

HIERARCHICAL GAMES FOR MACROECONOMIC

POLICY ANALYSIS

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Abstract

This paper analyses a new solution of non-cooperative games, i.e. the Closed Loop Stackelberg solution, which is shown to be very useful for macroeconomic policy analysis. The main features of the Closed Loop Stackelberg (CLS) solution are examined and it is shown that only if a certain degree of uncertainty is introduced into the model, can the CLS strategy be made credible. Furthermore, the relationship between the CLS strategy and other solutions of the game are explored and conditions for the optimality and time-consistency of the CLS strategy are provided.

HIERARCHICAL GAMES FOR MACROECONOMIC POLICY ANALYSIS1. INTRODUCTION

In recent years, a new mathematical problem has been proposed in the engineering literature. The main features of this problem, called closed-loop Stackelberg (CLS) problem, can be described in the following way.

Suppose the control problem can be described as a game between two decision-makers and suppose that one of the two players, called the leader, has the power to announce his strategy first and to make his strategy conditional on the other player's strategy. Can the leader announce a strategy such that the follower is induced to behave as if he were acting in the leader's interest? How can this strategy be computed? What are the properties of the solution of the game when such a strategy is announced by the leader? What are the properties of the leader's optimal strategy? These questions can be formalized into a mathematical problem and many papers have recently tried to provide a solution to this (CLS) problem. (1)

The features of the CLS problem are not completely new in economics. In fact, Chow (1981, ch. 17) provides an algorithm which can be used to compute the steady-state CLS solution of the game, if the follower's strategy is included into the state vector of the dynamic equation describing the economic system. However, several aspects of the CLS problem, not explored by Chow (1981),

were recently emphasized in the engineering literature, where other solutions of the problem have also been proposed. In particular, the existence, uniqueness, and time-consistency of the CLS solution have been analysed.

Furthermore, no application of the CLS solution to games representing economic problems has been provided, even if the typical behaviour of the policy-maker can easily be fitted into the CLS framework. The macroeconomic policymaker can indeed be assumed to be the leader of the game and to announce the optimal strategy by taking into account not only the follower's rational reaction set (as in Kydland, 1975, 1977, where the feedback Stackelberg solution is considered), but also the actual follower's decision. Therefore, the CLS solution determines that particular leader's strategy which induces the follower to behave as if he were cooperating. This provides a great advantage to the policy-maker.

If indeed the monetary authority or the government can announce a strategy which implies the minimization of their loss function with respect not only to their own decision variables, but also with respect to the private sector's decision variables, then it is obvious that, in general, a lower loss can be achieved, i.e. it becomes easier for the policy-maker to attain the desired targets.

Does such a strategy exist? The goal of this paper is to answer this question and to study the properties of different CLS strategies. Two main problems will be examined: first, under what conditions is a CLS strategy really effective, that is, when does

it imply a loss for the policy-maker which is lower than the loss implied by any other policy? Second, under what conditions is the CLS strategy credible?

The effectiveness of a CLS strategy depends indeed on a set of threats (or incentives) that the leader announces he will carry out any time the follower does not comply with the strategy selected for him by the leader.(2)

A well-defined CLS strategy must therefore be based upon a set of credible threats (or incentives) so that the leader does not have any incentive to depart from the announced strategy, whenever the follower does not act in accordance with his will. An effective and credible CLS strategy can easily be shown to be the policy-maker's best policy.

Suppose, for example, that the monetary authority controls the money stock and the private sector controls the price level. If the monetary authority can induce the private sector to keep the price level constant, then the money stock can be used to stimulate economic activity. In contrast, the standard optimal policy faces the typical trade-off between output and inflation and both targets cannot be attained.

This paper will therefore consider the effectiveness and credibility issues for general CLS strategies and will provide conditions for the existence of credible, effective CLS strategies. Section 2 will deal with static games, section 3 with repeated games, and section 4 with dynamic games. An increasing degree of complexity will be introduced and the new problems arising when repeated and dynamic games are considered

will be analyzed.

The structure of the CLS problem and its solution for linear quadratic static and dynamic games can be found in the engineering literature. However, this paper provides a critical evaluation of a relevant bulk of literature and an original analysis of the credibility problem. It will be shown that only if a certain degree of uncertainty is introduced into the model, can the CLS strategy be made credible. Furthermore, the relationship between the CLS strategy and other solutions of the game will be explored, and conditions for the optimality and time-consistency of a CLS strategy will be provided.

