$$



the right end of B
maximizes w over B



the peak maximizes w
over B

Thus the single peaked S.C.F. induced by the generalized majority relation M_v rely only upon its generalized Condorcet winner C_q :

$$S(u, B) = \operatorname{argmax}_B M_v(u) = \operatorname{proj}_B \{C_q(u)\} \quad \begin{array}{l} \text{all } u \in SP^n \\ \text{all subinterval } B \end{array}$$

By restricting feasible subsets of outcomes to subintervals, we reduce drastically the relevant dimension of the family of generalized majority relations: indeed the $(n-1)$ peaks q_1, \dots, q_{n-1} as well as the n agents peaks are enough to determine the social choice function. Moreover all that matters is the mapping $(p_1, \dots, p_n) \rightarrow CW_q(p_1, \dots, p_n)$ associating to any n -uple of individual peaks the collective peak from which social preferences go downward right and left. This informational feature is formally equivalent to the Nash's IIA axiom (Chapter 4.1) and a mild continuity axiom.

Lemma 3.

The single peaked s.c.f. S satisfies the Nash's IIA and continuity axioms if and only if it can be expressed as:

$$S(u, B) = \operatorname{proj}_B \{\sigma(u)\} \quad \text{all } u \in SP^n, \text{ all subintervals } B. \quad (1)$$

for some mapping σ associating to each single peaked profile u a collective peak $\sigma(u)$.

Nash's IIA: $\{B \subset B' \text{ and } S(u, B') \in B\} \Rightarrow \{S(u, B) = S(u, B')\}$

all profile u and subintervals B, B' .

Continuity Discrete Model: $\{\text{if } B \text{ and } B' \text{ differ by one outcome, } S(u, B) \text{ and } S(u, B') \text{ are equal or adjacent}\}$ $\forall u$ and subintervals B, B' .

Continuity Continuous Model: $\{S(u, B) \text{ is continuous w.r.t. both ends of } B\}$
all u and subinterval B .

Recall that in the unrestricted domain case the Nash's IIA axiom characterizes transitive rationalizability (Chapter 4.1). Here we must add the continuity axiom as a "price" for the restriction to subintervals.

When the Arrow's IIA axiom is added to the single peaked rationalizability expressed by equation (1) we get a full characterization of generalized Condorcet winners.

Theorem 1: (Moulin [1983] working paper) Continuous model. Given a single peaked s.c.f. S , the following two statements are equivalent

i) S is derived from some generalized Condorcet winner, CW_q with fixed ballots q_1, \dots, q_{n-1} :

$$S(u, B) = \text{proj}_B CW(p_1, \dots, p_n; q_1, \dots, q_{n-1}) \quad \text{all } u, B$$

ii) S satisfies: anonymity, efficiency, the Nash IIA and the continuity axioms, and finally the Arrow's IIA axiom over subintervals:

$$\{u \text{ and } u' \text{ coincide on } B\} \Rightarrow \{S(u, B) = S(u', B)\} \quad \text{all } u, u', B.$$

In the discrete model the same statement holds if in addition we add a minimal monotonicity axiom namely:

if u^r is a profile made up of r rightist and $(n-r)$ leftists, then the mapping $r \rightarrow S(u^r, A)$ is nondecreasing.

The above two theorems can be given a more compact formulation in the simpler context where the single peaked decision problem is the search of a mapping $(p_1, \dots, p_n) \rightarrow s(p_1, \dots, p_n)$ from n-uples of individual peaks into a collective peak. This formalism internalizes the Nash's IIA as well as the continuity axiom.

Theorem 2. (Moulin [1980], Border and Jordan(1983), Review of Economic Studies 50, 1, 153-170). Let s be a mapping from A^n into A and consider the following list of properties.

1) anonymity: $s(p_1, \dots, p_n)$ is symmetrical in p_1, \dots, p_n

2) efficiency:

$$\min_{i=1, \dots, n} p_i \leq s(p_1, \dots, p_n) \leq \max_{i=1, \dots, n} p_i, \text{ all } p$$

3) Arrow's IIA:

$$\text{proj}_B s(p_1, \dots, p_n) = s(p_1^B, \dots, p_n^B) \quad \text{all } p, \text{ all subintervals } B.$$

where a^B denotes $\text{proj}_B a$.

4) Strategyproofness:

$$s(p) \leq p_i \Rightarrow s(p_i', p_{-i}) \leq s(p) \quad \text{all } p$$

$$s(p) \geq p_i \Rightarrow s(p_i', p_{-i}) \geq s(p) \quad \text{all } i, \text{ all } p_i'$$

5) Uncompromisingness

$$s(p) \leq p_i' < p_i \Rightarrow s(p_i', p_{-i}) = s(p) \quad \text{all } p,$$

$$s(p) \geq p_i' > p_i \Rightarrow s(p_i', p_{-i}) = s(p) \quad \text{all } i, \text{ all } p_i'.$$

Then each combination 123, 124 and 125 characterizes the family of generalized Condorcet winners.

Concluding Remark

The domain $SP(A)$ of single peaked preferences, can be extended slightly (namely to the preferences with a single plateau, and strictly increasing--decreasing--before--after--the plateau). But not much if we want to preserve

the compatibility of Nash's and Arrow's IIA axioms. Indeed consider $QC(A)$ to be the domain of quasi-concave preferences on A namely $v \in QC(A)$ iff there exists a peak $p \in A$ (not necessarily unique) such that v is nondecreasing before p and nonincreasing after it. Then one proves (see Moulin [1983]) that NIIA and AIIA are not compatible on $QC(A)^n$.

Exercises on Chapter 5

Exercise 1. Uniform Condorcet Winners with Many Voters (Continuous Model).

Let P be a non atomic distribution on $[0,1]$ and assume that the population of n agents with peaks p_1, \dots, p_n grows in such a way that, in the limit, the distribution of agents is given by a probability distribution N . Prove that the limit of the associated uniform Condorcet winners is the unique solution a of the following system:

$$P([0,a]) + N([0,a]) \geq 1$$

$$P([a,1]) + N([a,1]) \geq 1$$

Exercise 2. Young's Reinforcement and Uniform Condorcet Winners.

Any nondecreasing function F from $[0,1]$ into itself is the inverse of the cumulative distribution of some (possibly atomic) probability distribution p . Observe that the positional dictatorships can be viewed as uniform Condorcet winners in this generalized sense, and that the uniqueness of the limit in Exercise 1 does not generalize.

- (1) Fix F as indicated and for any integer n denote by S^n the associated single peaked s.c.f.:

$$S^n(u,B) = \text{proj}_B \text{CW}(p_1, \dots, p_n, q_1^n, \dots, q_{n-1}^n) \quad \text{where } q_k^n = F\left(\frac{k}{n}\right).$$

Prove that $\{S^n\}_{n=1,2,\dots}$ satisfies Young's reinforcement axiom, namely:

$$S^n(u,B) = S^m(v,B) \Rightarrow S^{n+m}(u,v;B) = S^n(u,B) \quad \text{all integers } n, m, \text{ profiles } u, v \text{ and subintervals } B.$$

Interpretation of the axiom is provided in Chapter 7.2.

(2) Conversely, let $\{S^n\}_{n=1,2,\dots}$ be a family of generalized Condorcet winners s.c.f. (say that S^n has the fixed ballots $q_1^n \leq \dots \leq q_{n-1}^n$) satisfying Young's axiom. Prove the existence of F , nondecreasing from $[0,1]$ into $[0,1]$ such that $\{S^n\}_{n=1,2,\dots}$ are the associated uniform Condorcet winners.

CHAPTER 6. DIRECT REVELATION OF PREFERENCES

1. Game Forms and Strategy Proofness

A game form (in short g.f.) describes any voting rule for a fixed society and set of outcomes.

A game form distributes exhaustively the decision power among individuals by endowing each agent with a fixed message space and converting any bundle of agent's messages into a single outcome.

Definition 1. Given A , the set of outcomes, and N , the set of agents, a game form g is an $(N+1)$ -tuple $g = (X_i, i \in N; \pi)$, where:

(a) X_i is the strategy set (or message space) of agent i , and

(b) π is a (single valued) mapping from $X_N = \prod_{i \in N} X_i$ into A .

The mapping π describes the decision rule: if for all i agent i chooses strategy x_i , the overall strategy N -tuple is denoted $x = (x_i)_{i \in N}$ and the decision rule forces the outcome $\pi(x) \in A$.

Thus a game form is a more general object than a social choice function (decisive s.c. correspondence) inasmuch as the message space does not necessarily coincide with the set of individual preferences, and a less general object since the set A of feasible outcomes is no longer allowed to vary.

Any social choice function induces a game form with message space $L(A)$ (linear orderings of A) and decision rule $S(\cdot, A)$ (the mapping $u \rightarrow S(u, A)$). A game form, with message space $L(A)$ for every agent is called a direct game form.

Lemma 1 below reduces the search of strategy proof game forms to that of strategy proof voting rules. The full generality of the game form concept, which allows in particular the message space to be bigger than $L(A)$, is needed when we explore a more complex strategic behaviour than the dominating strategy equilibrium. This is indeed the case in Chapters 8 and 9.

Given a game form $g = (X_i, i \in N, \pi)$ we associate to every preference profile $u = (u_i)_{i \in N}$ the normal form game $g(u) = (X_i, u_i \circ \pi, i \in N)$, where agent i 's strategy is x_i and his or her utility level is $u_i(\pi(x))$. This game reflects the interdependence of the individual agents' opinions (utility) and their strategic abilities (agent i is free to send any message within X_i).

In this chapter we focus on those game forms such that for all profiles u every agent has a straightforward non-cooperative strategy whether or not he or she knows of the other agents' preference orderings. This is captured by the notion of dominating strategy.

Definition 2. Given A and N and a game form g we say that g is strategy-proof if for every agent i there exists a mapping from $L(A)$ into X_i denoted $u_i \rightarrow x_i(u_i)$ such that the following hold true:

$$\forall u_i \in L(A), \forall \hat{x}_i \in X_i, \forall y_i \in X_i: u_i(\pi(y_i, \hat{x}_i)) \leq u_i(\pi(x_i(u_i), \hat{x}_i)). \quad (1)$$

Given a direct game form S , we say that S is strategy-proof if for every agent i we have:

$$\forall u_i \in L(A), \forall \hat{u}_i \in L(A)^{N \setminus \{i\}}, \forall v_i \in L(A): u_i(S(v_i, \hat{u}_i)) \leq u_i(S(u_i, \hat{u}_i)). \quad (2)$$

Property (1) says that if agent i 's utility is u_i then strategy $x_i(u_i)$ is a best response to every possible strategic behaviour x_i of the other agents, in short x_i is a dominating strategy. Notice that $x_i(u_i)$ is a decentralized behaviour by agent i , who can simply ignore the utility of the other agents. By (2)

this information is worthless as long as he cannot communicate with his fellow agents (this is the basic informational assumption of the so-called non-cooperative context). Notice that due to Prisoner's Dilemma effect, strategyproofness of a game form does not imply efficiency of the corresponding decisions.

In the context of direct game forms, strategyproofness means that telling the truth is a dominating strategy at every profile and for all agents: hence we expect direct revelation of their preferences by the non-cooperative agents involved in a strategyproof direct game form.

The following result, a particular case of the "revelation principle," states that as far as strategyproofness is concerned, it is enough to look at direct game forms.

