THE DELEGATION AXIOM AND EQUAL-SHARING METHODS

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Abstract

A quasi-linear social choice problem amounts to select one (among finitely

many) indivisible public decision and a vector of monetary transfers among

agents to cover the cost of this decision. This decision is based upon

individual preferences which are assumed to be additively separable and linear

in money.

The delegation axiom is a consistency property for choice methods on

societies with variable size: the decision is not affected if we remove

an arbitrary agent under the condition that he should be guaranteed of his

original utility level and the cost to the remaining agents is modified

accordingly. Thus the utility level assigned by the social choice function

to agent i is the price at which the other agents are unanimously willing to buy agent i's share of the decision power.

A general characterization of choice methods satisfying this axiom is provided. Three subclasses of particular interest are characterized by additional milder axioms. Those are: i) equal sharing of the surplus left over some reference utility (e.g., the utility at a status quo decision, ii) utilitarian methods that merely select the efficient public decision and perform no monetary transfers, and iii) equal sharing of nonseparable cost that divide equally the surplus left over the utility derived from the pivotal mechanism (also known as the Vickrey-Clarke-Groves mechanism).

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1) Introduction

Think of any collective decision problem where the n concerned agents view the Pareto optimal utility vector $(U_1,\ldots,U_n)$ as the fair compromise. To select a decision achieving this utility vector, $(n-1)$ agents can safely delegate their share of the decision power to the n-th agent, say agent i, by restricting agent i's choice to those outcomes where, for all $j \neq i$, agent j's utility is at least U_j. That this one person decision problem achieves the original utility vector $(U_1,\ldots,U_n)$ is a well-known rephrasing of Pareto optimality.

The delegation axiom states a similar, more general property: suppose we decentralize the decision to the subcommittee $\{1,\ldots,t\}$, while restricting its choice set to those decisions where agents $t+1, t+2,\ldots,n$ enjoy at least the utility level $U_{t+1}, U_{t+2},\ldots,U_n$. The axiom requires that, for all t, the resulting decision is unaffected, namely the subcommittee picks a decision with associated utility vector $(U_1,\ldots,U_n)$. In other words delegating in those terms the choice problem to any subcommittee is inconsequential.

This axiom is very powerful in that it relates the decision problems of arbitrary societies with variable size. It was first introduced in the framework of bargaining theory by Harsanyi [1959] and later studied systematically by Lensberg [1982] [1983] (see also Lensberg and Thomson [1984]). The aim

of this paper is to explore its implications in quasi-linear problems.

A quasi-linear social choice problem amounts to select one (among finitely many) indivisible public decision and a vector of monetary transfers among agents to cover the cost of this decision. This decision is based upon individual preferences which are assumed to be additively separable and linear in money.

In this framework at least three families of arbitration methods have been recognized so far. Equally sharing the supply above a certain reference utility level (e.g., the utility derived from a particular status quo decision) is the first one. Although these methods are among the most common accounting devices when side-payments are available, their first systematical study in a quasi-linear model appears in Dubins [1977] who explored some of their strategical properties; more on those is to be found in Thomson [1979], Moulin [1983],[1984]. The second family rely upon a crude interpretation of utilitarianism: one rules out any redistributive transfers whenever there is a unique surplus-maximizing decision; this is simple to the point of naiveté; it is also highly vulnerable to strategical maneuvers (such as profitable disposal of one's own utility) and might be affected a lot by small changes in the preference profile (discontinuity). Nevertheless it meets some very demanding axioms, notably the delegation axiom.

The third type of arbitration methods (or social choice function) are value functions for the cooperative game associated with a quasi-linear problem. Within those, the delegation axiom uniquely selects a particular cost-sharing method, called the Equal Sharing of Nonseparable Costs. (Sraffin and Heaney [1981], Legros [1982]).

This paper axiomatically characterizes the three above families of social choce functions on the basis of the delegation axiom (which is satisfied by all three of them) and some additional milder properties (which are met specifically by only one of the three). It parallels an earlier paper (Moulin [1983]) which already proposes a characterization of the first two families on the basis of a different axiom, called No Advantageous Reallocations, ruling out profitable coalitional insurance against the public decision. Neither set of results (in the current paper and in Moulin [1983]) implies the other: see our comments at the end of Section 6.

In the quasi-linear context, most of the existing literature addresses the demand revealing mechanisms (notably the pivotal mechanism also known as the Vickrey-Clarke-Groves mechanism, and do not touch the cost-sharing issue. Our approach, on the contrary, emphasizes the redistributive role of monetary transfers. Although some of our axioms have a strategic flavor, they are mainly normative and the game of misrepresentation of preferences does not show up here.

The axiomatic derivation of solutions for cooperative games (with side payments) is another field that bears many similarities with this work. In fact our last result (Lemma 6) characterizes a particular value function for those games. It is strongly reminiscent of Sobolev's axiomatic characterization of the quasi-nucleolus (Sobolev [1975]) yet technically our result is trivial when compared to Sobolev's: see Section 8.

There are at least two important differences between our model of quasi-linear social choice and cooperative games with side-payments. Firstly we assume that the technology is indivisible: the cost of any public decision

is the same, whatever the coalition undertaking it. As a consequence, the cooperative games derived from a quasi-linear problem tend to be subadditive (instead of superadditive as is common in production economy games). Secondly the quasi-linear model distinguishes the utility genuinely derived from a public decision from the compensatory transfers: typically the utility enjoyed at a particular reference decision (e.g., status quo outcome) will determine the fair transfers. Accordingly our social choice functions depend on more than the set of feasible utility vectors: they take also into account the underlying public decisions gathered in the agenda.

2) Quasi-Linear Social Choice Functions

Let A be the set of public decisions among which n agents must jointly pick one. Decision a is public since no individual agent can be excluded from his consumption (although his or her opinion could be ignored in the choice process). It has a cost $c(a)$ that must be covered by a vector $t = (t_1,\ldots,t_n)$ of monetary transfers across agents. Thus an outcome is a pair (a,t) where $\sum_{i=1}^{n} t_i + c(a) = 0$.

Every agent i in $N = \{1,2,\ldots,n\}$ has quasi-linear preferences, described by a vector u_i in $\mathbb{R}^A$ so that his or her utility for outcome (a,t) is $u_i(a) + t_i$. Denote by $\mathbb{I}$ this vector in $\mathbb{R}^A$ whose all coordinates equal 1. Then two utility functions u_i, v_i such that $v_i = u_i + \alpha \mathbb{I}$ for some real number α (or $u_i(a) = v_i(a) + \alpha$, all $a \, \varepsilon \, A$), represent the same preferences over outcome set and will therefore be identified: property (2) states indeed that the zero of utility functions plays no role. Similarly consider two cost functions c, c' such that $c' = c + \alpha \mathbb{I}$ for some α . Then the additional fixed cost α from c to c' will be shared equally among agents (property (3)): in this way only incremental costs from one decision to another can matter: other cost estimates are typically harder to obtain.

Definition 1

Given a society $N = \{1,\ldots,n\}$ and a finite set A of public decisions, a social choice function S associates to any $(n+1)$-uple $(u_1,\ldots,u_n,c)$ made up of a preference profile and a cost function, a vector $S(u,c) = (S_1(u,c),\ldots,S_n(u,c))$ of utility levels: $S(u,c)$ is a vector in $\mathbb{R}^N$. We make the following assumptions on S:

· <u>Pareto optimality</u>:

for all u,c: $\sum_{i \in N} S_i(u,c) = \max_{a \in A} \{ \sum_{i \in N} u_i(a) - c(a) \}$ (1)

· <u>Anonymity</u>:   S is a symmetrical function of the variables $u_1, \ldots, u_n$.

· <u>Independence of the individual utilities' zero</u>

for all u,v,c ⎫ $u_i = v_i + \alpha\Pi$ and $u_j = v_j$ for all $j \neq i\} =>$ (2)

for all $i \in N$ ⎬

for all $\alpha \in \mathbb{R}$ ⎭ $\{S_i(v,c) = S_i(u,c) + \alpha$ and $S_j(v,c) = S_j(u,c)$

for all $j \neq i\}$

· <u>Independence of the cost function zero</u>:

for all u,c,c' ⎫ $\{c' = c + \alpha\Pi\} => \{S_i(u,c') = S_i(u,c) + \frac{\alpha}{n}$ for all $i \in N\}$ (3)

for all $\alpha \in \mathbb{R}$ ⎭

Notice that social choice functions retain only the final utility level $S_i(u,c)$

enjoyed by agent i. Implicit in the Pareto optimality axiom (1) we have that

the committee picks an efficient outcome (a^*, t^*) to achieve these levels:

$S_i(u,c) = u_i(a^*) + t_i^*$ where a^* is a solution of $\max_{A} \{ \sum_{i \in N} u_i - c \}$

and $\sum_{i \in N} t_i^* + c(a^*) = 0$.

If there is a unique efficient public decision (i.e. the problem

$\max_{A} \{ \sum_{i \in N} u_i - c \}$ has a unique solution) then (a^*, t^*) is unambiguously

determined by $S_1(u,c), \ldots, S_n(u,c)$. Otherwise, several outcomes (a,t) achieve

the utility vector $(S_1(u,c), \ldots, S_n(u,c))$. Choosing any one of these

is a merely technical issue, that the very formalism of social choice functions

ignores.