A list of remaining open problems will conclude the paper.

2. STATIC GAMES

Let us start the analysis by showing how the closed-loop Stackelberg (CLS) solution can be applied to static games. In this way, the main features of the CLS solution can be presented without resorting to the complex mathematical proofs which are necessary when dynamic games are considered. A simple example will also be used to clarify the relationship between the CLS optimal policy and other solutions of the game between the policy-maker and economic agents.

A. Closed-Loop Stackelberg Strategy

Let us assume that there are only two decision-makers: the leader (player 1) and the follower (player 2). Let x_i and $V_i = E[W_i(s_1, s_2, \epsilon)]$, $i = 1, 2$, be, respectively, the decision variables and the expected loss of the two players, where $s_i \in S_i$, $i = 1, 2$, is the strategy of each player and ϵ is a vector of random variables with given distribution representing the uncertainties introduced into the problem. The normal form of the game is therefore defined by the strategy sets S_i and the loss functions EW_i , $i = 1, 2$. The decision variables are related to the solution of the game in its normal form by $x_i = s_i(\theta_i)$, $i = 1, 2$, where θ_i represents the information set available to each player. Furthermore, let us define player- i 's rational reaction set $R_i = R_i(s_j)$ $i, j = 1, 2$, $i \neq j$, as

$$(2.1) \quad R_i(s_j) = \{\hat{s}_i \in S_i: E[W_i(\hat{s}_i, s_j, \epsilon)] \leq E[W_i(s_i, s_j, \epsilon)]\}$$

for all $s_i \in S_i$

Therefore $R_i(s_j)$ defines player-i's optimal reaction to player-j's strategy and can be determined by solving the following problem:

$$(2.2) \quad \min_{s_i} E[W_i(s_i, s_j, \epsilon)]$$

where $s_i \in S_i$ and $s_j \in S_j$, $i, j = 1, 2$. (3)

The optimal policy which is obtained by solving a standard control problem can be interpreted as the solution of the following problem

$$(2.3) \quad \min_{s_1} E[W_1(s_1, R_2(s_1), \epsilon)]$$

where the policy-maker is the leader of the game and the reaction function of the follower describes the economic system (see Chow, 1981, for a similar interpretation).

Before defining the closed-loop Stackelberg solution of the game, we have to determine the team solution. If the leader of the game can control both decision variable sets x_1 and x_2 , he can achieve the absolute minimum of his cost function by solving:

$$(2.4) \quad \min_{s_1, s_2} E[W_1(s_1, s_2, \epsilon)]$$

Under standard assumptions, the solution of this control problem exists and is defined by (s_1^t, s_2^t) .

The CLS problem can be described in the following way: find a strategy s_1^{cls} such that

$$(2.5.1) \quad s_2^t = \arg \min_{s_2} E[W_2(s_1^{\text{cls}}, s_2, \xi)]$$

$$(2.5.2) \quad s_1^{\text{cls}}(s_2 = s_2^t) = s_1^t$$

where it is important to stress that s_2 is included into the leader's information set. In other words, the CLS problem is solved if the leader can determine a strategy s_1^{cls} such that the follower is induced to behave as if he were minimizing the leader's loss function. The solution of the follower's problem given $s_1 = s_1^{\text{cls}}$, must indeed be $s_2 = s_2^t$, the leader's desired strategy: Moreover, when the follower plays s_2^t , the leader's rational reaction is $s_1^t \in R_1(s_2^t)$ so that the team solution is the outcome of the game.

Therefore, the solution of the CLS problem is a strategy s_1^{cls} such that the follower is induced to adopt in his own interest a strategy s_2^t which is the most desirable from the leader's point of view.