Lemma 1. Let g be a strategy-proof g.f. For all $i \in N$ and all $u_i \in L(A)$, let us denote by $D_i(u_i) \subset X_i$ the set of agent i 's dominating strategies:

$$\{x_i \in D_i(u_i)\} \Leftrightarrow \{x_i \in X_i \wedge \forall y_i \in X_i, u_i(\pi(y_i, x_i)) \leq u_i(\pi(x_i, x_i))\}.$$

Then for all profiles $u \in L(A)^N$ the set $\prod(D_i(u_i), i \in N)$ is a singleton and defines a strategy proof direct game form.

Let S be a direct game form, namely a single-valued mapping from $L(A)^N$ into A .

We say that S satisfies citizen sovereignty if $S(L(A)^N) = A$, i.e. if no outcome is a priori excluded by S . It is a very mild property, even weaker than the unanimity condition namely if a is on top of u_i for all $i \in N$, then $S(u) = a$.

We shall call voting rule a direct game form satisfying citizen sovereignty.

Theorem 1: Gibbard [1973] Satterthwaite [1973].

Let society N be finite and A (not necessarily finite) contains at least three distinct outcomes. Then a voting rule S is strategyproof if and only if it is dictatorial: there is an agent i (the dictator) whose top outcome is always elected: for all $u \in L(A)^N$ $S(u) = \text{top}(u_i)$.

The proof of Theorem 1 is technically equivalent to that of Arrow's theorem (see Exercise 1).

For binary choices (A is a doubleton) strategyproofness is equivalent to monotonicity (Chapter 1.2) and is therefore satisfied by many nondictatorial voting rules. As soon as three outcomes are on stage we cannot find a reasonable (e.g., anonymous) voting rule. This suggests two lines of investigation.

In the first one we insist on the strategy-proofness requirement, that is we want our mechanisms to allow "pure" decentralization of the decision process, therefore requiring that an agent's optimal strategy is unambiguous even if he or she ignores the other preferences, and is still unaffected if this agent happens

to know the preferences of some among his or her fellow agents. Then, by the Gibbard-Satterthwaite theorem we must restrict the domain of feasible profiles, just as standard assumptions of microeconomics severely restrict the possible configuration of the utility profile. This line will be explored in the next section.

Another way of escaping the Gibbard-Satterthwaite result is to weaken the equilibrium concept: not demanding that a dominating strategy equilibrium exists for all profiles still leaves room for patterns of behaviour that are, to a large extent, non-cooperatively decentralized (see Chapter 8). Or we can take a cooperative view of the decision-making mechanism so that specific equilibrium concepts will be in order (see Chapter 9).

2. Restricted Domains

Let $R(A)$ be the set of preferences preorderings on A (indifferences allowed). The voting rules assigning to each profile in $R(A)^N$ an outcome in A can be in particular restricted to $L(A)^N$. Thus Gibbard-Satterthwaite's result holds true as well if we allow individual preferences to vary in $R(A)$ (it is in fact a weaker statement).

When speaking of restricted domains we should pick several subsets D of $R(A)$ and assume that every agent's preferences stay in D .

Some restricted domains D allow us to overcome Arrow's impossibility theorem i.e., there exists on D^N a non dictatorial social welfare ordering satisfying the AIIA axiom and monotonicity. In that case we can just the same construct strategyproof voting rule.

Lemma 2.

Let $D \subset R(A)$ be a restricted domain and let $u \rightarrow R(u)$ be a decisive social welfare ordering on D , i.e. for each $u \in D^N$, $R(u)$ is a linear ordering on A . Consider the associated social choice function S .

$$S(u, B) = \underset{B}{\operatorname{argmax}} R(u) \text{ all } u \in D^N \text{ all } B \subset A$$

The two following statements are equivalent:

- i) S is monotonic and satisfies the AIIA axiom
- ii) for all B , $S(\cdot, B)$ is a strategyproof game form on D^N .

Under the assumptions of Lemma 2 even a stronger strategical feature hold, namely the sincere message is coalitionally strategyproof: no coalition of agents come together depart from their true message and improve upon (strictly) all preference levels in the coalition. In particular the elected outcome $S(u, B)$ is weakly Pareto optimal.

A first example where the above Lemma applies is the single peaked domain.

Let A be finite and linearly ordered as $A = \{a_1, \dots, a_p\}$. W.r.t. this ordering we speak of "right" or "left" and $D = SP(A) \subset R(A)$ is defined as in Chapter 5.

For all profile u define the rightist majority relation $M^r(u)$ by:

$$a M^r(u) b \text{ iff } |N(u, a, b)| \geq \frac{n}{2}$$

or :

$$|N(u, a, b)| = \frac{n}{2} \text{ and } a \text{ is right of } b.$$

Lemma 3.

For all singlepeaked profile $u \in SP(A)^N$, the rightist majority relation $M^r(u)$ is a (single peaked) linear order. The associated social choice function is monotonic and satisfies AIIA.

Example 2. Dichotomous Preferences and Approval Voting (Brams and Fishburn, 1978)

We say that a preordering on A is dichotomous if it contains at most two indifference classes. The set $D_i(A) \subset R(A)$ of these preorderings can be identified with $2^A \setminus \emptyset$ if we read $U_i \subset A$ as: agent i 's is indifferent over U_i and $A \setminus U_i$; he strictly prefers U_i to $A \setminus U_i$ ($U_i = A$ means overall indifference). We shall say that agent i approves of the outcomes in U_i . Now approval voting is the social welfare ordering induced by the number of agent of which a particular

outcome is approved of: $ap(a,U) = |\{i \in N/a \in U_i\}|$. If we break ties by a particular linear ordering $>$ of A we get a decisive social welfare ordering:

$$a R(U)b \text{ iff } ap(a,U) > ap(b,U)$$

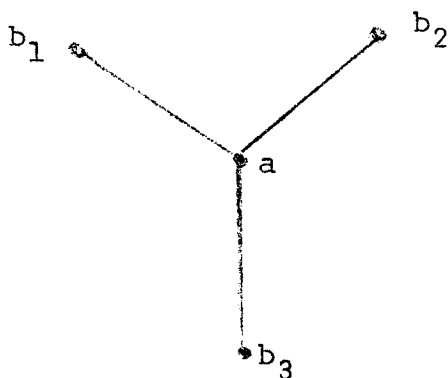
or

$$ap(a,U) = ap(b,U) \text{ and } a > b$$

This s.w.o. is monotonic and satisfies AIIA, hence it is strategyproof. One can state a converse property in the domain of dichotomous preferences: see Brams and Fishburn, 1978.

The general characterization of those domains on which the AIIA axiom is compatible with linear rationalization is quite difficult: the existing results are hard to interpret: see Kalai and Muller, JET, 16, 456-469 or the working paper by Muller and Satterthwaite [1983 Northwestern University].

On the other hand, some nondictatorial strategyproof voting rules can be obtained on domains where the Arrow's IIA axiom would not be compatible with linear rationalization: in other words Lemma 2 does not cover all interesting cases for strategyproofness. Take the following example: the set A is a 4-outcomes tree and D is the set of preferences that are single peaked w.r.t. this tree:



The set A

preference v is in D iff:

$$a = \operatorname{argmax}_A v \text{ or}$$

$$b_k = \operatorname{argmax} v \text{ and } v(a) > v(b_\ell), \\ v(b_m) \text{ where } \{k, \ell, m\} = \{1, 2, 3\}.$$

04

One checks easily that for any profile in D^N , a Condorcet winner exists (namely a , unless a majority of peaks are in one b_k) and that it is unique if $|N|$ is odd. This defines a strategyproof voting rule (for $|N|$ odd, otherwise add one phantom voter). Yet, the majority relation may be cyclic on D^N : take a profile where every agent's top is a ; this imposes no restriction on the preferences over $\{b_1, b_2, b_3\}$ hence a Condorcet paradox can arise.

Exercise 2 below investigates singlepeakedness on arbitrary trees. Our last example is one where, again, strategyproofness is possible but not Arrowian aggregation of preferences.

Example 3. House trading

Let $N = \{1, 2, \dots, n\}$ be the set of agents and $H = \{1, 2, \dots, n\}$ be a set of houses: initially agent i is endowed with house i . Any agent i can consume only one house and his preferences are described by a linear order u_i on H ; he does not care about other agents' allocation. We set $A = \Sigma(H)$ to be the set of permutations of H , describing reallocations as follows: for all $a \in A$: $a(i) = j$ means agent i receives house j . the restricted domain D of preferences over A is identified with $L(H)$ as follows:

$u_i \in L(H)$ represents the preferences over A :

$$u_i(a) \geq u_i(b) \text{ iff } u_i(a(i)) \geq u_i(b(i))$$

In this simple exchange economy the competitive equilibrium is unique and can be reached by a fairly simple algorithm, called the trading cycle algorithm (Gale):

- Let $\sigma \in H^H$ be constructed as: $\sigma(i) = \underset{H}{\operatorname{argmax}} u_i$ namely agent i 's best preferred house is $\sigma(i)$.
- Any cycle of $\sigma(i_1, \dots, i_K)$ where $\sigma(i_k) = i_{k+1}$, $K+1=1$ is a trading cycle: the corresponding exchange (which lifts the corresponding agents to their satiation level) is performed first. Then we repeat the same construction over the remaining agents and houses. Eventually we get an allocation $a^*(u)$ which is the unique competitive equilibrium as well as the core of our economy.

Lemma 4

The voting rule $u \rightarrow a^*(u)$ is strategyproof (even coalitionally strategyproof) on D^N .

Exercises on Chapter 6

Exercise 1. Proof of Gibbard-Satterthwaite's Theorem

Fix, N, A , with $|A| \geq 3$ and a strategyproof voting rule S on $L(A)^N$.

1. Prove that S satisfies the strong positive association property namely: for any two profiles u, u' , and outcome a :
 $\{S(u) = a \text{ and } (u_i(a) > u_i(b) \Rightarrow v_i(a) > v_i(b) \text{ all } i, b\} \Rightarrow S(v) = a$.

2. For all coalition T and outcomes a, b , set:

$$U(T, a, b) = \{u \in L(A)^N \text{ for } i \in T, a \text{ is on top of } u_i, b \text{ is second} \\ \text{for } i \in N \setminus T, b \text{ is on top of } u_i, a \text{ is second}\}$$

Prove that the 3 following statements are equivalent:

- i) $S(U(T, a, b)) = \{a\}$
- ii) $\exists u \in L(A)^N: N(u, a, b) = T \text{ and } S(u) = a$
- iii) $\forall u \in L(A)^N: (N(u, a, b) = T) \Rightarrow (S(u) \neq b)$

3. Define $W_{a,b}$ to be the set of coalitions T satisfying i)-iii) above. By copying the proof of Arrow's theorem (Theorem 1, Chapter 2) show that

- i) $W_{a,b}$ does not depend on a, b and
- ii) is an ultrafilter.

Conclude that S is dictatorial.

Exercise 2. Single Peaked Preferences on a Tree (Demange, Mathematical Social Sciences, 1982, 3, 4)

Given A , the finite set of outcomes, an (undirected) graph on A is a subset G of:

$$A \times A - \{(a,a), a \in A\}$$

such that $(a,b) \in G$ iff $(b,a) \in G$. A path between two distinct outcomes, a and b , is a sequence $\{a_0 = a, a_1, \dots, a_k, a_{k+1} = b\}$ such that $(a_i, a_{i+1}) \in G$ for $i = 0, 1, \dots, k$.

A graph G is a tree if there is a unique path between any two distinct outcomes. This implies that all a_i in any path (a_0, \dots, a_{k+1}) are distinct.