Notation

Given any $z \in \mathbb{R}^A$ we denote $z^{max} = \max_{a \in A} z(a)$.

Also for any coalition $T \subseteq N$ we write $S_T = \sum_{i \in T} S_i$, $u_T = \sum_{i \in T} u_i$, and so on . . .

Thus Pareto optimality ((1)) is compactly written as:

$$S_N(u,c) = (u_N - c)^{max} \quad \text{for all } u,c .$$

3) <u>The Delegation Axiom</u>

Let $S^1, \ldots, S^n, \ldots$ be a sequence of social choice functions, one for each size n of the society N. The delegation axiom is a consistency property between S^n and S^t for all $t \leq n$. It is enough to state it for $t = n - 1$:

<u>Delegation Axiom</u>

For all $n \geq 2$, all profile $u_1, \ldots, u_n$ and all cost function:

$$S_1^n(u_1, \ldots, u_n; c) = S_1^{n-1}(u_1, \ldots, u_{n-1}; c')$$

where $c'(a) = c(a) - u_n(a) + S_n^n(u, c)$, all $a \in A$ \qquad (4)

The issue is to delegate the decision process originally in the hands of the committee $\{1, \ldots, n\}$ to the subcommittee $\{1, \ldots, n-1\}$. Agent n simply resigns his rights under the condition that he should always end up with the utility level $S_n^n(u, c)$. This in turn affects the cost function of the subcommittee $\{1, \ldots, n-1\}$: to produce decision a, these agents must cover its cost $c(a)$ and pay a royalty t_n to agent n in such a way that $u_n(a) + t_n = S_n^n(u, c)$. Hence the modified cost function c' in formula (4).

Applying (4) repeatedly yields the delegation property for a subcommittee of arbitrary size:

for all t, n, $1 \leq t \leq n$, all $u_1, \ldots, u_n$ and all c.

$$S_1^n(u_1, \ldots, u_n, c) = S_1^t(u_1, \ldots, u_t, c')$$

where $c'(a) = c(a) - u_{T^c}(a) + S_{T^c}(a)$ all a

and $T^c = \{t+1, \ldots, n\}$

Its interpretation is similar.

Lemma 1

The sequence of social choice functions $S^1, \ldots, S^n, \ldots$ satisfies the delegation axiom <u>if and only if</u> there exists a real valued function $g(x,z)$ on the domain $[\mathbb{R}^A]^2$ satisfying i - ii

   i) $g(x + \alpha\amalg, z) = g(x,z) + \alpha$ for all $x, z \in \mathbb{R}^A$ and all $\alpha \in \mathbb{R}$

       $g(0,z) = 0$

  ii) $g(x, z + \alpha\amalg) = g(x,z)$ for all $x, z \in \mathbb{R}^A$ and all $\alpha \in \mathbb{R}$

<u>and</u> such that for all n, the s.c.f. S^n is worth:

$$S_i^n(u,c) = \frac{1}{n}(u_N - c)^{\max} + \frac{1}{n}\{(n-1)g(u_i, u_N - c) - \sum_{j \neq i} g(u_j, u_N - c)\} \qquad (5)$$

Proof:

Given g satisfying the invariance properties i-ii consider first the function S^n defined by (5). Routine checking shows that S^n is a social choice function (Definition 1). Let us check that the delegation axiom holds true.

Given $u_1, \ldots, u_n, c$, define c' as in (4) and compute

$$S_1^{n-1}(u_1, \ldots, u_{n-1}, c') = \frac{1}{n-1}(u_{N\backslash n} - c')^{\max} + \frac{1}{n-1}\{(n-2)g(u_1, u_{N\backslash n} - c')$$

$$- \sum_{2 \leq j \leq n-1} g(u_j, u_{N\backslash n} - c')\}$$

Since $u_{N\backslash n} - c' = u_N - c - S_n^n(u,c)\amalg$ and g satisfies ii we get

$$S_1^{n-1}(u_1, \ldots, u_{n-1}, c') = \frac{1}{n-1}(u_N - c)^{\max} - \frac{1}{n-1}S_n^n(u_1, \ldots, u_n, c) + \frac{1}{n-1}\{(n-2)g(u_1, u_N - c) -$$

$$\sum_{2 \leq j \leq n-1} g(u_j, u_N - c)\}$$

Set $z = u_N - c$ and for all $i = 1, \ldots, n$, $\gamma_i = g(u_i, z)$, next

replace in the above formula $S_n^n(u_1,\ldots,u_n,c)$ by means of (4):

$$S_1^{n-1}(u_1,\ldots,u_{n-1},c') = \frac{1}{n-1} z^{\max} - \frac{1}{n-1} \left[\frac{1}{n} z^{\max} + \frac{1}{n} \{(n-1)\gamma_n - \sum_{j \leq n-1} \gamma_j\}\right]$$

$$+ \frac{1}{n-1}\left[(n-2)\gamma_1 - \sum_{2 \leq j \leq n-1} \gamma_n\right]$$

$$= \frac{1}{n} z^{\max} + \frac{n-1}{n} \gamma_1 - \sum_{2 \leq j} \gamma_j \qquad \text{QED}$$

To prove the converse statement start with a sequence of s.c.f. $S^1,\ldots,S^n,\ldots$ satisfying the delegation axiom. For $n = 2$, the axiom is just Pareto optimality so apply it for $n = 3$. Formula (4) is written as

$$S_1^3(u,c) = S_1^2(u_1,u_2,c') = S_1^2(u_1,u_2,c-u_3) - \frac{1}{2} S_3^3(u,c)$$

for all $u = (u_1,u_2,u_3)$ and all c.

Define an auxiliary function R by:

$$S_1^2(u_1,u_2,c) = R(u_1,u_2,u_{12}-c) \iff R(x,y,z) = S_1^2(x,y,x+y-z)$$

We get now:

$$S_1^3(u,c) + \frac{1}{2} S_3^3(u,c) = R(u_1,u_2,u_{123}-c) \quad \text{all } u, \text{ all } c$$

Permuting the role of the agents yield similarly:

$$S_2^3(u,c) + \frac{1}{2} S_1^3(u,c) = R(u_2,u_3,u_{123}-c)$$

$$S_3^3(u,c) + \frac{1}{2} S_2^3(u,c) = R(u_3,u_1,u_{123}-c)$$

Summing up these 3 equations and using Pareto optimality we get

$$R(u_1,u_2,z) + R(u_2,u_3,z) + R(u_3,u_1,z) = \frac{3}{2} z^{\max}$$

$$\text{for all } u_1,u_2,u_3 \text{ and } z \text{ in } \mathbb{R}^A .$$

Thus for any <u>fixed</u> z , the function $f(x,y) = R(x,y,z) - \frac{1}{2} z^{max}$ satisfies the functional equation

$$f(u_1,u_2) + f(u_2,u_3) + f(u_3,u_1) = 0 \quad \text{all} \quad u_1,u_2,u_3 \tag{6}$$

which holds iff f can be written as $f(u_1,u_2) = \frac{1}{2} h(u_1) - \frac{1}{2} h(u_2)$ for some real valued function h . (To check this claim make $u_3 = 0$ in (6) so that $f(u_1,u_2) = \frac{1}{2} h(u_1) - \frac{1}{2} h'(u_2)$ for some h,h'. Applying (6) again gives $h = h'$). Hence R can be written as $R(x,y,z) = \frac{1}{2} z^{max} + \frac{1}{2} [h(x,z) - h(y,z)]$. We have just proved existence of a real valued function h defined on $(\mathbb{R}^A)^2$ such that

$$S_1^2(u_1,u_2,c) = \frac{1}{2}(u_{12}-c)^{max} + \frac{1}{2} \{h(u_1,u_{12}-c) - h(u_2,u_{12}-c)\}$$

$$\text{for all} \quad u_1,u_2,c \tag{7}$$

Setting now $g(x,z) = h(x,z) - h(0,z)$ yields another function for which S^2 is given by (7) (with g instead of h). In addition, $g(0,z) = 0$ for all z. We derive now the invariance properties i-ii for g from properties (2) and (3). Make first $u_2 = 0$ and write $S_1^2(u_1,0,c+\alpha\mathbb{I}) = S_1^2(u_1,0,c) - \frac{\alpha}{2}$ with the help of (7):

$$\frac{1}{2}(u_1-c)^{max} - \frac{\alpha}{2} + \frac{1}{2} g(u_1,u_1-c-\alpha\mathbb{I}) = \frac{1}{2}(u_1-c)^{max} + \frac{1}{2}g(u_1,u_1-c) - \frac{\alpha}{2}$$

Since this holds for all u_1,c, and α , property ii is proved. Similarly, writing $S_1^2(u_1+\alpha\mathbb{I},0,c) = S_1^2(u_1,0,c) + \alpha$ with the help of (7), buys us property i.