For example. let

$$(2.6) \quad s_1^{\text{cls}} = s_1^t + h(\theta_1, s_2, s_2^t)$$

where the function (\cdot) is defined as

$$(2.7) \quad h(\theta_1, s_2, s_2^t) = \begin{cases} 0 & \text{if } s_2 = s_2^t \\ s_1^P & \text{if } s_2 \neq s_2^t \end{cases}$$

where s_1^P must be such that

$$(2.8) \quad \arg \min_{s_2} E[W_2(s_1^{\text{cls}}, s_2, \varepsilon)] = \arg \min_{s_2} E[W_1(s_1^t, s_2, \varepsilon)]$$

If (2.8) is satisfied, we have $s_2^{\text{cls}} = s_2^t$ and (2.6)-(2.7) imply $s_1^{\text{cls}} = s_1^t$. The function $h(\cdot)$ is often called threat function since it defines a set of threats (or incentives) that are used by the leader in order to induce the follower to choose the strategy s_2^t . Another particular example is given by the linear function.

$$s_1^{\text{cls}} = s_1^t + P(s_2 - s_2^t)$$

where the matrix P penalizes any deviation of s_2 from s_2^t . In this last case, the CLS problem is solved by choosing a matrix P such that $R_2(s_1^{\text{cls}}) = s_2^t$. In section 4 we will discuss the advantage of choosing a nonlinear threat function with respect to the credibility of the leader's strategy.

A comparison between the control problem (2.3) and the CLS problem (2.5) is straightforward. If both problems are solvable, the solution provided by the CLS strategy is preferred by the leader since he achieves the absolute minimum of his loss function. However, the superiority of the CLS solution is based on a larger information set. In fact, in the control problem, the leader

announces his decision first, given his knowledge of the follower's reaction set (i.e. the follower's loss function and the initial conditions). In contrast, in the CLS problem the leader again announces his strategy first, but he also knows the follower's actual strategy. Therefore, the follower is supposed either to act before the leader or to declare his decision to the leader before the game starts. In this last case, the sequence of actions during the course of the game becomes irrelevant. The particular information assumption upon which the CLS solution is based might seem restrictive in a static game setting, but becomes very plausible when dynamic games are considered.

B. Inducibility and Credibility

The closed-loop Stackelberg strategy previously described is not well defined. Two important problems affect the existence of a CLS strategy:

(i) the follower may prefer to be penalized by the leader rather than adopt his desired strategy if the follower's loss is lower under the punitive strategy, i.e. if

$$(2.9) \quad E[W_2(s_1^t, s_2^t, \epsilon)] > E[W_2(s_1^{cls}(\bar{s}_2), \bar{s}_2, \epsilon)]$$

for some $\bar{s}_2 \in S_2$ and $\bar{s}_2 \neq s_2^t$

(ii) the threats announced by the leader and defined by the punitive strategy $s_1^{cls}(s_2)$ for $s_2 \neq s_2^t$, may not be credible if the leader finds it advantageous to follow a strategy which

differs from the announced one when $s_2 \neq s_2^t$, i.e. if

$$(2.10) \quad E[W_1(s_1^{cls}(\bar{s}_2), \bar{s}_2, \epsilon)] > E[W_1(R_1(\bar{s}_2), \bar{s}_2, \epsilon)] \\ \text{for some } \bar{s}_2 \in S_2 \text{ and } \bar{s}_2 \neq s_2^t$$

where $R_1(\bar{s}_2)$ defines the leader's rational reaction to \bar{s}_2 .

In order to study the first problem we assume that the leader commits himself to carrying out the declared strategy and we define the most punitive strategy s_1^{mp} as the leader's strategy which maximizes the follower's loss function any time he does not comply with the policy selected for him by the leader. In other words, s_1^{mp} is the solution of:

$$(2.11) \quad \max_{s_1} E[W_2(s_1, s_2, \epsilon)]$$

The follower's rational reaction to s_1^{mp} belongs to $R_2(s_1^{mp})$ so that we can conclude that, whatever strategy the leader announces, the follower can always unilaterally guarantee himself

$$(2.12) \quad B^{mp} = E[W_2(s_1^{mp}, R_2(s_1^{mp}), \epsilon)] \\ = \min_{s_2} \max_{s_1} E[W_2(s_1, s_2, \epsilon)]$$

Suppose the leader tries to achieve (s_1^*, s_2^*) . The most powerful CLS strategy he can announce is:

$$(2.13) \quad s_1^{\text{cls}} = \begin{cases} s_1^* & \text{if } s_2 = s_2^* \\ s_1^{\text{mp}} & \text{if } s_2 \neq s_2^* \end{cases}$$

Then, the following proposition can be proved:

Proposition 1: If the desired strategies (s_1^*, s_2^*) are such that $E[W_2(s_1^*, s_2^*, \varepsilon)] > B^{\text{mp}}$, the leader cannot induce the follower to choose s_2^* by using the CLS strategy defined by (2.13).