For a given tree G on A , and any two outcomes a or b in A , we denote by $P(a,b)$ the path between a and b . We say that outcome c belongs to $P(a,b)$ if it belongs to the range of sequence $P(a,b)$.

Now we say that a preordering $u \in R(A)$ with top outcome a is single peaked with respect to G if the following property holds:

$$\forall b, c \in A, b \neq c : [b \in P(a,c)] \Rightarrow [u(c) < u(b)] .$$

We denote by $SP_G \subset R(A)$ the single peaked preorderings w.r.t. G .

- 1) Check that single peaked preferences of Chapter 5 correspond to a "linear" tree.
- 2) For any society N with odd size, and any $u \in SP_G^N$, prove the

existence of a unique Condorcet winner $CW(u)$. Prove that it defines a strategyproof voting rule.

- 3) Assume $|N| \geq 5$ and prove that the majority relation $M(u)$ is cyclic (the strict majority relation has a 3-cycle) for some u in SP_G^N unless G is a linear tree.

Exercise 3. Manipulating the Housetrading Algorithm (Gale)

Although a coalition cannot manipulate the trading cycles algorithm by falsifying its reported preferences, it can do so by ex ante reallocations. Consider the following three agents house trading game:

Agents	Preferences	Initial endowment
1	$c > a > b$	a
2	$a > b > c$	b
3	$a > b > c$	c

Compute the competitive allocation. Next suppose that agents 1,2 exchange ex ante their endowments. Compute the new competitive allocation and observe that agent 1 gets the same house while agent 2 gets a better house.

Is it possible to design a configuration (with 3 agents or more) where a manipulation of this sort could be strictly profitable to all agents in the coalition?

CHAPTER 7. SCORING METHODS AND PROBABILISTIC VOTING

1. Scoring Social Welfare Orderings

The widely used scoring methods generate a large family of social welfare orderings. Throughout this chapter, we fix A with cardinality p and we represent a linear ordering $v \in L(A)$ as a fixed scale utility namely a bijection v from A into $\{0, \dots, p-1\}$. Thus $v(a) = k$ means that outcome a has rank $p-k$ in v (where ranks increase from the top outcome to the bottom outcome).

A vector of scores is an element s of \mathbb{R}^p such that:

$$s_0 \leq s_1 \leq \dots \leq s_{p-1}.$$

Given s , and a profile $u \in L(A)^N$, the score of outcome a at u is denoted $s(a;u)$ and defined by:

$$s(a,u) = \sum_{i \in N} s_{u_i(a)} \quad \text{all } a \in A, \text{ all } u \in L(A)^N$$

Equivalently we may denote by $v_k(a,u)$ the number of agents for which a is ranked $p-k$:

$$v_k(a,u) = |\{i \in N \mid u_i(a) = k\}| \quad \text{all } a \in A, \text{ all } u \in L(A)^N$$

Then we have

$$s(a,u) = \sum_{k=0}^{p-1} s_k \cdot v_k(a,u) = s \cdot v(a,u)$$

At profile u the vector of scores $s(u) = (s(a,u))$ defines a preordering of A (a is preferred to b iff its score is higher). Hence a social welfare ordering that we denote $u \rightarrow R^s(u)$. Two basic examples are the Borda s.w.o.: $s_k = k$, all k and the plurality s.w.o.: $s_0 = \dots = s_{p-2} = 0$, $s_{p-1} = 1$. We list now some facts about scoring s.w.o.s:

- They are anonymous and neutral.
- They are monotonic.
- They satisfy unanimity and strict monotonicity iff the sequence s_k is strictly increasing: in that case we say that s is a strict vector of scores.
- They do not satisfy the majority principle: elect a Condorcet winner whenever there is one.

To check the last claim we state a useful result. Say that the vector (v_0, \dots, v_{p-1}) stochastically dominates the vector (u_0, \dots, u_{p-1}) if we have

$$u_{p-1} \leq v_{p-1}$$

$$u_{p-1} + u_{p-2} \leq v_{p-1} + v_{p-2}$$

with at least one
strict inequality

$$u_{p-1} + \dots + u_0 \leq v_{p-1} + \dots + v_0$$

We denote $v > u$ the stochastic dominance relation.

Lemma 1:

For any strict vector of score and profile u , let a be a top outcome of the preordering $R^S(u)$. Then $v(a, u)$ is stochastically undominated

$$a \in \operatorname{argmax}_A R^S(u) \Rightarrow \{\text{for no } b \in A \quad v(a, u) \} < v(b, u)$$

Now one can construct a profile (for $|A| \geq 3$) where the unique Condorcet winner is stochastically dominated; the following example with 17 agents is due to Fishburn:

agents	6	3	4	4
outcomes	a	c	b	b
	b	a	a	c
	c	b	c	a

a is the unique Condorcet winner; it is stochastically dominated by b .

Although scoring methods are monotonic, some widely used combinations of them lead to paradoxical non-monotonicity properties.

The Paradox of Plurality with Run Off.

This voting method is widely used in various political issues: the first step is a plurality voting, where a candidate winning a strict majority of votes is elected. If no such outcome emerges, the two best-scoring candidates compete in a run-off, i.e. a binary majority contest.

Consider a situation where three candidates compete among voters and the profile is:

- 6 agents $u(a) > u(b) > u(c)$,
- 5 agents $u(c) > u(a) > u(b)$,
- 4 agents $u(b) > u(c) > u(a)$,
- 2 agents $u(b) > u(a) > u(c)$.

After the first round a and b (with plurality scoring 6 each above scoring 5 by c) go for a run-off where a passes b by 11 to 6.

Suppose now that the two agents with preferences $u(b) > u(a) > u(c)$ change their minds. They estimate that a is after all a better candidate than b and their new preference is therefore $u(a) > u(b) > u(c)$.

Although this change of opinion strictly improves upon the position of a, it actually prevents the election of a. That is to say, in the new profile a and c pass the first round (with plurality scoring 8 and 5, respectively) but a is knocked down by c 8 to 9 in the run-off.

Many practical multi-stage elections work by retaining the non-losing outcome from several successive scoring methods. Formally let s^1, \dots, s^K be a finite sequence of vectors of scores. Define the choice set by the following induction. $B_1 = A \setminus \operatorname{argmin} R^{s^1}$. Given B_{k-1} , restrict the vector s^k to its first $|B_{k-1}|$ components; then define $B_k = B_{k-1} \setminus \operatorname{argmin} R^{s^k}$ unless this set is empty, in which case $B_k = B_{k-1}$.

Fishburn observed that such methods will always lead to non-monotonic voting rules: his argument is explained in Exercise 3.

2. Young's Reinforcement Axiom

Think of the society N as variable and consider two disjoint populations N_1, N_2 with respective profiles u_1, u_2 over A (the outcome set is fixed). Then we have,

$$v_k(a, u_1) + v_k(a, u_2) = v_k(a, u_1 + u_2)$$

which implies $s(a, u_1) + s(a, u_2) = s(a, u_1 + u_2)$. Hence the following reinforcement property:

$$\begin{aligned} aR^S(u_1)b \text{ and } aR^S(u_2)b &\Rightarrow aR^S(u_1 + u_2)b \\ aP^S(u_1)b \text{ and } aR^S(u_2)b &\Rightarrow aP^S(u_1 + u_2)b . \end{aligned}$$

If both populations agree in comparing a pair of outcomes, the overall population has the same opinion.

Call choice correspondence a correspondence S from profiles u into A : $S(u)$ is the choice set at profile u . In particular a voting rule is just a single valued choice correspondence. The reinforcement property is stated for choice correspondences as follows:

$$S^{N_1}(u_1) \cap S^{N_2}(u_2) \neq \emptyset \Rightarrow S^{N_1 \cup N_2}(u_1+u_2) = S^{N_1}(u_1) \cap S^{N_2}(u_2)$$

(this formulation requires anonymity of S^N).

This says that an outcome which is best for society N_1 and best for society N_2 is best for society $N_1 \cup N_2$; also, no other outcome is best for $N_1 \cup N_2$.

Clearly the choice correspondence $S^N(u) = \underset{A}{\operatorname{argmax}} R^S(u)$ associated with a scoring s.w.o. satisfies the reinforcement axiom. A beautiful theorem shows that this property uniquely characterizes the scoring methods.

Theorem 1 (Young [1975])

Let, for all $n=1,2,\dots,S^n$ be an anonymous and neutral choice correspondence for societies of size n . Suppose the sequence $(S^n)_{n=1,2,\dots}$ satisfies the reinforcement property and the following continuity axiom:

continuity: $S^n(u) = \{a\} \Rightarrow$ for all $v \in L(A)^m$ there exists q such that $S^{m+qn}(v+nu) = \{a\}$

then there exists a vector of scores s such that for all n , S^n is the associated choice correspondence.

In fact, without the continuity axiom Young proves that reinforcement characterizes all choice methods derived from a finite repetition of distinct scoring methods: see Young [1975].

3. Supporting Size Methods

To any profile $u \in L(A)^N$ associate the weighted tournament

$$\tau(u) = [\tau(u,a,b)]_{a \neq b} : \tau(u,a,b) = |N(u,a,b)|$$

Fix a vector of weights $t_0 \leq t_1 \leq \dots \leq t_n$ and compute the weight of outcome a at profile u :

$$t(a,u) = \sum_{b \neq a} t_{\tau(u,a,b)}$$

This defines a social welfare ordering with the obvious properties of anonymity, neutrality and monotonicity. Whenever $t_0 < t_n$ it satisfies unanimity as well.

Typical examples are

• The Borda s.w.o. obtained for $t_{\ell} = \ell$, all $\ell = 0, \dots, n$.

• The Copeland s.w.o. obtained for:

$$\begin{cases} t_{\ell} = 0 & \text{if } \ell \leq \frac{n}{2} \\ t_{\ell} = 1 & \text{if } \ell > \frac{n}{2} \end{cases}$$

• The Kramer s.w.o. that compares the maximal size of objecting coalitions:

$$aK(u)b \text{ iff } \min_{a' \neq a} \tau(u,a,a') \geq \min_{b' \neq b} \tau(a,b,b')$$

see also its lexicographic refinement in Exercise 1.

Lemma 2.

The Borda s.w.o. is the unique social welfare ordering associated with a scoring method and with a supporting size method.

4. Probabilistic Voting

This section contrasts with the rest of Chapter 7 to 9. We enlarge the class of voting rules by allowing a random choice among the deterministic outcomes in A.

Definition 1: Given A and N, both finite, a probabilistic voting rule S is a (single-valued) mapping from $L(A)^N$ into P(A), the set of probability distribution over A. Thus S is described by p mappings S_a , $a \in A$ where $S_a(u)$ is the probability that a is elected

at profile u :

$$S_a(u) \geq 0, \quad \sum_{a \in A} S_a(u) = 1 \quad \text{for all } u \in L(A)^N.$$

Notice that individual messages are purely deterministic, even though collective decision is random.

Definition 2.