To summarize we have found g satisfying i-ii and formula (5) for $n=2$. Assume now by induction that (5) holds until $(n-1)$. Then the delegation axiom and property (3) allow us to derive S^n from S^{n-1}. Specifically (4) amounts to:

$$S_1^n(u,c) = S_1^{n-1}(u_1,\ldots,u_{n-1},c-u_n + S_n^n(u,c)\Pi) = S_1^{n-1}(u_1,\ldots,u_{n-1},c-u_n)-\frac{1}{n-1}S_n^n(u,c)$$

Fix u,c, set $\sigma_i = S_i^n(u,c)$ and $z = u_N - c$. By the induction assumption, the above formula yields:

$$\sigma_1 + \frac{1}{n-1}\sigma_n = \frac{1}{n-1} z^{max} + \frac{1}{n-1} \{(n-2)g(u_1,z) - \sum_{2\leq j\leq n-1} g(u_j,z)\}$$

A similar equation holds when agents 1, n are replaced by any pair i,j. Hence the system:

$$\sigma_i + \frac{1}{n-1}\sigma_j = \frac{1}{n-1} z^{max} + \frac{1}{n-1} \{(n-2)\gamma_i - \sum_{k\neq i,j}\gamma_k\} \text{ , where } \gamma_i = g(u_i,z)$$

with unique solution $\sigma_i = \frac{1}{n} z^{max} + \frac{1}{n} \{(n-1)\gamma_i - \sum_{j\neq i}\gamma_j\}$ \qquad QED

Lemma 1 is a complete characterization of the sequences of social choice functions satisfying the delegation axiom. A first corollary is that the whole sequence is determined as soon as S^2 is known. Yet S^2 can not be taken arbitrarily. Since the function g is arbitrary except for the mild invariance properties i-ii, the class of s.c.f.s encompassed by formula (5) is still fairly large. In the next four sections we impose various additional assumptions that in turn narrow considerably the choice of the function g.

4) <u>Equal Sharing From an Individual Reference Level</u>

The <u>No Disposal of Utility</u> axiom

$$\{u_1 \le v_1 \text{ \underline{and}} \quad u_i = v_i \quad \text{for all } i \ge 2\} \Rightarrow S_1(u) \le S_1(v)$$

for all profiles u, v and cost function c.

The NDU axiom rules out profitable destruction of utility contingent upon the final public decision: by lowering my utility for some of the public decisions I cannot raise my eventual utility level. This axiom is a mild requirement, as we feel that voluntary losses of utility are failry easy to achieve; in particular they do not require any misrepresentation of preferences. (Of course, if true disposal of utility is profitable, then fake disposal--i.e. by reporting a utility lower than the true one--is even <u>more</u> profitable.)

The <u>Cost-Monotonicity</u> Axiom

$$\{c \le c'\} \Rightarrow \{S_i(u,c') \le S_i(u,c)\} \quad \text{all } i, \text{ all } u, \text{ and } c, c'.$$

This means that the agents unanimously want to improve the technology for producing the public decisions at stake. Symmetrically, no agent can gain by raising the cost of a decision even if that move improves his relative bargaining position by lowering the aspiration level of his fellow agents (e.g. he made more costly a decision that he hates but everybody else loves).

Again we argue that manipulating the arbitration by adding artificial overhead costs is neither difficulty nor uncommon. The cost monotonicity axiom rules out the profitability of such manipulations.

To describe the consequences of the two above axioms (together with the Delegation axiom) we denote by R the following set of real valued functions $\tilde{g}$ with domain $\mathbb{R}^A$.

$$\tilde{g} \in \mathbb{R} \quad \text{iff} \quad \begin{cases} \tilde{g} \text{ is monotonic: } x \leq y \implies \tilde{g}(x) \leq \tilde{g}(y) \quad \text{all } x,y \in \mathbb{R}^A \\ \tilde{g} \text{ is translation invariant: } \tilde{g}(x+\alpha \mathbb{I}) = g(x) + \alpha \quad \text{all } x \in \mathbb{R}^A \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{all } \alpha \in \mathbb{R} \end{cases}$$

Equivalently, $\tilde{g} \in R$ iff we have:

$$y^{\min} \leq \tilde{g}(x+y) - \tilde{g}(x) \leq y^{\max} \quad \text{all } x,y \in \mathbb{R}^A \qquad (8)$$

or simply

$$\tilde{g}(x+y) - \tilde{g}(x) \leq y^{\max} \quad \text{all } x,y \in \mathbb{R}^A \qquad (9)$$

Checking the equivalence of these 3 statements is straightforward, and therefore omitted. Typical elements of R are $\tilde{g}(x) = x^{\max}$, $\tilde{g}(x) = x^{\min}$, $\tilde{g}(x) = x \cdot \sigma = \sum_{a \in A} x_a \sigma_a$ where σ belongs to the unit simplex of $\mathbb{R}^A$ ($\sigma_a \geq 0$ for all a and $\sum_{a \in A} \sigma_a = 1$), as well as any convex combinations of those. Given a utility profile $u_1, \ldots, u_n$ and a function $\tilde{g} \in R$ we interpret $\tilde{g}(u_i)$ as agent i's reference utility level. Our first class of equal sharing s.c.f.s work by starting from the reference utility vector $(\tilde{g}(u_1), \ldots, \tilde{g}(u_n))$ and dividing equally the surplus (or deficit) left above (or below) that reference utility vector, namely

$$\Delta = (u_N - c)^{\max} - \sum_{i \in N} \tilde{g}(u_i)$$

Hence the s.c.f

$$S_i^n(u,c) = \tilde{g}(u_i) + \frac{1}{n}\Delta = \frac{1}{n}(u_N - c)^{\max} + \frac{1}{n}\{(n-1)\tilde{g}(u_i) - \sum_{j \neq i} \tilde{g}(u_j)\} \qquad (10)$$

Any such s.c.f. is a particular case of those described in Lemma 1, namely by making the function $g(x,z)$ independent upon z and monotonic in x. Therefore any sequence of s.c.f S^n associated with some reference level function $\tilde{g}$ satisfies the delegation axiom. Clearly it satisfies also No Disposal of Utility and Cost monotonicity. The converse statement is true as well.

Theorem 1

A social choice functions satisfies Delegation, No Disposal of Utility and Cost Monotonicity if and only if it is derived from a reference level function $\tilde{g}$ via formula (10). Then it is called <u>equal sharing from the reference level</u> $\tilde{g}$.

Proof

<u>Step 1</u>: For a fixed size n of the population, let S be a s.c.f. satisfying No Disposal of Utility (NDU) and Cost-monotonicity (CM).

Fix a profile u and a cost function c, next observe:

$$u_N - (u_N-c)^{max}\Pi \leq c$$

Hence, by CM and property (3):

for all $i \in N$, $S_i(u,c) \leq S_i(u, u_N-(u_N-c)^{max}.\Pi) = S_i(u,u_N) + \frac{1}{n}(u_N-c)^{max}$

On the other hand by efficiency (1)

$$S_N(u,c) = (u_N-c)^{max} = S_N(u,u_N-(u_N-c)^{max}.\Pi)$$

Therefore

$$S_i(u,c) = S_i(u,u_N) + \frac{1}{n}(u_N-c)^{max} \quad \text{for all } i, u, \text{ and } c. \qquad (11)$$

<u>Step 2</u>: Suppose now that the sequence S^n satisfies the Delegation Axiom, No Disposal of Utility and Cost Monotonicity. By Lemma 1, S^n can be written as (5) for some function $g(x,z)$ satisfying i-ii. Taking (11) into account this gives:

$$(n-1)g(u_i,u_N-c) - \sum_{j\neq i} g(u_j,u_N-c) = (n-1)g(u_i,0) - \sum_{j\neq 1} g(u_j,0) \quad \text{for all } u,c$$

which in turn means that $g(x,z) - g(x,0)$ does not depend on x. Hence we can simply replace $g(x,z)$ by $g(x,0) = \tilde{g}(x)$ in formula (5). This proves (10).

We already know (by property i Lemma 1) that $\tilde{g}(x+\alpha\mathbb{I}) = \tilde{g}(x) + \alpha$ all x,α.

It remains to check that $\tilde{g}$ is monotonic. Suppose it is not. Then we

can find u_1, v_1 both in $\mathbb{R}^A$ and an outcome $a \in A$ such that:

$$u_1(a) < v_1(a), \quad u_1(b) = v_1(b) \quad \text{all} \quad b \neq a \quad \underline{\text{and}} \quad \tilde{g}(u_1) > \tilde{g}(v_1).$$

Next choose a function u_2 such that both $u_1 + u_2$ and $v_1 + u_2$ reach

their maximum at some $b \neq a$. We have

$$S_1^2(u_1,u_2,0) = \frac{1}{2}[u_1(b) + u_2(b)] + \frac{1}{2}[\tilde{g}(u_1) - \tilde{g}(u_2) >$$

$$\frac{1}{2}[v_1(b) + u_2(b)] + \frac{1}{2}[\tilde{g}(v_1) - \tilde{g}(u_2)] = S_1^2(v_1,u_2,0)$$

thus contradicting NDU.   QED

5) Equal Sharing Above a Convex Status Quo

Among the social choice functions characterized in Theorem 1, we will now discriminate on the basis of the utility level guaranteed to each individual agent. Given an agent i endowed with a utility function u_i, and given a cost function c, we think of agent i as risk-avert and completely ignoring other agents' utility functions. This agent will therefore compare various social choice functions on the basis of his or her <u>guaranteed utility level</u> namely

$$h^n(u_i,c) = \inf S_i^n(u_i,u_{-i},c)$$ where the infimum is taken over all

$(n-1)$- uples $u_{-i} \in L(A)^{N \setminus \{i\}}$. Suppose a particular public decision $a^* \in A$ is viewed as a status quo (initial situation) from which the surplus (if any) is to be divided in some fair (not necessarily equal) way. Then we would restrict attention to those social choice functions where no player ends up below his or her status quo utility level. This leads to the following axiom:

<u>Individual Rationality Above a^*:</u>

$$h^n(u_i,c) \geq u_i(a^*) - \frac{c(a^*)}{n}$$ for all n, u_i and c.