Proof: From (2.12), the follower can guarantee himself a loss which is lower than the loss he attains when he adopts the leader's desired strategy.

Therefore, the following definition of inducible region is implied by the above proposition:

$$(2.14) \quad IR^{\text{mp}} = \{s_1 \in S_1, s_2 \in S_2: E[W_2(s_1, s_2, \cdot)] \leq B^{\text{mp}}\}$$

In other words, the inducible region defines all points in the strategy space that the follower prefers to the conflict with the leader, because they imply lower losses than the disagreement loss B^{mp} .

The definition of s_1^{mp} also implies:

Proposition 2: If (s_1^*, s_2^*) cannot be induced by s_1^{cls} as defined by (2.13), it cannot be induced by any other CLS strategy.

Proof: Obvious, since s_1^{mp} defines the most punitive (and effective) strategy.

This proposition also implies that IR^{mp} is the largest inducible region and that the minimum loss the leader can achieve without conflict with the other player is

$$(2.15) \quad \min_{(s_1, s_2) \in \text{IR}^{\text{mp}}} E[W_1(s_1, s_2, \varepsilon)]$$

The previous conclusions hold if we assume either that the leader commits himself to carrying out his declared threat strategy or that the leader's threats are credible (the credibility issue will be discussed later). Under this assumption, it is also possible to prove:

Proposition 3: If the inducible region contains the team solution (s_1^t, s_2^t) , the leader can achieve the global minimum of his loss function and his announced strategy is time-consistent.

Proof: If $(s_1^t, s_2^t) \in \text{IR}^{\text{mp}}$, the solution of (2.15) is the team solution, i.e.

$$(s_1^t, s_2^t) = \arg \min_{(s_1, s_2) \in \text{IR}^{\text{mp}}} E[W_1(s_1, s_2, \varepsilon)]$$

which, by definition, provides the absolute minimum of the leader's loss function. This implies that there exists no other strategy s_1^* such that the leader can attain a lower loss after

having observed the follower's decision. Therefore, the actual strategy coincides with the announced strategy and no time-inconsistency problem arises.

As stated above, these results depend largely on the assumption either that the leader commits himself to carrying out the announced strategy or that the leader's threats are credible.

Suppose they are not. Then, the follower knows that any time he chooses $s_2 \neq s_2^t$, the leader's optimal strategy will belong to the rational reaction set $R_1(s_2)$ so that the follower's optimal strategy when the leader's threats are not credible is defined by:

$$(2.16) \quad \bar{s}_2 = \arg \min_{s_2} E[W_2(R_1(s_2), s_2, \epsilon)]$$

Consequently, the leader's optimal reaction will be $\bar{s}_1 = R_1(\bar{s}_2)$, so that $E[W_1(\bar{s}_1, \bar{s}_2, \epsilon)]$ represents, by definition of rational reaction set, the leader's minimum loss when the follower adopts the strategy $s_2 = \bar{s}_2$. Therefore, the leader will carry out the announced strategy, i.e. will adopt $s_1^{cls}(\bar{s}_2)$, if and only if

$$(2.17) \quad E[W_1(s_1^{cls}(\bar{s}_2), \bar{s}_2, \epsilon)] = E[W_1(\bar{s}_1, \bar{s}_2, \epsilon)]$$

If we assume that the minimum problem (2.1) has a unique solution, and if the leader is not committed to carrying out his threats, so that the announced strategy must be credible, then (2.17) implies that the only CLS outcome of the game is (\bar{s}_1, \bar{s}_2) .

The following proposition summarizes the previous analysis:

Proposition 4: If the leader of the game is not committed to carrying out his threats, the only strategy he can induce the follower to adopt is $s_2 = \bar{s}_2$ where

$$(2.18) \quad \bar{s}_2 = \arg \min_{s_2} E[W_2(R_1(s_2), s_2, \varepsilon)]$$

Therefore, the leader can achieve the absolute minimum of his loss function if and only if $s_2^t = \bar{s}_2$.

Notice that the solution (\bar{s}_1, \bar{s}_2) is nothing more than the standard Stackelberg solution of the game when the follower becomes the leader and the leader becomes the follower.