The probabilistic voting rule S is strategy-proof if for all profile u and all cardinal representation U of this profile:

$$U_i(a) > U_i(b) \iff u_i(a) > u_i(b) \quad \text{all } i, a, b$$

then the following inequality holds true

$$\sum_{a \in A} U_i(a) S_a(u) \geq \sum_{a \in A} U_i(a) S_a(u_i^!, u_i) \quad \begin{array}{l} \text{all } i \in N \\ \text{all } u_i^! \in L(A) \end{array}$$

Notice that the cardinal representation U_i of u_i cannot be part of agent i 's message: he is only allowed to reveal his (linear) ordering of the deterministic outcomes. It is only natural to enlarge the class of probabilistic voting rules so as to include any mapping S from $(R^A)^N$ into $P(A)$ and state the strategyproofness property as follows:

$$\sum_{a \in A} U_i(a) S_a(U) \geq \sum_{a \in A} U_i(a) S_a(U_i^!, U_i) \quad \text{all } i, U, U_i^!$$

Unfortunately, the set of such strategyproof cardinal probabilistic voting rules is unknown.

Back to Definition 1 we consider two subclasses of probabilistic voting rules inspired respectively by scoring and supporting size systems:

First take a vector of scores such that:

$$0 \leq s_0 \leq \dots \leq s_{p-1} \quad \text{and} \quad s_0 + \dots + s_{p-1} = \frac{1}{n}$$

and define the scoring voting rule S^s as follows:

$$S_a^s(u) = \sum_{i \in N} s_{u_i(a)} \quad \text{all } a, u$$

(remember that the range of u_i is $0, \dots, p-1$).

Next take a vector of weights t such that

$$0 \leq t_0 \leq \dots \leq t_n ; t_\ell + t_{n-\ell} = \frac{2}{p(p-1)} \quad \text{all } \ell = 0, \dots, n$$

and define the supporting size voting rule S^t as follows

$$S_a^t(u) = \sum_{b \neq a} t_{\tau(u,a,b)} \quad \text{all } a, u$$

Lemma 3.

These voting rules are all anonymous neutral and strategyproof.

Theorem 2. Gibbard [1978] Barbera [1979]

A probabilistic voting rule S which is anonymous, neutral and strategyproof is a convex combination of one scoring voting rule and one supporting size voting rule.

Exercises on Chapter 7

Exercise 1. Lexicographic Scoring and Supporting Size Systems

1. Consider first the maximin scoring s.w.o. R_0

$$a R_0(u) b \Leftrightarrow \min_{i \in N} u_i(a) \geq \min_{i \in N} u_i(b)$$

Interpretation of the associated choice correspondence is as follows:

Each agent proposes first his top outcome: if they are not unanimous, each proposes next his two best outcomes: if some outcome is proposed by all agents it is elected and the process ends. If

not, the process goes on with each agent proposing his three best outcomes, and so on . . .

Prove that R_0 is derived from some vector of scores. Similarly prove that the Kramer s.w.o. is a supporting size method.

2. Next consider the lexicographic refinement of R_0 namely R_{00}

$$aR_{00}b \Leftrightarrow (v_0(a), v_1(a), \dots, v_{p-1}(a)) \leq (v_0(b), \dots, v_{p-1}(b))$$

where \leq is the lexicographic ordering of \mathbb{R}^p .

Consider also the lexicographic refinement of the Kramer s.w.o.:

$$aK_{00}b \Leftrightarrow [\tau(a, a')]_{a \neq a'}^* \geq [\tau(b, b')]_{b \neq b'}^*$$

where for all $z \in \mathbb{R}^{p-1}$, $z^* \in \mathbb{R}^{p-1}$ is obtained by rearranging the coordinates of z in increasing order.

Prove that R_{00} (resp. K_{00}) is a scoring s.w.o. (resp. a supporting size s.w.o.).

Exercise 2.

Any scoring s.w.o.: $u \rightarrow R^S(u)$ satisfies the following weak AIIA axiom. For all outcomes a, b, c, d and profiles u, u' we have:

$\{(c, d) \cap \{a, b\} = \emptyset \text{ and } u \text{ and } u' \text{ differ only inasmuch as } c, d \text{ have been permuted in some agents' opinion}\} \Rightarrow \{aR^S(u)b \Leftrightarrow aR^S(u')b\}$

Give an example of an s.w.o. satisfying weak AIIA that is not a scoring s.w.o.

Exercise 3. The Paradox of Repeated Elimination of the Losing Outcomes

Let $A = \{a, b, c\}$ and consider the following 27 agents profile

agents	6	4	6	2	6	3
outcomes	a	b	b	c	c	a
	b	a	c	b	a	c
	c	c	a	a	b	b

1) Consider any pair s^1, s^2 of vectors of scores such that $s_1^i > s_2^i$ and prove that the successive elimination of losing outcomes as described in Section 1 enforces the election of a .

2) Consider a new profile, where the only change is that a and b are exchanged in three of the four bac rankings and in the two cba rankings. Prove that the same voting rule elects now c.

Exercise 4.

Consider the probabilistic Borda voting rule:

$$S_a(u) = \frac{2}{p(p-1)n} \sum_{i \in N} u_i(a) .$$

Show that it is a (probabilistic) supporting size voting rule. Conversely which probabilistic voting rules are together scoring and supporting size methods?

Chapter 8. Sophisticated Voting

In this chapter and the following, the set A of feasible outcomes is fixed, so that the dilemma social welfare ordering versus Arrow's IIA axiom vanishes.

1. Voting by Veto

and

Given a game form/a fixed profile, a dominating strategy of a particular agent i is an optimal decentralized behaviour for this agent whatever the information he or she possesses on the other agents' preferences. If i is completely informed of the whole profile or if he is only aware of his own preference ordering, he will still use his dominating strategy as long as cooperation with the rest of the society is not possible. This is how strategy-proof game forms achieve full decentralization of collective decisions: not creating any incentive for individual agents to acquire information about their respective preferences.

In most familiar voting methods, however, an agent can profitably use information about other agents' opinions. A simple example makes this intuition precise.

Example 1: Voting by Successive Veto. Let the set $A = \{a, b, c, d\}$ of candidates contain four outcomes and society $N = \{1, 2, 3\}$ is made up of three agents. The following game form is in order: each agent successively vetoes one among the non-vetoed candidates. The (necessarily unique) remaining candidate is elected. Suppose next that the following profile holds:

u_1	u_2	u_3
a	b	d
b	a	b
c	d	a
d	c	c

Agent 3 clearly has a dominating strategy. When facing a pair $\{\alpha, \beta\}$ of candidates he vetoes the one he prefers less. This "sincere" strategy is (non-cooperatively) optimal whatever the utility, and therefore the strategy, of the other two agents. On the other hand, agents 1 and 2 have no dominating strategy. Actually, every strategy of agent 2, where he never vetoes his best preferred candidate among the three left by agent 1, is undominated. Hence, he has 2^4 undominated strategies among 3^4 possible strategies. Finally, among the 4 possible strategies of agent 1 not one is dominated (exercise: check that "veto a" is not a dominated strategy). Thus, if he ignores the others' utilities, no good choice emerges and he will presumably use the secure strategy: veto d, the unique best choice of a risk-averse agent. After d is vetoed, the preferences of agents 2 and 3, restricted to a,b,c, do coincide and b will be elected by any pair of undominated strategies.

Let us assume now that agent 1 is aware of the whole profile: in particular he knows that by vetoing d he guarantees the election of b. Vetoing b might be a better strategy for agent 1. The restrictions of u_2 and u_3 to a,c,d are:

u_2	u_3
a	d
d	a
c	c

The two undominated strategies of agent 2, namely to veto c or d, yield the election of d and a, respectively. If agent 2 ignores agent 3's preferences, he will risk-aversedly veto c, and d will be the final outcome--a complete failure for agent 1's ruse!

If, on the contrary, agent 2 is completely informed, he can anticipate the output of his own veto, therefore optimally vetoing d to enforce the election of a . This ultimately justifies that agent 1's best strategy is to veto b .

Notice that no explicit communication between agents is needed during this play. The crucial assumption is completeness of the information (I know your preferences, you know that I know your preferences, I know that you know...). Then by mutually anticipating their respective strategies, the agents non-cooperatively pick a unique equilibrium strategy. The resulting strategy n -uple is called the sophisticated equilibrium (or subgame perfect equilibrium): we shall define it formally in section 2. As an introduction to the general results about sophisticated voting, we study the voting by veto game forms. We fix A and N with respective cardinalities p and n . Let $v_i, i \in N$ be nonnegative integers such that:

$$\sum_{i \in N} v_i = p - 1 .$$

We set $v = (v_i)_{i \in N}$ and we denote by $\Sigma(v)$ the set of those mappings σ from $\{1, \dots, p-1\}$ into N such that

$$\forall i \in N: \sigma^{-1}(i) \text{ has cardinality } v_i .$$

An element $\sigma \in \Sigma(v)$ is a finite sequence with length $(p-1)$ and values in N which entirely specifies the elimination process. More precisely, to σ we associate the game form g_σ defined (in extensive form) as follows:

First, agent $\sigma(1)$ eliminates one outcome within A , say a_1 .

Next, agent $\sigma(2)$ eliminates one outcome within $A \setminus \{a_1\}$, say a_2 .

⋮

At step k , agent $\sigma(k)$ eliminates one outcome within $A \setminus \{a_1, \dots,$

$a_{k-1}\}$, say a_k .

⋮

The finally elected outcome is the remaining element of

$$A \setminus \{a_1, \dots, a_{p-1}\} .$$

For instance,

$$= \underbrace{(11\dots 122\dots 23\dots)}_{v_1} \quad \underbrace{(n-1nn\dots n)}_{v_n}$$

means that agent 1 exercises first all his veto power, next agent 2, and so on... On the other hand

$$d = (123123123\dots)$$

means that agents 1, 2 and 3 successively veto one outcome at a time.

Clearly all game forms g are neutral but not anonymous (however d is "more anonymous" than σ as the results below will make clear). Two different non-cooperative behaviours can be explicitly computed for the game form g . At one extreme the sincere (risk-averse) strategy is used by an agent who has no information on the other agents' preferences and is only aware of his or her own preference ordering. Thus, if at step k agent i must eliminate one outcome among $A \setminus \{a_1, \dots, a_{k-1}\}$ he will eliminate his worst outcome. Hence, if $u \in L(A)^N$ is a given profile and every agent plays sincerely, the elected outcome, denoted by $\text{sin}(\sigma, u)$, is given by the following algorithm:

a_1 is the worst outcome of $u_{\sigma(1)}$ among A ,

a_2 is the worst outcome of $u_{\sigma(2)}$ among $A \setminus \{a_1\}$,

\vdots

a_k is the worst outcome of $u_{\sigma(k)}$ among $A \setminus \{a_1, \dots, a_{k-1}\}$, (1)

\vdots

a_{p-1} is the worst outcome of $u_{\sigma(p-1)}$ among $A \setminus \{a_1, \dots, a_{p-2}\}$,

$\text{sin}(\sigma, u) = A \setminus \{a_1, \dots, a_{p-1}\}$.

At the other extreme, sophisticated behaviour is expected from non-cooperative agents completely informed of the overall profile. Let us denote by $\text{sop}(\sigma, u)$ the outcome elected by the sophisticated equilibrium of g_σ at profile u . Our first theorem says that the sophisticated equilibrium strategies as well as the sophisticated outcome $\text{sop}(\sigma, u)$ can be computed by the reverse algorithm of (1).