Notice that the cost of the status quo decision is equally shared.

If we want to allow for a more symmetrical role of the public decisions we can deduce the guaranteed utility level from a fixed convex combination of decisions. This was originally suggested by Dubins [1977] for those problems where one want to treat all decisions in A on the same foot (neutrality): if A has p elements we mean the status quo utility level $\frac{1}{p} \sum_{a \in A} (u_i(a) - \frac{c(a)}{n})$ for agent i. In general let σ be a <u>convex decision</u>, namely an element of the unit simplex in $\mathbb{R}^A$. $\sum_{a \in A} \sigma_a = 1$ and $\sigma_a \geq 0$ for all a . We define

Individual Rationality Above σ

$$h^n(u_i,c) \geq (u_i - \frac{c}{n}) \cdot \sigma = \sum_{a \varepsilon A} (u_i(a) - \frac{c(a)}{n}) \sigma_a \quad \text{for all } n, u_i \text{ and } c$$

One can interpret σ as a probability distribution: the status quo (or disagreement) outcome amounts to draw a decision according to distribution σ and to share equally its cost. But this requires interpreting utility functions in the Von Neumann and Morgenstern sense, which is by no means necessary in our entirely deterministic model.

Theorem 2: There is exactly one sequence of social choice functions $s^1, \ldots, s^n, \ldots$ satisfying the Delegation axiom and Individual rationality above σ, namely:

$$_\sigma s_i^n(u,c) = \frac{1}{n}(u_N - c)^{\max} + \frac{1}{n} \{(n-1)u_i \cdot \sigma - \sum_{j \neq i} u_j \cdot \sigma\} \quad \begin{array}{l} \text{for all } n, \text{ all agent } i, \\ \text{all profile } u \text{ and cost} \\ \text{function } c \end{array} \quad (12)$$

or equivalently

$$_\sigma s_i^n(u,c) = u_i \cdot \sigma + \frac{1}{n} \{(u_N - c)^{\max} - u_N \cdot \sigma\} \quad \begin{array}{l} \text{for all } n, \text{ all agent } i, \text{ all} \\ \text{profile } u \text{ and cost function } c \end{array}$$

We call it <u>equal sharing above the convex decision</u> σ.

The Equal sharing above σ-social choice functions are a particular case of equal sharing s.c.f. where the reference level $\tilde{g}$ is linear, namely $\tilde{g}(u_i) = u_i \cdot \sigma$. Notice that No Disposal of Utility and Cost Monotonicity are met by these s.c.f.s; yet they are not necessary in the characterization result.

<u>Proof</u>: The s.c.f. defined by (12) satisfies the Delegation axiom by Theorem 1 (since $\tilde{g}(u_i) = u_i.\sigma$ belongs to R) and Individual Rationality above σ since it can be written as:

$$_\sigma S_i^n(u,\sigma) = (u_i - \frac{c}{n}).\sigma + \frac{1}{n}\{(u_N-c)^{max} - (u_N-c).\sigma\}$$

where the term between brackets is nonnegative, being the joint surplus left over the convex decision σ . Conversely let $S^1,\ldots,S^n,\ldots$ be a sequence of s.c.f. satisfying Delegation: by Lemma 1 they are given by (5) for some function $g(x,z)$ satisfying i-ii. We apply Individual Rationality above σ by setting $z = u_N - c$:

$$\{S_1^n(u,c) \geq (u_1 - \frac{c}{n}).\sigma \text{ for all } u,c\} <=> \{n \cdot S_1^n(u,z) \geq (nu_1 + z - u_N).\sigma$$

$$\text{for all } u,z$$

Using (5) and developing this gives:

$$(n-1)g(u_1,z) - \sum_{j=2}^{n} g(u_j,z) \geq [(n-1)u_1 - \sum_{j\geq 2} u_j].\sigma + z\cdot\sigma - z^{max}$$

Setting $h(x,z) = g(x,z) - x\cdot\sigma$ this is equivalent to $(n-1)h(u_1,z) - \sum_{j=2}^{n} h(u_j,z)$

$$\geq z\cdot\sigma - z^{max}$$

which holds for all n,z and u. Choose now $u_2=\ldots=\frac{u}{n}$, divide both sides by (n-1) and let n go to infinity; we get:

$$h(u_1,z) - h(u_2,z) \geq 0 \quad \text{all } u_1,u_2,z$$

i.e. $h(x,z)$ does not depend on x so that $g(x,z) = x.\sigma + h(0,z)$.

Adding to g a function of the variable z only leave invariant (5) so we can

take $g(x,z) = x \cdot \sigma$ afterall. $\qquad$ QED

Theorem 2 pinpoints the biblically simple social choice functions ${}_{\sigma}S^n$ by invoking the <u>exogenous</u> status quo σ . We characterize now these methods by emphasizing certain features of their guaranteed utility levels: existence of a convex status quo is then derived <u>endogenously</u> from these new axioms.

We work in the class of equal-sharing methods characterized by Theorem 1, namely equal sharing above a reference level $\tilde{g} \, \varepsilon \, R$. We observe that some of these s.c.f.s provide no guarantee <u>at all</u> to risk-avert individuals in the sense that $h^n(u_i, c) = -\infty$! This is true for instance if $\tilde{g}(u_i) = u_i^{max}$ and $n \geq 3$.

To chech this claim, observe that the difference $u_2^{max} + u_3^{max} - (u_{23})^{max}$ can be made arbitrarily large (if A has cardinality at least two). Then compute:

$$S_1^n(u_1, u_2, u_3, 0, \ldots, 0, c) = \frac{1}{n}(u_{123} - c)^{max} + \frac{1}{n}\{(n-1)u_1^{max} - u_2^{max} - u_3^{max}\}$$

$$\leq \frac{1}{n}\{u_{23}^{max} - u_2^{max} - u_3^{max}\} + \frac{1}{n}\{(u_1 - c)^{max} + (n-1)u_1^{max}\}$$

As a matter of exercise, the reader can check that for $n = 2$ this s.c.f. has $h^2(u_1, c) = \frac{1}{2}[u_1^{max} + (u_1 - c)^{min}]$.

The trouble with low guaranteed utility level is that at some profiles an individuals would prefer to drop out of the committee and pay his fair share of the cost of whatever decision the remaining committee selects: this will happen for sure if an agent's final utility level falls short of $(u_i - \frac{c}{n})^{min}$ where z^{min} is a short hand for $\min_a z(a)$. Hence our next axiom:

Minimal Individual Rationality

$$h^n(u_i,c) \geq (u_i - \frac{c}{n})^{min} \quad \text{for all n, } u_i, \text{ all c} \tag{13}$$

We view it as a necessary stability requirement in problems where the agents can potentially withdraw from the committee.

Lemma 2

Let $\tilde{g} \in R$ be a reference level function and $S^1,\ldots,S^n,\ldots$ be the associated sequence of social choice functions (by (5)). Then $\tilde{g}$ is concave <u>if and only if</u> S^n satisfies Minimal Individual Rationality for all n. In this case we have

$$h^n(u_i,c) = \frac{n-1}{n}(\tilde{g}(u_i) - \tilde{g}(\frac{c-u_i}{n-1})) \quad \text{for all } n, u_i \text{ and c} \tag{14}$$

Thus $\tilde{g}$ is concave if and only if the secure utility level to any agent is at least what he expects when paying only his or her fair share of the public decision (and not influencing the choice of that decision anymore). The proof of Lemma 2 and 3 is postponed until the Appendix.

One alternative requirement about the guaranteed utility level $h^n(u_i,c)$ is that they should not depend upon the size of the committee in the following sense

for all n $\quad h^n(u_i,nc) = h^2(u_i,2c)$, all u_i and all c $\tag{15}$

The point here is that the guaranteed utility level to agent i depends only upon the individual utility and the per capita cost function, but not on the size of the committee of which i is a member. It is meaningless to compare guaranteed utility levels in two committees with different size but the same global cost function since this induces a genuine economy of scale--as decisions are publicly consumed-- .

Lemma 3

Let $S^1, \ldots, S^n, \ldots$ be the equal sharing from the reference level $\tilde{g}$ s.c.f. Then its guaranteed utility level is size-independent (property (15)) _if and only if_ $g(x) = x \cdot \sigma$, all x for some convex decision σ.

Combining theorem 1 and Lemma 3: the family of equal sharing above a convex status quo s.c.f. ((12)) is _characterized_ by four axioms namely Delegation, No Disposal of Utility, Cost Monotonicity and Guaranteed utility level size-independent.