Proposition 4 also implies that the interior of the inducible region determined by a set of credible threats is empty. Define indeed the follower's maximum loss when the CLS strategy is credible as \bar{B} , where

$$\bar{B} = E[W_2(R_1(s_2), s_2, \varepsilon)]$$

Then the inducible region is redefined as

$$\bar{IR} = \{s_1 \in S_1, s_2 \in S_2 : E[W_2(s_1, s_2, \varepsilon)] \leq \bar{B}\}$$

However, by definition \bar{s}_2 is the best follower's strategy when the leader adopts $R_1(s_2)$ so that \bar{B} is also his minimum loss and no rational follower can be induced to adopt a strategy $s_2 \neq \bar{s}_2$. Therefore, the interior of \bar{IR} is empty and $\bar{IR} = (\bar{s}_1, \bar{s}_2)$.

The conclusion that can be derived from the previous analysis is that a credible CLS solution for a static game either does not exist or coincides with the Stackelberg solution with a reversed role for the two players. However, this disappointing conclusion will be shown not to hold when repeated and dynamic games will be considered. In fact, under suitable assumption, credible CLS strategies which do not belong to the leader's rational reaction set will be determined, and it will be shown that the leader can achieve the absolute minimum of his loss function when the inducible region contains the team solution.

We want to emphasize that the credibility problem that we have discussed in this section is slightly different from the credibility problem which arises when the optimal policy is time inconsistent. Suppose indeed that the leader is committed to carrying out his threats and that the follower knows that. Then, the follower will choose $s_2 = s_2^*$, where s_2^* is the strategy desired by the leader. If $s_2^* = s_2^t$, the leader's rational reaction is s_1^t and the leader achieves the minimum of his loss function (Proposition 3). Suppose, however, that the team solution does not belong to the inducible region, i.e.

$$E[W_2(s_1^t, s_2^t, \varepsilon)] > B^{mp}$$

If there exists a strategy s_1^* such that

$$E[W_2(s_1^*, s_2^t, \varepsilon)] \leq B^{mp}$$

the leader's best CLS solution will be

$$s_1^{\text{cls}} = \begin{cases} s_1^* & \text{if } s_2 = s_2^t \\ s_1^{\text{mp}} & \text{if } s_2 \neq s_2^t \end{cases}$$

The leader is committed to carrying out his threats so that the follower will choose $s_2 = s_2^t$. However, the leader, after having observed $s_2 = s_2^t$, will choose $s_1 = s_1^t$ instead of the announced policy $s_1 = s_1^*$. (4)

This is the time-inconsistency problem as it is presented in the traditional control and rational expectations literature. (5)

As proved by Proposition 3, the time-inconsistency problem (and the following credibility problem) affects the CLS solution of the game only if the inducible region does not contain the team solution. In contrast, we will show that the standard optimal control policy is often time-inconsistent, even when the CLS policy is time-consistent.

C. An Example

Let the game be described by the following matrix:

		Player 2		
		(0,5)	(2,2)	(3,4)
Player 1		(1,0)	(2,5)	(4,0)
		(5,3)	(1,1)	(5,4)

When the leader follows the most punitive strategy, the lowest loss the follower can secure for himself is $B^{\text{MP}} = 4$, so that the inducible region is:

$$IR^{\text{MP}} = \{(2,1) (3,1) (1,2) (2,3) (3,2) (1,3) (3,3)\}$$

In contrast, if the leader's threats are credible, the follower can lose only $\bar{B} = 1$, so that \bar{IR} might be $\bar{IR} = \{(2,1) (3,2) (2,3)\}$. However, the choice $u_1 = 2$ is not the leader's rational choice when either $u_2 = 1$ or $u_2 = 3$. Therefore, \bar{IR} contains only $(3,2)$. In contrast, IR^{MP} includes several possibilities. In particular, the leader can achieve $W_1 = 1$ by inducing $u_2 = 1$ through the following strategy:

$$(2.15) \quad s_1^{\text{CLS}} = \begin{cases} 2 & \text{if } u_2 = 1 \\ 2 & \text{if } u_2 = 2 \\ 1 & \text{if } u_2 = 3 \end{cases}$$

However, this policy is not time-consistent since when the follower has chosen $u_2 = 1$, the leader has an incentive to pick $u_1 = 1$ in order to achieve $W_1 = 0$, the minimum loss.