Theorem 1. (Mueller, 1978; Moulin, 1979). For any element σ of $\Sigma(v)$ let us denote by $\bar{\sigma}$ the symmetrical mapping of σ :

$$\forall k = 1, \dots, p-1: \bar{\sigma}(k) = \sigma(p-k).$$

Then we have

$$\forall u \in L(A)^N: \text{sop}(\sigma, u) = \text{sin}(\bar{\sigma}, u).$$

In other words, $\text{sop}(\sigma, u)$ is computed by the following algorithm:

b_1 is the worst outcome of $u_{\sigma(p-1)}$ among A ,
 b_2 is the worst outcome of $u_{\sigma(p-2)}$ among $A \setminus \{b_1\}$,
 \vdots
 b_k is the worst outcome of $u_{\sigma(p-k)}$ among $A \setminus \{b_1, \dots, b_{k-1}\}$,
 \vdots
 b_{p-1} is the worst outcome of $u_{\sigma(1)}$ among $A \setminus \{b_1, \dots, b_{p-2}\}$,
 $\text{sop}(\sigma, u) = A \setminus \{b_1, \dots, b_{p-1}\}$.

Notice that the above algorithm also gives the sophisticated equilibrium strategy of every agent. For instance, agent $\sigma(1)$ must perform the entire computation of b_1, \dots, b_{p-2} in order to obtain that outcome b_{p-1} that he is to eliminate at the first step.

As an immediate corollary of theorem 1, observe that if $\sigma = \bar{\sigma}$ then both the unanimously sophisticated behaviour of the agents and their unanimously sincere behaviour yield the election of the same outcome for all profile. There are plenty of such mappings

σ , as the reader can easily verify: if p is even and $n = 2$, voting by alternating veto is an example:

$$\sigma = (121212\dots121) \quad \begin{cases} 1 \text{ appears } \frac{p}{2} \text{ times} \\ 2 \text{ appears } \frac{p}{2} - 1 \text{ times} \end{cases}$$

In voting by alternating veto, each of the 2 agents successively eliminates one outcome: this yields a game form as close as anonymity as we can do within the family of voting by veto methods (see exercise 1). However, many more game forms exist that achieve, through sophisticated voting, an efficient, neutral and almost anonymous social choice function. We illustrate this point for $n=2$ by means of the rawlsian voting rules.

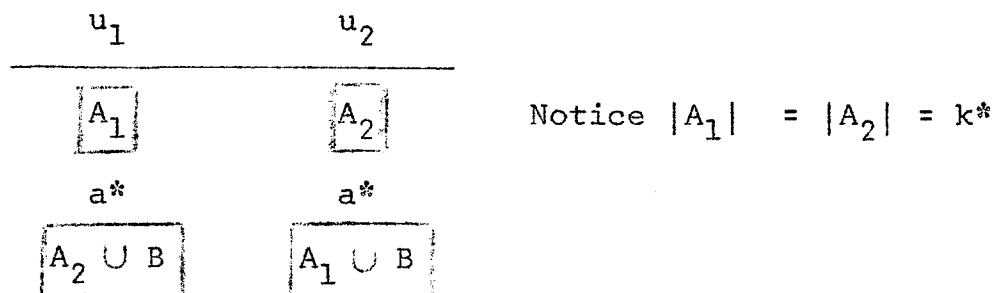
Implementing Two-Person Rawlsian Voting Rules

Given A and $N = \{1,2\}$ we represent an individual preference by a fixed scale utility function u_i , namely a one-to-one mapping from A into $\{0, \dots, (p-1)\}$. Denote by $R(u_1, u_2)$ the rawlsian outcomes at profile (u_1, u_2) namely

$$a \in R(u_1, u_2) \Leftrightarrow a \text{ is solution of } \max_{b \in A} \min \{u_1(b), u_2(b)\}$$

We define k^* by $p-1-k^* = \max_A \min\{u_1, u_2\}$. Typically if profile (u_1, u_2) has a unique rawlsian outcome a^* then $A \setminus \{a^*\}$ is partitioned into:

$A_1 \cup A_2 \cup B$ where A_i are the outcomes u_i -superior to a^* and B are the outcomes Pareto inferior to a^* :



Another possibility that we discard for simplicity is when there are two rawlsian candidates (see Exercise 4).

Observe first that voting by alternating veto does not implement a rawlsian voting rule, as the following profile shows:

u_1	u_2	
A_1	A_2	
a	a	for instance take
A_2	B	$ B = A_1 $ but this
B	A_1	is not necessary.

One can implement a rawlsian voting rule by either one of the following two methods:

Voting by successive approval (Laffond)

There are 2 agents and p outcomes. Each agent, starting with agent 1 successively approves of one outcome (an agent must approve of a different outcome at each play). As soon as one outcome is approved by both players it is elected.

Here a prudent, maximinimizer agent, approves first his top outcome, next his second preferred outcome and so on.... If both agents are prudent then a^* is elected.

On the contrary, sophisticated agents approve first alternatively one outcome from B until B is exhausted, whereafter they approve of A_1 until a^* is finally approved by both. Such delaying strategy allows optimal exploitation of the opponent's mistakes.

Another method is:

Conditional Auction.

For the sake of simplicity we assume that $p = 2p' - 1$ is odd.

Step 1: Each agent i simultaneously bids an integer λ_i , $1 \leq \lambda_i \leq p$, reports a utility v_i and a number $\epsilon_i \in \{0, +1\}$ (to be used to break possible ties). The agent whose bid is highest becomes the leader in Step 2. In case of a tie, a "matching pennies" rule is used from the messages ϵ_i , $i=1,2$: agent 1 (resp. 2) is declared leader if $\epsilon_1 + \epsilon_2 = +1$ (resp. if $\epsilon_1 + \epsilon_2 = 0$ or 2).

Step 2: Agent i , the leader, picks the final outcome a under the sole constraint $\lambda_j \leq v_j(a)$ (where j is the non-leader agent). Agent i 's (essentially unique) prudent (maximin) strategy is to announce $\lambda_i = p'$ and to reveal as v_i his true utility u_i (with an arbitrary ϵ_i). By such a message he is guaranteed of the utility level p' : if i is not leader in Step 2, this follows from the definition of our mechanism. If i is leader in Step 2, he faces a constraint such as $v_j(a) \geq \lambda_j$ where $\lambda_j \leq p'$, thus forbidding at most $(p'-1)$ candidates: hence at least one a can be found satisfying the constraint and moreover $u_i(a) \geq p'$. We let the reader check that any other message guarantees a lower utility level.

Now sophisticated agents are easily shown to use the following pair of mirror strategies (each player announcing a mirror image of the other's utility):

$$x_1 = (k^*, \bar{u}_2, \epsilon_1) \quad x_2 = (k^*, \bar{u}_1, \epsilon_2)$$

where \bar{u}_i is the reverse of u_i and ϵ_i is arbitrary.

Indeed, the leader in Step 2--say agent i --faces the constraint $\bar{u}_i(a) \geq k^*$ ($=$) $u_i(a) \leq u_i(a^*)$. Strikingly enough, both voting by alternating veto and the two above methods implement the same outcome with two equally prudent or two equally sophisticated voters. Yet a sophisticated (completely informed) agent can extract a lot more utility from a prudent (uninformed) agent.

2. Dominance Solvable Game Forms

We formalize the strategic mutual anticipation exemplified above.

Notation: Undominated strategies. Given A and N , both finite, let $g = (X_i, i \in N; \pi)$ be a game form and $u \in L(A)^N$ be a fixed profile. For any subsets $Y_i \subset X_i$, $i \in N$, and any agent $j \in N$, we denote by $\mathcal{D}_j(u_j; Y_i, i \in N)$ the set of agent j 's undominated strategies when the strategy spaces are restricted to Y_i , $i \in N$. Thus, x_j belongs to $\mathcal{D}_j(u_j; Y_i, i \in N)$ if and only if

$$x_j \in Y_j \text{ and for no } y_j \in Y_j \left\{ \begin{array}{l} \forall x_j^{\hat{}} \in Y_j: u_j(\pi(x_j, x_j^{\hat{}})) \leq (\pi(y_j, x_j^{\hat{}})), \\ \exists x_j^{\hat{}} \in Y_j: u_j(\pi(x_j, x_j^{\hat{}})) < u_j(\pi(y_j, x_j^{\hat{}})) \end{array} \right.$$

Definition 1. Given a game form g and a profile $u \in L(A)^N$ the successive elimination of dominated strategies is the following N -tuple of decreasing sequences: X_j^t , $j \in N$, $t \in N$:

$$X_j^0 = X_j; \quad X_j^{t+1} = \mathcal{D}_j(u_j; X_i^t, i \in N) \subset X_j^t.$$

We say that g is dominance-solvable at u if there is an integer t such that $\pi(X_{i \in N} X_i^t)$ is a singleton, denoted $S(u)$. We say that g

is dominance solvable if it is so at every profile. In that case we say that g sophisticatedly implements the social choice function S .

The crucial behavioural assumptions underlying the concept of sophisticated voting are complete information (every agent is aware of the whole profile) and non-cooperation (agent i 's strategic choice is made independently of the other agents' choices). Hence, a dominance-solvable game form is a decentralization device to the extent that it gives an incentive to the agents to acquire information on their mutual preferences, which in turn yields the selection of an unambiguous outcome. Thus, it is a realistic equilibrium concept only when the relevant information can be obtained and non-cooperative behaviour can be enforced. This restriction did not apply to strategy-proof game forms.

Actually, at any fixed profile, dominance-solvability is a generalization of strategy-proofness: if every agent has a dominating strategy in the normal form game $(X_i, u_i \circ \pi, i \in N)$, then X_i^1 is made up of agent i 's dominating strategies and $\pi(X_{i \in N}^1)$ is a singleton. The key observation is that dominance-solvable game forms are a rich class of decision-making procedures (allowing great flexibility in the distribution of power) and therefore implement a great variety of social choice functions. By weakening strategy-proofness into dominance-solvability we convert the Gibbard-Satterthwaite impossibility theorem into a wide possibility result. A large family of dominance solvable game forms is generated by finite game trees: these generalize the binary game trees of Chapter 3.2 in the following way:

Consider any finite tree (not necessarily a binary one) and assign to each nonterminal node m a strategy-proof game form bearing on the successors of m . Formally, a finite tree is a pair $\Gamma = (M, \sigma)$, where M is the finite set of nodes and σ associates to each node its nearby predecessor. We require that σ satisfy the following properties:

- (a) There is a unique node m_0 subject to $\sigma(m_0) = m_0$. It is the origin of Γ .
- (b) There is an integer l subject to $\sigma^l(m) = m_0$ for all $m \in M$.
- (c) The smallest such integer is the length of Γ .

A node m subject to $\sigma^{-1}(m) = \emptyset$ is called a terminal node of Γ , and their set is denoted Z . For a non-terminal node m , we call $\sigma^{-1}(m)$ the set of successors of m .

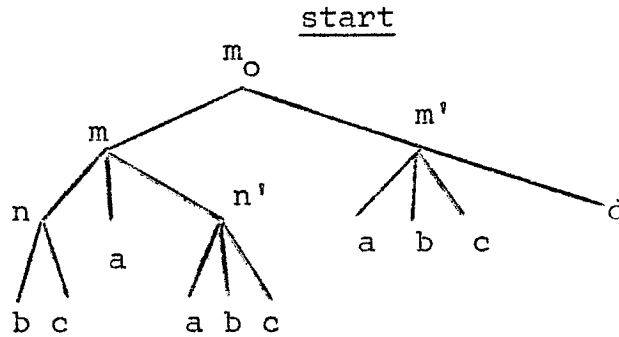
Given a set A of outcomes, we construct a game form on A by assigning to each terminal node an element of A and to each non-terminal node a game form bearing on its successor nodes. Formally, let θ be a mapping from Z onto A , and for all $m \in M \setminus Z$, let $g_m = (X_i^m, i \in N; \pi^m)$ be a game form on $\sigma^{-1}(m)$. (Thus, $\pi^m \prod_{i \in N} X_i^m = \sigma^{-1}(m)$.) Then the game form associated with (M, σ, θ, g) is described as follows:

• A strategy x_i of agent i associates to each non-terminal node $m \in M \setminus Z$ an element x_i^m of X_i^m . Their set is denoted X_i .