Remark 1

Among the equal sharing methods characterized in Lemma 2 ($\tilde{g}$ is in R and concave) those with a fixed convex status quo ($g(x) = x \cdot \sigma$ for some σ) endow coalitions with a guaranteed utility level that merely add up the guaranteed utility levels to each agent in the coalition.

Let t, $2 \leq t \leq n-1$, be the size of our coalition and define

$$h_t^n(u_1, \ldots, u_t, c) = \inf_{u_{t+1}, \ldots, u_n} \sum_{i=1}^{t} S_i^n(u_1, \ldots, u_n; c)$$

Clearly for all social choice functions and all u, c,

$\sum_{i=1}^{t} h^n(u_i; c) \leq h_t^n(u_1, \ldots, u_t; c)$: coalition formation cannot conflict with individual guarantees. Now suppose $S^1, \ldots, S^n, \ldots$ are derived from a concave reference level function $\tilde{g} \in R$ so that (Lemma 2) individual guaranteed utility levels are given by (14). A similar computation (omitted for the sake of brevity) yields

$$h_t^n(u_1, \ldots, u_t, c) = \frac{n-t}{n} \left[\sum_{i=1}^{t} \tilde{g}(u_i) - t \tilde{g}\left(\frac{c - u_T}{n-t}\right) \right]$$

Thus equality $\sum\limits_{i=1}^{t} h^n(u_i, c) = h_t^n(u_1, \ldots, u_t, c)$ holds iff $\tilde{g}$ is linear, $\tilde{g}(x) = x \cdot \sigma$ for some convex status quo σ, in which case

$$h_t^n(u_1, \ldots, u_t, c) = (u_T - \frac{tc}{n}) \cdot \sigma \quad .$$

6) Utilitarianism

The previous section was concerned with the minimal utility level that an agent is guaranteed of when entering the arbitration room. Here we look symmetrically at the maximal aspiration level that an agent can dream of achieving. Specifically, we consider:

The No Free Lunch Axiom:

for all n, all u and all c: $S_i^n(u,c) \leq (u_i - \frac{c}{n})^{max}$

The upper bound to your aspirations is not better than being able to choose dictatorially the public decision while equally sharing its cost. In particular an agent deriving no individual surplus from the control of the public decision ($u_i(a) - \frac{c(a)}{n}$ is independent of $a \in A$) receives no share of the collective surplus.

Typically none of the equal sharing methods in sections 4, 5 satisfies the No Free Lunch axiom: think of a costless problem with two agents and a status quo a*. Although agent 1 is unconcerned: $u_1(a) = u_1(a*)$ all a, he is entitled to one-half of the surplus $u_2^{max} - u_2(a*)$ generated by player 2:

$$a* S_1(u_1,u_2) = u_1(a*) + \frac{1}{2} \{u_2^{max} - u_2(a*)\}$$

When combined with the Delegation Axiom and Minimal Individual Rationality the above axiom characterizes a narrow set of arbitration methods.

Definition 2

Let τ be a mapping from $\mathbb{R}^A$ into $\mathbb{R}$ satisfying

i) for all z, $\tau(z)$ is in the unit simplex of $\mathbb{R}^A$:

$\tau(z)_a \geq 0$ all a and $\sum_{a \in A} \tau(z)_a = 1$ 　　　　　　(16)

ii) for all z, a, if $\tau(z)_a > 0$ then $z(a) = z^{max}$

iii) invariance: $\tau(z + \alpha \mathbb{1}) = \tau(z)$ for all $z \in \mathbb{R}^A$, $\alpha \in \mathbb{R}$

Thus $\tau(z)$ is a convex combination of those public decisions maximizing z.

To each such function τ we associate a sequence of social choice functions $\tau_S^1,\ldots,\tau_S^n,\ldots$ defined by:

$$\tau_{S_i^n}(u,c) = (u_i - \frac{c}{n})\cdot\tau(u_N - c) \qquad \text{all i, u and c} \qquad (17)$$

We call τ_S^n an __utilitarian__ social choice function.

These methods (originally introduced in Moulin [1983]) essentially prevent the social planner to play any redistributive role by means of the private transfers: indeed when only one efficient public decision is at hand (only one a maximizes joint surplus $u_N - c$) then formula (17) means that this decision is undertaken and its cost is equally shared among agents. Only when more than one efficient decision exists, does the utilitarian social planner have some room for arbitration: he can enforce any convex combination of the utility vectors associated with the various efficient decisions; the coefficients of this combination, however, depend only upon the joint surplus $u_N - c$ (from which he can infer nothing of the relative individual preferences).

Theorem 3

Given any τ satisfying (16), the associated sequence of utilitarian social choice functions $\tau_S^1,\ldots,\tau_S^n,\ldots$ satisfies the Delegation axiom, Minimal Individual Rationality and the No Free Lunch axiom. Conversely, any sequence of social choice functions satisfying these 3 axioms is obtained in this way.

Despite its statement as a positive characterization result, we want to interpret Theorem 3 as a negative result. For a utilitarian social choice

function satisfies neither the No Disposal of Utility nor the Cost-Monotonicity properties.

To check that NDU is violated, choose u, c with two efficient decisions a, b such that $u_1(a) < u_1(b)$, $c(a) = c(b)$. Taking a small $\varepsilon > 0$ set:

u_1', u_1'' by $u_1'(a) = u_1(a) - \varepsilon < u_1(a) + \varepsilon = u_1''(a)$ ⎫
⎬
$u_1'(d) = u_1(d) = u_1''(d)$ all $d \neq a$ ⎭

At (u_1', u_{-1}, c) decision b is uniquely efficient while at (u_1'', u_{-1}, c) decision a is unquely efficient. Thus agent 1 suffers a <u>loss</u> when his utility increases from u_1' to u_1''. Similarly define c', c" by

$c'(a) = c(a) - \varepsilon < c''(a) = c(a) + \varepsilon$ ⎫
⎬
$c'(d) = c(d) = c''(d)$ all $d \neq a$ ⎭

Then agent 1 enjoys a <u>gain</u> when the cost function <u>increases</u> from c' to c". Thus utilitarian social choice functions are highly vulnerable to voluntary cuts in one's own utility <u>and</u> to tactical increase of costs; this makes them very unappealing as arbitration methods. Hence the negative reading of Theorem 3: any social choice function satisfying Delegation and No Disposal of Utility (or Delegation and Cost-Monotonicity) implies some free lunches and/or violates Minimal Individual Rationality.

<u>Proof of Theorem 3</u>

Fix τ satisfying (16) and define $g(x,z) = x \cdot \tau(z)$. Then g satisfies i-ii from Lemma 1. Let us compute the associated s.c.f. by (5):

$$S_1^n(u,c) = \frac{1}{n}(u_N - c)^{max} + \frac{1}{n}\left[(n-1)u_1 - \sum_{i \geq 2} u_i\right] \cdot \tau(u_N - c) \quad \text{for all } n, u, c.$$

From (16) we have $z \cdot \tau(z) = z^{max}$ for all z, hence:

$$S_1^n(u,c) = \frac{1}{n}[(u_N-c) + (n-1)u_1 - \sum_{i\geq 2} u_i]\cdot\tau(u_N-c) = (u_1 - \frac{c}{n})\cdot\tau(u_N-c)$$

Thus equation (11) defines a sequence of s.c.f.s satisfying the Delegation Axiom. They satisfy Minimal Individual Rationality and No Free Lunch, too: (16) implies $x^{min} \leq x\cdot\tau(z) \leq x^{max}$ for all x,z hence $(u_1 - \frac{c}{n})^{min} \leq$ $(u_N - \frac{c}{n})\cdot\tau(u_N-c) \leq (u_1 - \frac{c}{n})^{max}$.

We prove now the converse statement. Let $g(x,z)$ satisfying i-ii Lemma 1 and $...,S^n,...$ be the associated sequence of s.c.f.s by (5). The No Free Lunch property is written as

$$(n-1)g(u_1,u_N-c) - \sum_{i\geq 2} g(u_i,u_N-c) \leq (nu_1-c)^{max} - (u_N-c)^{max} \quad \text{for all n,u,c,}$$

Setting $z= u_N - c$ this is equivalent to

$$(n-1)g(u_1,z) - \sum_{i\geq 2} g(u_i,z) \leq ((n-1)u_1-u_{N\backslash 1} + z)^{max} - z^{max} \qquad (18)$$
$$\text{for all n,u,z}$$

Similarly the Minimal Individual Rationality axiom is written as:

$$(n-1)g(u_1,z) - \sum_{i\geq 2} g(u_i,z) \geq ((n-1)u_1-u_{N\backslash 1} + z)^{min} - z^{max} \qquad (19)$$

Applying (18) whenever $u_1 = \frac{1}{n-1} u_{N\backslash 1}$ gives

$$g(\frac{1}{n-1} \sum_{i\geq 2} u_i,z) \leq \frac{1}{n-1} \sum_{i\geq 2} g(u_i,z) \quad \begin{array}{l}\text{all n, all z}\\\text{all } u_2,...,u_n.\end{array} \qquad (20)$$