This is not the case if the team solution belongs to the inducible region. Suppose that the element $(1,1)$ of the matrix is replaced by $(1,5)$ and the element $(2,1)$ by $(0,0)$. Then the strategy (2.15) becomes time-consistent. Therefore, when $(s_1^t, s_2^t) \in IR^{\text{MP}}$, the CLS strategy is time consistent but, being defined by the inducible

region IR^{MP} , it is based on a set of non-credible threats.

We can conclude that two conditions must be satisfied for a closed-loop Stackelberg strategy to be credible:

- (i) the inducible region must contain the team solution;
- (ii) the leader is committed to his declared strategy. This could be the result of a binding contract, an institutional arrangement or the minimization of a long-term loss.

Of course, this last possibility (which might be introduced through a reputation mechanism) cannot be explored by using static games. Therefore, the next sections will discuss the CLS solution for repeated and dynamic games.

3. REPEATED GAMES

The simplest way of introducing a multi-stage control problem is to assume that the game between the policy-maker and economic agents is repeated a finite (or infinite) number of times. Each stage depends on the previous ones only as far as the players are not memoryless, i.e. the information set at time t , where $1 \leq t \leq N$ and $N \leq \infty$ defines the length of the game, contains the decisions of the players in the previous stages.

A multi-stage framework gives us the possibility of providing new insight into the credibility problem. In the previous section we have shown that the ability to raise threats can greatly reduce the leader's loss, provided that the follower is convinced that the leader is really committed to his threat if the circumstances arise in which he claims he would use it. However, we have seen that the action following from the execution of threats is generally not optimal with respect to the leader's loss function at the time of their realization. If the game is repeated N times, however, the leader may find it advantageous to carry out his threats in the first stages of the game in order to induce the follower to adopt the desired strategy in the following periods. In other words, the punitive strategy, though irrational in a single play of a game, may well be rational in repeated play. The reason is that a carried-out threat enhances the leader's credibility in doing the apparently irrational thing in a single play so that, over the long run, the leader may develop a sufficiently fearsome reputation to deter future undesired

actions by the follower, Thereby, while losing in the short run, the leader can gain over time. This argument, however, can be shown to be correct only under special assumptions.

Let $V_i = E[W_i(s_1, s_2, \xi)]$, $i = 1, 2$, be the players' loss and let (s_1^*, s_2^*) be the leader's desired strategies. Suppose $(s_1^*, s_2^*) \in IR^{mp}$. The inducibility of (s_1^*, s_2^*) implies

$$(3.1) \quad V_2^* \leq V_2^p$$

where V_i^* , $i = 1, 2$, is player- i 's loss when the solution of the game is (s_1^*, s_2^*) and V_i^p , $i = 1, 2$, are the losses when the leader's threats are actually carried out. Furthermore, (\bar{s}_1, \bar{s}_2) is the solution of the game when the follower assumes that the leader's reaction will belong to his rational reaction set at any stage of the game, i.e. $s_1 \in R_1(s_2)$ for any $s_2 \in S_2$. The relative losses are (\bar{V}_1, \bar{V}_2) .

The normal form of the game can be described in the following way. The follower can choose between the leader's desired strategy s_1^* and his optimal strategy \bar{s}_2 when he does not think the leader is committed to carrying out the announced threats. The follower will adopt s_1^* if he believes the leader's threats. He will choose \bar{s}_2 , otherwise.

In contrast, the leader's reaction is s_1^* whenever $s_2 = s_2^*$, but the leader can choose between his punitive strategy s_1^{mp} and his single-play rational strategy \bar{s}_1 , when $s_2 = \bar{s}_2$. The following matrix describes the outcomes of the game

(3.2)

		Follower	
		s_2^*	\bar{s}_2
Leader	\hat{s}_1	v_1^*, v_2^*	\bar{v}_1, \bar{v}_2
	s_{1mp}	/	v_1^p, v_2^p

$$\text{where } \hat{s}_1 \in R_1(s_2) \text{ and } \hat{s}_1 = \begin{cases} s_1^* & \text{if } s_2 = s_2^* \\ \bar{s}_1 & \text{if } s_2 = \bar{s}_2 \end{cases}$$