• For each strategy N -tuple $x = (x_i, i \in N)$ we define $\pi(x) = \theta(m_T)$, where m_T is the first terminal node of the sequence

$$\left\{ \begin{array}{l} m_0, m_1 = \pi^0(x^0), \dots, m_{i+1} = \pi^i(x^i), \dots \\ \text{where } x^m = (x_i^m, i \in N). \end{array} \right.$$

Let us visualize a typical game form associated to a finite tree: σ associates to each node the nearby upper node; at nodes m_0 and n a majority vote is taken to pick one successor; and at m (resp. at m' , n') agent 1 (resp. agent 2) selects dictatorially the successor node.



Theorem 2. (Moulin, 1979). Under the above notations and assumptions suppose that for each non-terminal node m , the game form g_m is strategy-proof. Then the game form associated with (Γ, θ, g) is dominance-solvable.

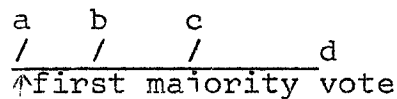
In view of the Gibbard-Satterthwaite theorem, if the game form g_m is strategy-proof, then either g_m is binary, i.e., m has at most one successor, or g_m is dictatorial. Therefore the two main examples of dominance-solvable game forms derived from theorem 2 are the voting by binary choices at one extreme and the "voting by dictatorial choices" at the other extreme.

Voting by binary choices are derived from the binary game trees of Chapter 3.2. The idea is to decide on the successor of any node by an anonymous voting such as a q -quota majority game; $q > \frac{n}{2}$. (Of course, we could use a nonanonymous simple game but the privilege of voting by binary choices is to allow for a strict anonymity.) Take the game tree at the beginning of Chapter 3.2. The game form starts from the origin of the tree and decides each move to the left by

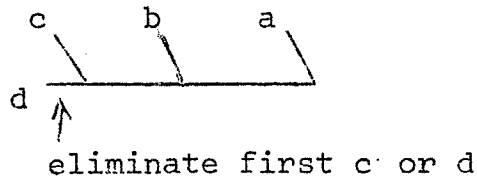
the q-quota game: a tie breaking device (e.g. favoring the downward branch) must be attached to the tree in case no successor receives the quota .

From the proof of theorem 2 follows that sophisticated voting will implement by this game form the binary social choice functions derived from this tree. Thus the elected outcome always belongs to the top cycle.

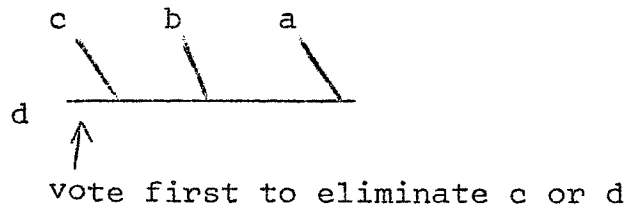
Notice the opposite orientations of the voting rule and its implemented s.c.f., e.g. voting by successive amendments:



implements the sequential elimination aggregator:



Similarly voting by successive elimination



implements the provisional winner aggregator (see Chapter 3.2).

The second important subclass of game trees voting methods emerges when each game form π is dictatorial: of this type are the voting by veto game forms, as well as the successive approval voting rule (Example 2). Here is one more example.

Kingmaker

Agent 1 pinpoints to one other agent, say i_2 who must name a successor agent in $N \setminus \{1, i_2\}$, say i_3 who must name a successor in $N \setminus \{1, i_2, i_3\}$ and so on... until only one agent is left who dictatorially chooses the final outcome. Computing the sophisticated outcome of this method does not seem tractable, yet a centrist voter (assuming for instance that preferences are single-peaked) has a clear advantage.

3. Sophisticated Implementation

A sophisticated social choice function is one that is sophisticatedly implemented by some finite dominance solvable game form (requiring finiteness of the message space is consistent with our assumption, throughout this chapter, that A and N are both finite). Thus, it is a s.c.f. that results from the non-cooperative behaviour of completely informed agents in some procedure. Optimally, we would like to characterize the set \mathcal{S} of sophisticated s.c.f.s. This would indeed explain much of the collective implications of non-cooperation. Actually, we know only a few facts about \mathcal{S} : it is very big (theorem 2), but many familiar s.c.f.s are outside \mathcal{S} (corollary of theorem 3). We establish a necessary condition for an s.c.f. to be sophisticated.

Notation. Given a utility $u \in L(A)$ and a coalition $T \subset N$,

we denote by \bar{u} the reverse of u : $u(a) < u(b)$ iff $\bar{u}(b) < \bar{u}(a)$, and by $[u]_T \in L(A)^T$ the profile of coalition T of which every coordinate is u .

Definition 2. Let A , N and a s.c.f. S be given. We will say that S is tight if the following property holds:

$$\begin{aligned} \forall u \in L(A), \forall T \subset N: u(S(v_T, [\bar{u}]_{T^c})) &\leq u(S([u]_T, [\bar{u}]_{T^c})) \\ &\leq u(S([u]_T, v_{T^c})) \\ \text{all } v_T \in L(A)^T, \text{ all } v_{T^c} \in L(A)^{T^c}. \end{aligned} \quad (2)$$

This means that for those profiles for which society splits into two homogeneous and antagonistic coalitions (members of T all having utility u , members of T^c all having utility \bar{u}) then the sincere messages $[u]_T$, $[\bar{u}]_{T^c}$ form a saddle-point of the two-person zero sum game where T and its complement T^c are the two players with respective utilities u and \bar{u} . Hence, for this very peculiar sort of profile, truth is an optimal strategy for each coalition. It is a "maximin" strategy as well:

$$u(S([u]_T, [\bar{u}]_{T^c})) = \inf_{v_{T^c}} u(S([u]_T, v_{T^c})) = \sup_{v_T} \inf_{v_{T^c}} u(S(v_T, v_{T^c})).$$

These equations are an easy consequence of (2): it is well known that in the two-person zero sum games where a saddle point does exist, optimal strategies and maximin (prudent) strategies do coincide. Notice that it does not follow from (2) that the truth is a dominating strategy of either T or T^c .

Theorem 3. Moulin [1983].

A sophisticated voting rule is tight.

Corollary.

For $n = |N|$ large enough and a vector of scores

$$s_0 < s_1 < \dots < s_{p-1}$$

no associated scoring voting rule (i.e. a single valued selection of the choice set of that scoring s.w.o.) is sophisticated. For instance no Borda voting rule is sophisticated.

Unfortunately the set of sophisticated voting rules is mostly unknown. Here are some open problems:

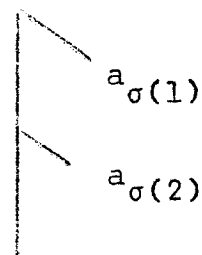
- if n is a prime integer strictly greater than p , there exists an anonymous, neutral and efficient sophisticated voting rule (see Moulin [1983]). What if n has no prime factor less than or equal to p (see Exercise 3, Chapter 1)?
- It is known that some sophisticated voting rules cannot be derived from a game tree; how to characterize those derived from a game tree?

Exercises on Chapter 8

Exercise 1. A sophisticated efficient anonymous and neutral s.c.f. (Kim and Roush, 1980). We suppose $|A| = p = 3$ and $|N| = n = 5$. We denote by σ an arbitrary ordering of A , written as a one-to-one mapping from $\{1,2,3\}$ into A . Next G_σ denotes the voting by successive amendments corresponding to σ (fig. 5.8).

For all $i \in N$, let H_i be the following voting rule: agent i picks one among the four other agents--say j --then agent j chooses an ordering σ of A , finally G_σ is played.

The overall procedure is any voting by binary choices over H_1, \dots, H_5 . Prove that this procedure is d-solvable and implements the plurality with a run-off, s.c.f. (see Chapter 7), and hence an efficient anonymous and neutral s.c.f.



Exercise 2. Voting by Unanimous Approval.

Let A be fixed with cardinality p and let $A = \{a_1, \dots, a_p\}$ be an ordering of

A . The agents vote first to elect a_1 or reject. Unanimity is required to enforce a_1 . If at least one agent's vote against a_1 we go to the second round of voting where a_2 is on the floor. unanimity being needed to enforce a_2 , and so on.

This procedure is pictured as a binary tree (identical to the tree of voting by successive amendments) where to each node is attached a non-neutral strategy-proof game form. Prove that the sophisticatedly implemented s.c.f. is as follows.

We set $p_0 = p$, $p_1 = \sup\{k \in \{1, \dots, p\} / a_k \text{ is Pareto superior to } a_p\}, \dots, p_{t+1} = \sup\{k \in \{1, \dots, p_t\} / a_{p_{t+1}} \text{ is Pareto superior to } a_{p_t}\}$. Then $S(u) = a_{p_t}$, where p_t is the first integer such that no p_{t+1} exists. In particular $S(u)$ is an efficient, anonymous and monotonic s.c.f.

Exercise 3. Voting by Alternating Veto. Society contains two agents $N = \{1, 2\}$ who alternately eliminate one among the existing outcomes. For a given cardinality p of A , two voting by veto methods are in order:

$$\sigma = (12121\dots)$$

where agent 1 starts the eliminating process, and

$$\sigma_2 = (2121\dots)$$

where agent 2 starts the eliminating process.

(1) For any given $p \geq 2$, find an example of a profile $u \in L(A)^2$ such that

$$\text{sop}(\sigma_1, u) \neq \text{sop}(\sigma_2, u).$$

(2) Prove that, when sophisticated voting is in order, it is always an advantage to start to eliminate first:

$$\text{setting } a_i = \text{sop}(\sigma_i, u) : u_i(a_i) > u_i(a_j), \text{ for } \{i, j\} = 1, 2.$$

When sincere voting is in order, it is an advantage or a disadvantage to start first?

(3) Prove that the outcomes of sophisticated voting a_i , $i=1,2$, either coincide or are two consecutive Pareto optima. There is no outcome c such that

$$u_1(a_2) < u_1(c) < u_1(a_1)$$

and

$$u_2(a_1) < u_2(c) < u_2(a_2).$$

Exercise 4. More on Implementing Rawlsian s.c.f.s

a) Analyze the two voting rules (voting by successive approval and conditional auction) when two rawlsian outcomes are at hand. Show that the sophisticated equilibrium outcome is one of them.

b) Consider voting by variable veto:

Step 1. Each agent selects a subset of outcome A_i and a number $\varepsilon_i \in \{-1, +1\}$ to break possible ties.

Step 2. If $|A_i| < |A_j|$ then j is leader; he must select the final outcome within $A \setminus A_i$.

If $|A_1| = |A_2|$ a matching pennies rule decides upon the leader.

Prove that the sophisticated equilibrium outcome is a Rawlsian one. What about prudent behaviour?

c) Consider the following voting rule:

Step 1.

Agent 1 selects an integer k , $0 \leq k \leq p - 1$.

Step 2.

Agent 2 selects one from two possible g.f. $G_1(k)$ or $G_2(k)$:

$G_1(k)$: first agent 1 vetoes k candidates and then agent 2 vetoes $p - 1 - k$ of the remaining candidates.