Similarly from (19) we get

$$g(\frac{1}{n-1} \sum_{i\geq 2} u_i,z) \geq \frac{1}{n-1} \sum_{i\geq 2} g(u_i,z) + \frac{1}{n-1} (z^{min} - z^{max}) \quad \text{all n,z and } u_2...u_n.$$

Use now a replication argument (replacing (n-1) by k(n-1) and each u_i by k identical replicas) to derive from the latter property

$$g(\frac{1}{n-1} \sum_{i>2} u_i, z) \geq \frac{1}{n-1} \sum_{i>2} g(u_i, z) \quad \text{all n, all z, all } u_2, \ldots, u_n \quad (21)$$

Combining (20) and (21) we get equality in both: together with $g(0,z) = 0$ this guarantees that $g(x,z)$ is a linear function of x, provided we make sure that g is continuous in x. But this follows from (18) for n=2:

$$g(u_1, z) - g(u_2, z) \leq (u_1 - u_2 + z)^{max} - z^{max} \leq (u_1 - u_2)^{max} \quad (22)$$

So we have proved that $g(x,z) = x \cdot \tau(z)$ for some mapping τ from $\mathbb{R}^A$ into itself. From i) in Lemma 1 we deduce $\Pi \cdot \tau(z) = 1$ and from (22) we get $\tau(z) \geq 0$ (since g must be montonic in x). Also from ii) in Lemma 1 follows $\tau(z + \alpha\Pi) = \tau(z)$ for all real α. Thus the proof of Theorem 3 is complete if we establish (16)ii.

Applying (22) to $u_1 = 0$ and $u_2 = z$ gives $z^{max} \leq g(z,z) = z \cdot \tau(z)$. On the other hand $z \cdot \tau(z) \leq z^{max}$ since $\tau(z)$ is in the simplex of $\mathbb{R}^A$. Thus $z \cdot \tau(z) = z^{max}$ which implies (16)ii.   QED

Remark 2

The results of sections 5 and 6 bear close resemblance to those in Moulin [1983]. There the powerful axiom is the following property:

No Advantageous Reallocation (NAR)

for all coalition $T \subseteq N$
for all profiles u,v and $\{u_j = v_j$ for all $j \notin T$ and $u_T = v_T\} =>\{$No $S_i(u,c)<S_i(v,c$
cost function c   for all $i \in T$ $\}$

The idea here is to rule out coalitional insurance taking the form of monetary transfers contingent upon the final public decision. When we adapt the results in the mentioned paper to quasi-linear problems with cost functions it

turns out that:

i) The NAR axiom and Individual Rationality above σ characterize the
s.c.f. $_\sigma S$ (corollary to theorem 2 in [1983].

ii) The NAR axiom and Non Disposal of Utility and Cost monotonicity
characterize the family of equal sharing above a convex status quo
(this is an easy consequence of Theorem 1 in [1983])

iii) The NAR axiom and No Free Lunch characterize a family of utilitarian
s.c.f. slightly more general than those of Definition 2 (by adapting
theorem 3 in [1983]).

However, there is no obvious relation between the Delegation axiom and
the No Advantageous Reallocation axiom.

7) Equal Allocation of Nonseparable Costs

We propose here one more sequence of social choice functions satisfying the delegation axiom, yet quite different from the s.c.f.s encountered so far. This method emerged originally in the cost-sharing literature (Ransmeier [1942]) and was recently investigated in the framework of cooperative games with side-payments (Straffin and Heaney [1981] Legros [1983]); see also Moulin [1981] where it is the equilibrium outcome of an auction mechanism suited to quasi-linear problems.

Definition 3: Let A and N be given. To any utility profile u and cost function c , associate the following game in characteristic form:

$$\text{for all } T \subseteq N \quad v(T) = (u_T - c)^{\max} \tag{23}$$

Define the separable cost to agent i as $s_i = v(N) - v(N \backslash i)$. The Equal allocation of nonseparable cost social choice function (in short EANS) amounts to share equally the surplus (or deficit) left above (or below) the utility vector $(s_1, \ldots, s_n)$. It is denoted S*:

$$S_i^*(u,c) = s_i + \frac{1}{n}\left[v(N) - \sum_{j=1}^{n} s_j\right] = \frac{1}{n}\left[v(N) + \sum_{j=1}^{n} v(N \backslash j)\right] - v(N \backslash i) \tag{24}$$

To interpret this s.c.f. suppose first c = 0 (dropping the cost sharing issue). Then $s_i = u_N^{\max} - u_{N \backslash i}^{\max}$ is the utility level allocated to agent i by the familiar demand revealing pivotal mechanism (also called Vickrey-Clarke-Groves mechanism): every agent is taxed the amount of the external effect that he inflicts on the other agents. Then $v(N) - \sum_{j=1}^{n} s_j \geq 0$ is a nonnegative surplus that the EANS s.c.f. proposes to share equally.

Of course if cost is no longer zero, there is no restriction on the sign of $v(N) - \sum_{j=1}^{n} s_j$. This is because the value of a coalition T is taken to be its maximal joint surplus in the event that T chooses dictatorially the public decision while covering its _full_ cost (see formula (23)). This specific choice is necessary for the delegation axiom to be satisfied, as the proof of our next result makes clear.

Lemma 4

The sequence of Equal Allocation of Nonseparable cost s.c.f.s satisfies the delegation axiom as well as No Disposal of Utility and Minimal Individual Rationality. On the other hand it violates Cost-Monotonicity and No Free Lunch.

Proof

Set $g^*(x,z) = z^{max} - (z-x)^{max}$. Then g^* satisfies i-ii in Lemma 1. We claim that the associated sequence of s.c.f.s (by (5)) are precisely the EANS: the routine proof is omitted. To check No Disposal of Utility observe in (24) that $v(N)$ as well as $v(N\backslash j)$ are nondecreasing functions of u_i for all $j \neq i$, while $v(N\backslash i)$ is independent of u_i .

To check Minimal Individual Rationality, rewrite the inequality $S_i^*(u,c) \geq (u_i - \frac{c}{n})^{min}$ as

$$v(N) + \sum_{j\neq i} v(N\backslash j) \geq (n-1)v(N\backslash i) + (nu_i - c)^{min}$$

Pick an arbitrary $a \in A$ and compute

$$(n-1)(u_{N\backslash i}-c)(a) + (nu_i - c)(a) = (u_N - c)(a) + \sum_{j\neq i} (u_{N\backslash j}-c)(a) \leq v(N) + \sum_{j\neq i} v(N\backslash j)$$

Choosing a such that $(u_{N\backslash i}-c)(a) = v(N\backslash i)$ yields the desired inequality by:

$$(n-1)v(N\backslash i) + (nu_i-c)^{min} \leq (n-1)v(N\backslash i) + (nu_i-c)(a) = (n-1)(u_{N\backslash i}-c)(a)$$

$$+ (nu_i-c)(a)$$

We check now that Cost Monotonicity is violated even for n=2. Namely:

$$S_1^*(u_1,u_2,c) = \frac{1}{2}[v(12)+v(1) - v(2)] = \frac{1}{2}[(u_{12}-c)^{max} + (u_1-c)^{max} - (u_2-c)^{max}]$$

When the function c increases, it may well be that $(u_{12}-c)^{max}$ and $(u_1-c)^{max}$ stay put while $(u_2-c)^{max}$ strictly decreases: take a profile where the functions $(u_{12}-c)$ and (u_1-c) both reach uniquely their maximum at a while the maximum of u_2-c is reached at b ; then increases a little bit c(b) while c(a) stays put.

Suppose finally that No Free Lunch holds for n=2. This would imply for all u_1,u_2,c:

$$S_1^*(u_1,u_2,c) \leq (u_1 - \frac{c}{2})^{max} \; (=) \; (u_{12}-c)^{max} + (u_1-c)^{max} \leq (u_2-c)^{max} + (2u_1-c)^{max}$$

For any real valued functions w_i, i=1,...,4 defined on A and such that $w_1 + w_2 = w_3 + w_4$ one can find u_1,u_2,c such that

$$w_1 = u_{12}-c, \; w_2 = u_1-c, \; w_3 = u_2-c, \; w_4 = 2u_1-c$$

Thus we would have for all w_i, i=1,...,4.

$$w_1 + w_2 = w_3 + w_4 \Rightarrow w_1^{max} + w_2^{max} \leq w_3^{max} + w_4^{max} \qquad \text{QED}$$

In order to single out the sequence of EANS social choice functions among those satisfying the Delegation Axiom we shall use two more axioms. The No Transfer Paradox axiom:

for all n, profile u, cost function c and all increment function $\delta \in \mathbb{R}_+^A$:

$$\{v_1 = u_1 - \delta; \ v_2 = u_2 + \delta \text{ and } \ v_j = u_j \quad \text{all } j \geq 3\} \quad \Rightarrow \{S_1(u,c) \geq S_1(v,c)\}$$

This axiom (originally introduced in Moulin [1983]) says that giving away some utility to a fellow agent (this gift taking the form of a monetary transfer contingent upon the final public decision) can never be profitable to the donor. Equivalently we want that such a gift is never harmful to the receiver: the NTP axiom can be written as

$$\{v_1 = u_1 - \delta; \ v_2 = u_2 + \delta \text{ and } \ v_j = u_j \quad \text{all } j \geq 3\} \Rightarrow \{S_2(u,c) \leq S_2(v,c)\}$$

All social choice functions encountered so far, equal sharing above a reference utility level, utilitarian s.c.f.s as well as the Equal Allocation of Non-Separable Costs, share the No transfer Paradox property.