$G_2(k)$: first agent 2 vetoes k candidates and then agent 1 vetoes $p - 1 - k$ of the remaining candidates.

Intuitively, agent 1 cannot select too high a value for the integer k (for $k = p - 1$, agent 2 is a dictator when choosing $G_2(p-1)$) nor can he or she select too low a value (for $k=0$, agent 2 is again a dictator when choosing $G_1(0)$).

Prove that the sophisticated equilibrium outcome is Rawlsian, whereas prudent behaviour is not.

CHAPTER 9. Cooperative Voting

1) Cooperative Instability and Strong Equilibrium

We describe cooperative behavior by the concept of strong equilibrium.

Definition. Given a game form $g = (X_i, i \in N; \pi)$ and a profile $u \in L(A)^N$ we shall say that $x = (x_i)_{i \in N}$ is a strong equilibrium if the following holds:

$$\forall T \subset N, \forall y_T \in X_T: \text{No} \{ \forall i \in T: u_i \pi(y_T, x_{-T}^c) > u_i \pi(x) \} .$$

We denote by $SE(g; u)$ the set of strong equilibriums of g at u .

In words, x is a strong equilibrium if no coalition of players can jointly deviate, this deviation being profitable if the players outside the coalition do not react. This passive behavior from the non-deviating players must be seen as a threat successfully deterring the complement coalition. The fact that this threat is merely "no move" makes it particularly easy to carry out. Let us remark finally that a strong equilibrium must be in particular a Nash equilibrium and a Pareto optimum (as follows when T is a singleton coalition or is the grand coalition).

Definition. We shall say that game form g is stable if for all profile $u \in L(A)^N$ it has at least one strong equilibrium: for all u , $SE(g, u) \neq \emptyset$.

To any game form g we associate the simple game $W(g)$ of its winning coalitions, namely:

$$T \in W(g) \iff \forall a \in A \exists x_T \in X_T : \forall x_{-T} \in X_{-T} : \pi(x_T, x_{-T}) = a .$$

A first necessary condition for a game form g to be stable is that the dominance relation derived from its simple game at any profile is acyclic, since we have

Lemma

For any game form g and profile u a strong equilibrium outcome belongs to the core of game $W(g)$ at u :

$$\pi(SE(g,u)) \subset C_{W(g)}(u) \quad \text{for all profile } u$$

(notations as in Chapter 2.2).

Thus, for a game form to be stable, it is necessary that its simple game g satisfies the premises of Nakamura theorem (theorem 3, Chapter 2): if g is stable then the Nakamura number of its simple game $W(g)$ is strictly greater than $|A|$.

In other words the Condorcet paradox can be thought of as expressing cooperative instability of any game form where any strict majority is winning (as soon as A has at least 3 outcomes and $|N|$ is 3 or ≥ 5). The family of voting by veto games do not share that instability.

2) Cooperation in Voting by Veto

Cooperative stability allows to endow every coalition (however small) with some decision power. Consider a voting by veto game form as described in Chapter 7:

$v = (v_i, i \in N)$ is the vector of veto power: $\sum_{i \in N} v_i = p - 1$ (where $p = |A|$); for any $\sigma \in \Sigma(v)$, denote by g_σ the associated game form (see Chapter 7.1).

Any such game form is stable. Moreover its strong equilibrium outcomes are easily described. As far as cooperative stability is concerned it turns out that the particular choice of an ordering σ of the agents is irrelevant: the vector v of individual veto power only matters.

Given v and a coalition T , observe that by acting cooperatively, the agents in T can veto any subset B of outcomes as long as $|B| \leq \sum_{i \in T} v_i$. The righthand side of this inequality, we call the veto power of coalition T and we denote it $v(T)$. Intuition suggests, and theorem 1 below confirms, that an outcome a can possibly be stable with respect to cooperative behavior in (g, u) only if

$$\text{No } \begin{cases} \exists T \subset N \\ \exists B \subset A \end{cases} : |B| = p - v(T) \text{ and } \begin{cases} \forall i \in T \\ \forall b \in B \end{cases} : u_i(b) > u_i(a).$$

The above property defines the veto core $C_v(u)$ of v associated with u .

The following notation will be useful:

$$\begin{aligned} \text{for all } T \subset N, a \in A, u \in L(A)^N : \text{Pr}(T, a, u) \\ = \{b \in A / \forall i \in T \ u_i(a) < u_i(b)\}. \end{aligned}$$

Thus, the veto core $C_v(u)$ is equivalently defined by

$$C_v(u) = \{a \in A / \forall T \subset N : |\text{Pr}(T, a, u)| \leq p - v(T) - 1 = \sum_{i \notin T} v_i\}.$$

thus an outcome a is not stable if some coalition T can find a subset B of A such that:

(i) a coalition T can force the final outcome in B by vetoing its complement B^c ; and

(ii) all members of coalition T have an advantage to do so because every agent within T strictly prefers every outcome in B to a .

Theorem 1. Given v and a profile $u \in L(A)^N$, the set of strong equilibrium outcomes of the game (g_σ, u) is non-empty and does not depend on the particular choice of σ within $\Sigma(v)$. It coincides with the veto core $C_v(u)$:

$$\forall a \in A \{a \in C_v(u)\}$$

\Leftrightarrow a strong equilibrium x of (g_σ, u) exists, s.t. $a = \pi(x)$.

Moreover the veto core $C_v(u)$ equals the set of sophisticated equilibrium (resp. sincere) outcomes when σ varies over $\Sigma(v)$:

$$C_v(u) = \{\text{sop}(\sigma, u) / \sigma \in \Sigma(v)\} = \{\text{sin}(\sigma, u) / \sigma \in \Sigma(v)\}.$$

This theorem provides two different characterizations of the veto core $C_v(u)$. On the one hand it is the set of strong equilibrium outcomes of every game form g_σ (provided that $\sigma \in \Sigma(v)$): as is apparent from the proof of the theorem, one can describe explicitly a strong equilibrium strategy n -tuple to "implement" any given element of $C_v(u)$. Actually other (direct) game forms are available to implement the veto core by strong equilibrium (see the self-implementation property below). On the other hand $C_v(u)$ is just the set of sincere (resp. sophisticated) outcomes of g_σ when the ordering σ of the veto algorithm (but not the overall veto function) varies: thus outcome in $C_v(u)$ are easy to compute.

The above theorem also raises a remarkable property of the voting by veto method: the non-cooperative outcome always is coalitionally stable as well. This is not to say that a sincere (resp. sophisticated) strategy n -tuple is immune against coalitional manipulation--which would indeed contradict the Gibbard-Satterthwaite theorem! More modestly we say that any outcome that results from the sincere (resp. sophisticated) strategy n -tuple also results from a strong equilibrium (in general different) strategy n -tuple. This consistency property was first introduced by Peleg (in a slightly different context), see Peleg [1978].

The choice correspondence C_V (it is a s.c.c. where the set of feasible outcomes is kept fixed) has a number of appealing properties: it is neutral and efficient; it is strongly monotonic (see Moulin 1983, Ch. 3) as well as inclusion minimal with that property. Finally it is self-implementable in the following sense: take any single valued selection S of $C_V: S(u) \in C_V(u)$ all u ; thus S is just a voting rule, to which is associated a direct game form (where the message space of any agent is $L(A)$). Then this game form does implement C_V by means of strong equilibrium.

This fact is an easy consequence of the above theorem. For any $a \in C_V(u)$ we can find a labelling $\tau : A \rightarrow N$ such that $\tau^{-1}(i)$ has cardinality v_i for all $i \in N$ and moreover:

$$\forall i \in N \quad \forall b \in \tau^{-1}(i) \quad u_i(b) < u_i(a)$$

Then any message n -uple v where $\tau^{-1}(i)$ is at bottom of v_i is a strong equilibrium of S with associated outcome a .

3) Rational Voting by Veto

If p is small relative to n , the indivisibility of veto power in game forms g_σ described above does not allow for an anonymous allocation of decision power. In fact one can enlarge the class of voting by veto game forms by a replication argument which in turn endows each individual agent with a rational veto power:

We fix A and N with respective cardinalities p and n . Let $\mu_i, i \in N$, and $\lambda_a, a \in A$, be some non-negative integers such that

$$1 + \sum_{i \in N} \mu_i = \sum_{a \in A} \lambda_a .$$

Now consider a set A_λ made up of λ_a replicas of outcome a , for all $a \in A$. Formally:

$$A_\lambda = \{(a, z) / a \in A, 1 \leq z \leq \lambda_{a, z} \text{ integer}\} .$$

Next, let us denote by $\Sigma(u)$ the set of all finite sequences of N such that i appears exactly μ_i times for all i .

Formally:

$$\Sigma(\mu) = \{\sigma : \{1, \dots, \bar{\mu}\} \rightarrow N / |\sigma^{-1}(i)| = \mu_i\} ,$$

where $\bar{\mu} = \sum_{i \in N} \mu_i$.

To every particular element $\sigma \in \Sigma(\mu)$ we associate the following game form, denoted $g_{\sigma, \lambda}$:

First, agent $\sigma(1)$ eliminates one outcome within A_λ , say (a_1, t_1) .

Next agent $\sigma(2)$ eliminates one outcome within $A_\lambda \setminus \{(a_1, t_1)\}$, say (a_2, t_2) .

⋮

At step k agent $\sigma(k)$ eliminates one outcome within

$$A_\lambda \setminus (a_1, t_1) \dots (a_{k-1}, t_{k-1}) , \text{ say } (a_k, t_k) .$$

⋮

After step $\bar{\mu}$ there remains exactly one non-eliminated outcome within A , say (a, t) : then a is the finally elected outcome.

The additional property of $g_{\sigma, \lambda}$, as compared with the voting by integer veto g_σ , is what we are now able to adjust at will the resistance of the various outcomes of A to elimination:

for if $\lambda_a > \lambda_b$ it is strictly easier for any agent or coalition

of agents to eliminate b rather than a . Game forms $g_{\sigma, \lambda}$ are not neutral in general.

General voting by veto shares most strategic properties of voting by integer veto.

First the sincere elimination algorithm results when each agent kills a replica of the outcomes he likes least among those still alive. Then sophisticated behavior is given by the elimination algorithm of the reverse ordering $\bar{\cdot}$. (thus copying theorem 1, Chapter 7).

Going now to the cooperative behavior, we observe that given (λ, μ) the possibility for coalition T to veto the subset B of outcomes is no longer determined by the cardinality of B alone: the veto power of $g_{\sigma, \lambda}$ is not neutral in general.

Actually, we have that T can veto B if and only if

$$\sum_{i \in T} \mu_i = \mu(T) \geq \lambda(B) = \sum_{a \in B} \lambda_a .$$

Thus the veto core associated to a pair λ, μ is denoted

$C_{\lambda, \mu}$:

$$C_{\lambda, \mu}(u) = \{a \in A / \forall T \subset N : \lambda(\Pr(T, a, u)) + \mu(T) \leq \bar{\mu}\} .$$

Theorem. Given λ, μ and a profile $u \in L(A)^N$, the set of strong equilibrium outcomes of the game $(g_{\sigma, \lambda}, u)$ is non-empty and does not depend on the particular choice of σ within $\Sigma(\mu)$. It coincides with the veto core $C_{\lambda, \mu}(u)$ defined by (43). Moreover, it equals the set of sophisticated equilibrium (resp. sincere) outcomes when σ varies over $\Sigma(\mu)$:

$$C_{\lambda, \mu}(u) = \{\text{sop}(\sigma, \lambda; u) / \sigma \in \Sigma(\mu)\} = \{\text{sin}(\sigma, \lambda; u) / \sigma \in \Sigma(\mu)\} .$$

Notice that $C_{\lambda, \mu}$, however, is no longer neutral, nor self-implementable, nor inclusion minimal strongly monotonic.

4(The Proportional Veto Core)

Whenever n and p are relatively prime, we can find two integers λ, μ such that

$$1 + n\mu = p\lambda$$

thus allocating μ veto tokens to each voter and setting the veto price of any outcome at λ is a rational voting by veto method.

The point here is that allocation of veto power is both anonymous and neutral. Even when n, p are not relatively prime, we can meet these two requirements:

Definition: Given A the set of outcomes and society N , with respective cardinalities p, n , an anonymous veto function is a nondecreasing function v from $\{1, \dots, n\}$ into $\{0, \dots, p-1\}$, where $v(t) = k$ is interpreted as: any coalition with size t can veto any subset with at most k outcomes.

Definition:

Fix A, N , and an anonymous veto function v . Given a profile $u \in L(A)^N$, the core of v associated with u is denoted by $C_v(u)$. By definition it contains a iff for all coalition T we have

$$|\text{Pr}(T, a, u)| + v(t) \leq p - 1$$

where $t = |T|$ (same notation as in Section 2).

Definition: We say that an anonymous veto function $v \in V$ is stable if for every profile $u \in L(A)^N$ the associated core is non-empty:

$$\forall u \in L(A)^N : C_v(u) \neq \emptyset .$$

For instance the veto function of a q -majority game ($v_q(t) = 0$ if $t < q$, $v_q(t) = p - 1$ if $q \leq t$) is stable if and only if $q > n - \frac{n}{p}$, that is to say if the associated simple game yields an acyclic dominance relation (see Nakamura's theorem, Chapter 2).

An easy necessary condition for stability of a veto function is as follows:

$$t_1 + \dots + t_r \leq n \Rightarrow v(t_1) + \dots + v(t_r) \leq p - 1$$

Indeed if this fails, we can find a partition T_1, \dots, T_r of N and a partition B_1, \dots, B_r of A such that coalition T_1 can veto B_1, \dots, T_r can veto B_r . It is not difficult, then, to find a profile u at which the v -core is empty.

Proportional Veto

For fixed n, p the proportional veto function is defined by

$$\bar{v}_{n,p}(t) = [p \cdot \frac{t}{n}] - 1 \quad \text{all } t=1,2,\dots,n$$

where $[x]$ is the smallest integer bounded below by x . Hence $\bar{v}_{n,p}(t)$ is the greatest integer strictly less than $p \cdot \frac{t}{n}$.

Theorem 2. Moulin [1981] For all n, p (i) The proportional veto function $\bar{v}_{n,p}$ is stable. For every profile u the associated core is called the veto core associated to u and denoted $C_{n,p}(u)$.

(ii) An anonymous veto function is stable if and only if it is bounded above by the proportional veto function

$$\forall v \in V \{v \text{ stable}\} \Leftrightarrow \{\forall t = 1, \dots, n: v(t) \leq \bar{v}_{n,p}(t)\}.$$

Thus the proportional distribution of power (as precisely described by the proportional veto function $\bar{v}_{n,p}$) is the optimal distribution of coalitional power if one wants firstly to guarantee the stability of at least one outcome and secondly to make the set

of stable outcomes as small as possible. Namely $v \leq \bar{v}_{n,p}$ implies that $C_{n,p}(u)$ is a subset of $C_v(u)$ for all profile u .

To allocate veto power across coalitions is to arbitrate a trade-off: if we give too much veto power, cooperatively stable outcomes will disappear, yet if we give too little, stable outcomes will be too many. In the anonymous and neutral case, this dilemma has a unique solution, that we call the minority principle:

any coalition should be given a veto power nearly proportional to its size (Theorem 2 above makes this statement precise). The point is that this method gives some decision power (taken to be a right-to veto some outcomes) to all coalitions, however small.

In this way is met an ethical requirement that appears as an early objection to the majority principle, namely the need to prevent minorities of being crushed by the antagonistic majority:

"cependant une minorite ne peut pas etre a la merci d'une majorite: la justice, qui est la negation de la force, veut que la minorite ait ses garanties" (Proudhon).

To illustrate, let us oppose on several examples the minority principle embodied into the proportional veto core, and the majority principle that suggests to choose the Condorcet winner whenever there is one.

Suppose first n, p are large (so that $\bar{v}(t) = p \frac{t}{n}$ is a good approximation of $\bar{v}_{n,p}$ and society splits into two homogeneous and antagonistic coalitions

t agents having preferences: $1 < 2 < \dots < p$

$(n-t)$ agents having preferences $1 > 2 > \dots > p$

Then the Condorcet winner is just 1 or p according to $t > n-t$ or $n-t > t$. In contrast, the proportional veto allows the t rightist to veto (roughly) the $(p \cdot \frac{t}{n})$ outcomes with smallest index, while the $(n-t)$ leftist veto nearly all remaining outcomes: the corresponding core is in general a singleton (it is a doubleton only if $p \cdot \frac{t}{n}$ is an integer) namely $[p \cdot \frac{t}{n}]$. This solution clearly reflects a compromise oriented view of collective choice (observe that it coincides with the uniform Condorcet winner of Chapter 5). As another example take $n = 3$ and $p = 4$, so that the proportional veto is simply $\bar{v}_{3,4}(t) = t, 1 \leq t \leq 3$. Consider the following profile:

u_1	u_2	u_3
d	a	a
c	d	c
b	b	b
a	c	d

Here a is the Condorcet winner, whereas the proportional veto core is $C_{3,4}(u) = (b)$ since outcomes a, c and d are blocked respectively by agents 1, 2 and 3. In words, b is the only candidate that no voter regards as extremist.

Our last example is one where the veto core is poorly decisive: we choose $n = 5$ and $p = 6$ so that the proportional veto function is $\bar{v}(t) = t, 1 \leq t \leq 5$.

a	a	a	f
b	c	d	e
c	d	e	d
d	e	b	c
e	b	c	b
f	f	f	a
u_1	u_2	u_3	u_4, u_5

Here a is a Condorcet winner since it is the top candidate of a majority coalition $\{1,2,3\}$. The minority coalition $\{4,5\}$, however, has a veto power of 2 that it can use to prevent the election of both a and b. On the other hand, the majority $\{1,2,3\}$ although endowed with a right to veto any three outcomes will not use it since the agents in $\{1,2,3\}$ (although they agree on their top candidate a and their bottom candidate f) disagree on the ranking of the middle candidates (b,c,d,e). Only candidate f is therefore vetoed (by any single agent 1, 2 or 3) and the veto core is then $C_{5,6}(u) = \{c,d,e\}$.

5) Effectivity Functions

Definition 1. Given A and N an effectivity function is a binary relation defined on coalitions and subsets of outcomes (it is then a subset of $2^N \times 2^A$) denoted "eff" and satisfying the following properties.

(i) Monotonicity with respect to coalitions:

$$T \text{ eff } B \text{ and } T \subset T' \Rightarrow T' \text{ eff } B, \text{ for all } T, T' \text{ and } B.$$

(ii) Monotonicity with respect to subsets of outcomes:

$$T \text{ eff } B \text{ and } B \subset B' \Rightarrow T \text{ eff } B', \text{ for all } T, B \text{ and } B'.$$

(iii) Boundary conditions:

$$T \text{ eff } A, \text{ for all } T,$$

$$N \text{ eff } B, \text{ for all } B.$$

An effectivity function is a model of the distribution of power among agents and coalitions. Given A the set of outcomes, and N the society, we say that coalition $T \subset N$ is effective for the subset B of A if T can force the final decision within B or, equivalently, can veto the subset $B^c = A \setminus B$ of A . The main examples are simple games, anonymous veto functions, and additive effectivity functions (see below).

Definition 7. Given an effectivity function eff , and a profile $u \in L(A)^N$ the core $C(\text{eff}, u)$ is the following--possibly empty--subset of A :

$$\forall a \in A \{a \in C(\text{eff}, u)\} \stackrel{\text{def}}{\iff} \{\forall T \subset N: \{\text{No } T \text{ eff } \Pr(T, a, u)\}\}.$$

We shall say that eff is a stable effectivity function if the associated core is non-empty for all profiles:

$$\text{eff is stable} \stackrel{\text{def}}{\iff} \{\forall u \in L(A)^N C(\text{eff}, u) \neq \emptyset\}.$$

The core of an effectivity function at a particular profile is the set of cooperatively stable outcomes: an outcome a is unstable if a coalition T can force the final outcome within B , where all members of T strictly prefer every outcome in B to a . Since N is effective for any singleton, it follows that the core of any e.f. at any profile contains only Pareto optimal outcomes.

Definition. An effectivity function eff is said to be maximal if we have

$$T \text{ eff } B \iff \text{No}(T^c \text{ eff } B^c)$$

(where $T^c = N \setminus T$ and $B^c = A \setminus B$).

To any game form g on A, N we associate its effectivity function eff_g as follows:

$$T \text{ eff}_g B \text{ iff } \{\exists x_T \in X_T, \forall x_{T^c} \in X_{T^c} \pi(x_T, x_{T^c}) \in B\}$$

Theorem 3. Moulin Peleg [1982]

If g is a stable game form, its effectivity function eff_g is stable and maximal. Moreover

$$\pi(\text{SE}(g, u)) \subset C(\text{eff}_g, u) \text{ all profile } u$$

Conversely the core correspondence of any stable effectivity function can be implemented by strong equilibrium.

Hence the study of stable g.f. is (almost) equivalent to that of stable and maximal effectivity functions.

Definition

Say that an effectivity function eff is convex if it satisfies for all $T_i, B_i \ i=1,2$ $\{T_i \text{ eff } B_i \ i=1,2\} \Rightarrow \{T_1 \cup T_2 \text{ eff } B_1 \cap B_2$
and/or $T_1 \cap T_2 \text{ eff } B_1 \cup B_2$

Theorem 4 Peleg [1982]

A convex effectivity function is stable. A stable and maximal effectivity function is convex.

Corollary 1. Additive Effectivity Functions

Given two probability distributions λ and m on A and N , respectively:

$$\ell = (\ell_a)_{a \in A}, \quad \ell_a > 0; \quad \sum_{a \in A} \ell_a = 1,$$

$$m = (m_i)_{i \in N}, \quad m_i > 0; \quad \sum_{i \in N} m_i = 1,$$

we denote by $\text{eff}_{\ell, m}$ the following effectivity function:

$$T \text{ eff}_{\ell, m} B, \text{ iff } m(T) + \ell(B) > 1.$$

We say that an effectivity function is additive if it coincides with $\text{eff}_{\ell, m}$ for some probability distributions ℓ and m . Clearly these effectivity functions are convex. Hence they are stable. However only those derived from a rational allocation of veto power (see above Section 3) are maximal.

Corollary 2. Veto Functions

A neutral effectivity

function is described as a veto function $v : 2^N \rightarrow \{0, \dots, p-1\}$

where $T \text{ eff } B$ iff $v(T) \geq p - |B|$ and, in addition, v is nondecreasing.

A veto function is a convex effectivity function iff it satisfies:

$$v(T_1 \cup T_2) + v(T_1 \cap T_2) \geq v(T_1) + v(T_2); \text{ all } T_1, T_2.$$

It is a maximal effectivity function iff

$$v(T) + v(T^c) = p - 1 \quad \text{all } T.$$

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