Lemma 5

Let $g(x,z)$ satisfying i-ii Lemma 1 and $S^1,..,S^n,...$ be the associated sequence of s.c.f.s by (5). Then the three following statements are equivalent:

   i) g is nondecreasing in its first variable:

   $$x \leq x' \Rightarrow g(x,z) \leq g(x',z) \quad \text{all } x,x',z \in \mathbb{R}^A$$

   ii) for all n, the s.c.f. S^n satisfies the No Transfer

   Paradox

   iii) For all n, the s.c.f. S^n is such that for all profile u,

   cost function c and increment function $\delta \in \mathbb{R}_+^A$

   $$\{c' = c + \delta; \ v_1 = u_1 + \delta ; \ v_j = u_j \quad \text{all } j \geq 2\} \Rightarrow \{S_1(u,c) \leq S_1(v,c')\} \quad (24)$$

Property iii is easily interpreted: if the cost of a decision increases by what amounts to a private income to agent i, this income cannot be taxed more

than 100% in the redistributive process.

Proof of Lemma 5

In view of formula (5), the No Transfer Paradox is written as

for all u,c and $\delta \geq 0$:

$$\frac{1}{n}\{(n-1)g(u_1,u_N-c) - g(u_2,u_N-c)\} \geq \frac{1}{n} \{(n-1)g(u_1-\delta,u_N-c) - g(u_2+\delta,u_N-c)\}$$

that is to say

$$(n-1)g(x,z) - g(y,z) \geq (n-1)g(x-\delta,z) - g(y+\delta,z)$$

This inequality holds if g is nondecreasing in x. Conversely, apply it

to $x = y + \delta$ to show that g is nondecreasing. The proof of i)<=>iii)

is a straightforward application of (5).   QED

Our last axiom deals with the tiny sub-class of problems where all agents

but one are <u>unconcerned</u> namely indifferent among all public decisions:

$u_2 = u_3 = \ldots = u_n = 0$. Then every s.c.f. bearing upon society $\{2,\ldots,n\}$

would select the cheapest public decision and share its cost equally (by

anonymity and efficiency: Definition 1) whence the final utility $-\frac{1}{n} c^{min}$

to every agent. The axiom requires that none of the unconcerned agents will

suffer if one single <u>concerned</u> agent joins the society:

for all u,c $\{u_2 = \ldots = u_n = 0\}$ => $\{S_i(u,c) \geq - \frac{c^{min}}{n}$ all $i \geq 2$ $\}$ (25)

Notice that if the above inequality is strict the concerned agent 1 is actually

subsidizing the others, which makes sense since agent 1 needs the consent of

his partners $\{2,\ldots,n\}$ to enforce a noncheapest decision. On the other

hand, if decisions are not discriminated by costs (cost function is constant)

then the rationale for agent 1 subsidizing the others disappears, which suggests

the property

for all u,c $\{u_2 = \ldots = u_n = c = 0\}$ => $\{S_i(u,c) = 0$ (26)

all $i \geq 2$ $\}$

Theorem 4

There is exactly one sequence of social choice functions atisfying i) the Delegation Axiom, ii) No Transfer Paradox, iii) properties (25) and (26). It is the sequence of Equal allocation of Nonseparable costs.

Proof

Given a function $g(x,z)$ satisfying i-ii) in Lemma 1 the associated sequence $\ldots, S^n, \ldots$ of s.c.f.s satisfies (25) if and only if

$$\frac{1}{n}(u_1-c)^{max} - \frac{1}{n}g(u_1,u_1-c) \geq -\frac{1}{n}c^{min} \qquad \text{for all} \quad u_1, c$$

which is rewritten as

$$g(x,z) \leq z^{max} - (z-x)^{max} \qquad \text{all } x,z \qquad (27)$$

Similarly property (26) amounts to

$$g(z,z) = z^{max} \qquad \text{all } z \qquad (28)$$

We check now that there is only one function g satisfying (27) and (28) and nondecreasing in x, namely $g^*(x,z) = z^{max} - (z-x)^{max}$. This will prove our theorem in view of Lemma 5. From (27) and (28) follows

$$g(x,z) - g(z,z) \leq -(z-x)^{max}$$

On the other hand for any fixed z the mapping $\phi(x) = g(x,z)$ is nondecreasing and translation invariant (i)-lemma 1). Therefore ϕ satisfies (8) (see the description of R in section 4). In particular

$$-(z-x)^{max} = (x-z)^{min} \leq g(x,z) - g(z,z)$$

so the above inequality is an equality. QED

Remark 3

Actually the EANS social choice function satisfies a property which is stronger than both (25) and (26). Namely:

for all u,c $\{u_2 = \ldots = u_n = 0\}$ $\Rightarrow \{S_i(u,c) = -\dfrac{c^{min}}{n}$ all $i \geq 2$ $\}$ (29)

Moreover, the Delegation Axiom and property (29) together characterize the sequence of EANS s.c.f.s (these two claims are routinely checked via formula (5) and Lemma 1). However we feel that condition (29) is less easily interpreted.

8) Value Functions and the Delegation Axiom

One striking feature of the EANS social choice functions is that they
are expressed as a value function of the game in characteristic form (23).
This implies in particular that Pareto inferior decisions (decision a is
Pareto inferior to b if $u_i(a) \leq u_i(b)$ all i $\underline{and}$ $c(b) \leq c(a)$) do not matter:
removing them from the decision set if of no consequence to anybody; symmetrically,
adding foolish decisions to the agenda is no possible trick to influence the
overall decision. We ask now if other value functions could be derived
from the delegation axiom and answer negatively in Lemma 6 below. Given N,
a cooperative game (with side-payments) is any mapping v associating to
each coalition $T \subseteq N$ a real number $v(T)$. A $\underline{value\ function}$ in the context
of our problem is any mapping ϕ associating to each game v a vector $\phi(v)$
in $\mathbb{R}^N$ and satisfying i) anonymity, ii) efficiency ($\sum_{i \in N} \phi_i(v) = v(N)$)
and iii) two invariance properties that parallel those in Definition 1 namely
for all games v,w:

for some $\lambda \in \mathbb{R}$ {$w(T) = v(T) + \lambda$ for all T} => {$\phi_i(w) = \phi_i(v) + \frac{\lambda}{n}$

for all i }

for some $\lambda \in \mathbb{R}$ } {$w(T) = v(T) + \lambda$ for all T containing i } =>{$\phi_j(w) = \phi_j(v)$
and some $i \in N$ $w(T) = v(T)$ for all T not containing i

for all $j \neq i$}

To each such value function ϕ and each set A we associate the social choice
function defined by:

$$S_i(u,c) = \phi_i(v) \text{ where } v(T) = (u_T - c)^{max} , \text{ all } T \subseteq N \qquad (30)$$

We let the reader check that properties i to iii in the Definition of ϕ
ensure that S is a social choice function (Definition 1).

Lemma 6

There is exactly one sequence of social choice function, derived (via formula (30)) from a sequence of value functions (one for each size n of N), and satisfying the Delegation Axiom. It is the sequence of Equal Allocation of Nonseparable costs s.c.f.s.

Proof

Although a proof can be worked out by using once more Lemma 1, we prefer a more specific argument. Let $\phi^1, \ldots, \phi^n, \ldots$ be a sequence of value functions. The definition of value functions (property i to iii) leaves no freedom of choice for ϕ^1 and ϕ^2 :

$$\phi^1(v) = v(1)$$

$$\phi_1^2(v) = \frac{1}{2}[v(12) + v(1) - v(2)]$$

Now apply the delegation axiom to the sequence of social choice functions derived from $\phi^1, \ldots, \phi^n, \ldots$ Specifically consider formula (4). If v is the game derived from $(u_1, \ldots, u_n, c)$ then the game derived from $(u_1, \ldots, u_{n-1}, c^1)$ is v':

$$v'(T) = (u_T - c)^{max} = (u_{T+n} - c)^{max} - S_n^n(u, c) = v(T+n) - S_n^n(u, c).$$

Thus, by the invariance property of ϕ^{n-1}, the delegation axiom amounts to:

for all n, all v: $\phi_1^n(v) = \phi_1^{n-1}(v^{(n)}) - \frac{1}{n-1}\phi_n^n(v)$

where we set $v^{(n)}(T) = v(T+n)$ all $T \subseteq \{1, \ldots, n-1\}$

This yields a system of equations

$$\phi_i^n(v) + \frac{1}{n-1}\phi_j^n(v) = \phi_i^{n-1}(v^{(j)}) \quad \text{all } v, \text{ all } i,j \quad 1 \leq i, j \leq n$$

This system determines the value function ϕ^n once ϕ^{n-1} is given. Thus by an induction argument already encountered at the end of the proof of Lemma 1, we conclude

$$\phi_i^n(v) = \frac{1}{n}[v(N) + \sum_{j=1}^{n} v(N \backslash j)] - v(N \backslash i) \qquad\qquad \text{QED}$$

The paper by Sobolev [1982] bears close resemblance to the above proof. However Sobolev has a different definition of the game v' facing agents $\{1,\ldots,n-1\}$ once player n delegates his decision power. Specifically, we define $v'(T) = v(T+n) - \phi_n^n(v)$ while he sets $v'(T) = \max \{v(T), v(T+n) - \phi_n^n(v)\}$: coalition T has a choice to incorporate player n or not. This difference has far reaching technical consequences: while our Lemma 6 is quite elementary his characterization of the quasi-nucleolus solution is anything but simple.

References

Harsanyi, J. C., 1959. A bargaining model for the Cooperative n-person Game in Tucker and Luce (eds.) Contribution to the theory of games IV, Annals of Math Studies 40, Princeton University Press.

Legros, P., 1982. The Nucleolus and the cost allocation problem, mimeo, Northwestern University, Evanston.

Lensberg, T., 1982. Stability and the Nash Solution, mimeo, Norwegian School of Economics and Business Administration, Oslo.

Lensberg, T., 1983. Stability and Collective Rationality, mimeo, Norwegian School of Economics and Business Administration, Oslo.

Lensberg, T. and W. Thomson, 1984. Lecture notes on the axiomatic theory of bargaining with a variable population, mimeo, University of Rochester.

Moulin, H., 1981. Implementing Just and Efficient Decision-Making, Journal of Public Economics, 16, 193-213.

Moulin, H., 1983. Egalitarianism and Utilitarianism in Quasi-linear Bargaining, forthcoming in Econometrica.

Moulin, H. 1984. The Conditional Auction for Sharing a Surplus, The Review of Economic Studies, LI, 157-170.

Ransmeier, J. S., 1942. The Tennessee Valley Authority, Vanderbilt University Press, Nashville, Tennessee.

Sobolev, A. I., 1975. The characterization of the Nucleolus by Functional Equations, mimeo, Leningrad.

Straffin, P. D. and J. P. Heaney, 1981. Game Theory and the Tennessee Valley Authority, International Journal of Game Theory, 10, 1, 35-43.

Thomson, W., 1979. Maximin Strategies and Elicitation of Preferences in Aggregation and Revelation of Preferences, J. J. Laffont, ed., Studies in Public Economics, North-Holland.

APPENDIX

Proof of Lemma 2

Suppose first $\tilde{g}$ is concave. We prove formula (14):

For all n, u, c, the concavity of $\tilde{g}$ implies:

$$\frac{1}{n-1} \sum_{i \geq 2} \tilde{g}(u_i) \leq \tilde{g}(\frac{u_{N\backslash 1}}{n-1}). \quad \text{Therefore}$$

$$S_1^n(u,c) \geq \frac{1}{n}(u_N - c)^{max} + \frac{n-1}{n}\tilde{g}(u_1) - \frac{n-1}{n}\tilde{g}(\frac{u_{N\backslash 1}}{n-1}) \qquad (31)$$

On the other hand from (9) we derive

$$(x-y)^{max} - (n-1)\tilde{g}(\frac{x}{n-1}) \geq -(n-1)\tilde{g}(\frac{y}{n-1}) \qquad \text{all } x,y$$

Applying this with $x = u_{N\backslash 1}$, $y = c - u_1$ in (31):

$$S_1^n(u,c) \geq \frac{n-1}{n}\{\tilde{g}(u_1) - \tilde{g}(\frac{c-u_1}{n-1})\} \qquad \text{all } n, u, c \qquad (32)$$

which is just one-half of (14). To prove the other half we fix n, u_1, c

and we construct u_i, $2 \leq i \leq n$ such that (14) actually is an equality.

Taking $u_i = \frac{c-u_1}{n-1}$, $2 \leq i \leq n$ does the job. Thus (14) is proved. This

implies at once (13) in view of (8) since

$$\tilde{g}(u_i) - \tilde{g}(\frac{c-u_i}{n-1}) \geq (u_i - (\frac{c-u_i}{n-1}))^{min} = \frac{n}{n-1}(u_i - \frac{c}{n})^{min}$$

It remains to prove that if $S^1, \ldots, S^n, \ldots$ derived from $\tilde{g} \in R$ all

satisfy (13) then g is concave. By assumption

$$S_i^{n+1}(u,c) \geq (u_i - \frac{c}{n+1})^{min} \qquad \text{all } n, i, u, c$$

Apply this to arbitrary $u_1, \ldots, u_n$, $i = n+1$, $u_i = \frac{1}{n} \sum_{j=1}^{n} u_j$ and

$c = \frac{n+1}{n} (\sum_{j=1}^{n} u_j)$, through formula (10). Since $u_N - c = 0$ and $u_i = \frac{c}{n+1}$,

we get

$$\frac{n}{n+1} \, \tilde{g}(\frac{1}{n} \sum_{j=1}^{n} u_j) - \frac{1}{n+1} \sum_{j=1}^{n} \tilde{g}(u_j) \geq 0$$

This gives us the concavity inequality for all convex combinations with

rational coefficients. As $\tilde{g}$ is continuous ((8)) we conclude that it is

concave. $\hspace{6cm}$ QED

Proof of Lemma 3

Firstly the equal sharing above the convex decision σ has guaranteed

utility level $h^n(u_i, c) = (u_i - \frac{c}{n}) \cdot \sigma$: at any profile u and cost function

c such that $u_N - c = 0$ we have indeed $S_j^n(u,c) \geq (u_j - \frac{c}{n}) \cdot \sigma$ for all j,

whereas $\sum_{j \in N} S_j^n(u,c) = 0 = \sum_{j \in N} (u_j - \frac{c}{n}) \cdot \sigma$ so that equality holds everywhere.

The converse statement is more difficult. We start with a function

$\tilde{g} \in R$ and denote $S^1, \ldots, S^n$, the associated s.c.f.s (by (10)). We assume

property (15) and must prove that $\tilde{g}$ takes the form $\tilde{g}(x) = x \cdot \sigma$ for some

σ in the unit simplex of $\mathbb{R}^A$. To save space we simply sketch the proof.

First by the monotonicity and translation invariance of $\tilde{g}$ one computes

$$h_1^2(u_1, c) = \inf_{u_2} \frac{1}{2} [(u_{12} - c)^{max} + \tilde{g}(u_1) - \tilde{g}(u_2)] = \frac{1}{2}[\tilde{g}(u_1) - \tilde{g}(c - u_1)] \qquad (33)$$

the infimum being reached at $u_2 = c - u_1$.

Next we observe that for all social choice functions and any n :

$$\sum_{i=1}^{n} h^n(u_i, nc) \leq (u_N - nc)^{max}$$

Using (15) this gives

$$\frac{1}{n} \sum_{i=1}^{n} h^2(u_i, 2c) \leq (\frac{1}{n} u_N - c)^{max} \qquad \text{all } n, u, c$$

Fix $x \in \mathbb{R}^A$: the above inequality implies

$$\sup\{ \sum_{i=1}^{K} \lambda_i h^2(u_i, c)/x = \sum_{i=1}^{K} \lambda_i u_i \} \leq (x-c)^{max} \qquad \text{all } x, c$$

The supremum being taken over K and all convex coefficients

$(\lambda_1, \ldots, \lambda_K)$ such that $\sum_{i=1}^{K} \lambda_i u_i = x$. Thus the left hand side term is

$\tilde{h}^2(x, 2c)$ where $\tilde{h}^2$ is the concave hull of h^2 with respect to x .

$$\tilde{h}^2(x, 2c) \leq (x-c)^{max}$$

Fix c: the left hand term is concave, the right hand term is convex and

they both vanish at $x = c$. Therefore we can separate them by a linear

function: for some $\sigma(c) \in \mathbb{R}^A$: $\tilde{h}^2(x, 2c) \leq (x-c) \cdot \sigma(c) \leq (x-c)^{max}$ \qquad (34)

It is easy to check that $\sigma(c)$ is in the unit simplex since h^2, as well as $\tilde{h}^2$,

are monotonic in x and translation invariant.

Combining (33) and (34) we have finally:

$$\frac{1}{2}[\tilde{g}(x) - \tilde{g}(2c-x)] \leq (x-c) \cdot \sigma(c) \qquad \text{all } x, c$$

Replacing x by $2c-x$ yields the reverse inequality so this is an equality.

The left hand term is a nonincreasing function of c, so the right hand

side must be:

for all $d \in \mathbb{R}_+^A$, all $x, c \in \mathbb{R}^A$: $(x-c-d).\sigma(c+d) \leq (x-c).\sigma(c)$

which is rewritten as

$(x-c)\cdot\{\sigma(c+d) - \sigma(c)\} \leq d.\sigma(c+d)$

Fix c and d. This takes the form

$x\cdot \alpha \leq \beta$ all x

where α is a vector in $\mathbb{R}^A$ and β a real number. This implies $\alpha = 0$ hence $\sigma(c)$ is independent of c after all:

$\frac{1}{2}\tilde{g}(x) - \tilde{g}(2c-x) = (x-c)\cdot\sigma$ all x,c

Taking $c = \frac{x}{2}$ and recalling $\tilde{g}(0) = 0$ we are home.