Equation (3.1) and the definition of \bar{s}_2 imply:

$$(3.3) \quad \bar{v}_2 \leq v_2^* \leq v_2^p$$

Furthermore, in the previous section we have shown that the leader's threats are not credible if

$$(3.4.1) \quad \bar{v}_1 < v_1^p$$

Therefore, (3.4.1) and the definition of desired strategies (team solution, if possible) imply:

$$(3.4.2) \quad v_1^* \leq \bar{v}_1 < v_1^p$$

Inequalities (3.3)-(3.4) imply that the dominant strategy for the leader is \hat{s}_1 so that the solution of the game is (\bar{s}_1, \bar{s}_2) . Indeed, when the leader plays \hat{s}_1 , the follower's dominant strategy becomes \bar{s}_2 . This solution, obtained by recursively eliminating any dominated strategy, is called the d-solution by Moulin (1981).

Therefore, we have proved again the main result of the previous section. However, when the game is repeated N times, it can be argued that it pays to the leader to loose $V_1^P - \bar{V}_1$ in some early stage of the game in order to get $\bar{V}_1 - V_1^*$ in the following periods. This argument is based upon the assumption that the leader establishes his reputation by punishing the follower so that in the following periods the follower will never choose a strategy which differs from s_2^* . However, if the game is deterministic, this argument is not correct. The structure of the game is indeed equivalent to the Selten's Chain Store Paradox. It was proved by Selten (1978) that the only perfect equilibrium of a game described by the normal form (3.2) is (\bar{s}_1, \bar{s}_2) at each stage of the game. The proof starts from the last period by showing that at $t = N$ the leader has no incentive to punish the follower when $s_{2N} \neq s_N^*$ since no remaining period exists where the leader can get $\bar{V}_1 - V_1^*$. Therefore, at $t = N$ we surely have $s_{2N} = \bar{s}_{2N}$. But then, at $t = N - 1$, the leader has no effect on the last stage ($s_{2N} = \bar{s}_{2N}$). Therefore, at $t = N - 1$, we surely have $s_{2N-1} = \bar{s}_{2N-1}$. This argument can be repeated at each stage, thus proving that $\{(\bar{s}_{1t}, \bar{s}_{2t}); t = 1 \dots N\}$ is the solution of the repeated game. This is the unique perfect Nash (and Stackelberg; see Tirole, 1983) equilibrium of the game.

However, recent papers by Rosenthal (1981), Kreps-Wilson (1982a), Milgrom-Roberts (1982), have shown that other solutions of the game can be determined when some uncertainty is introduced into the model.

Let us rewrite the normal form of the game in the following way:

(3.5)

		Follower	
		s_2^*	\bar{s}_2
Leader	\hat{s}_1	$\frac{\bar{v}_1 - v_1^*}{v_2^p - \bar{v}_2}, 0$	$0, \frac{v_2^* - \bar{v}_2}{v_2^p - \bar{v}_2}$
	s_1^{mp}	/	$\frac{\bar{v}_1 - v_1^p}{v_2^p - \bar{v}_2}, \frac{v_2^* - v_2^p}{v_2^p - \bar{v}_2}$

where inequalities (3.3) and (3.4) have been used to transform the normal form (3.2) defined by the players' losses into the normal form (3.5) defined by the players' payoffs. The positive quantity $v_2^p - \bar{v}_2 > 0$ has been used to normalize the payoffs of each player.

Furthermore, let us assume:

A.1 The follower is uncertain whether the punitive action will be carried out at stage t of the game. Since the leader will punish the follower only when $v_1^p \leq \bar{v}_1$, an equivalent assumption is that the follower is not certain about the payoffs of the leader. (6)

A.2 The CLS strategy does not satisfy condition (2.17). In other words, the strategy that the leader wants the follower to adopt does not coincide with s_2 .

These assumptions and the normal form (3.5) imply that the results derived in Kreps-Wilson (1982a) can be applied to determine under what conditions the CLS solution is actually a possible solution

of the game. The following inequalities are indeed assumed to hold:

I1. $\bar{V}_1 - V_1^* \geq 0$ by definition of desired (team) solution;

I2. $V_2^P - V_2^* > 0$ by definition of inducible region;

I3. $1 > \frac{V_2^* - \bar{V}_2}{V_2^P - \bar{V}_2} \equiv b > 0$ by assumption A.2 and the definition of inducible region;

I4. $V < V_1^P$ by assumption A.1.

By defining $a = (\bar{V}_1 - V_1^*) / (V_1^P - \bar{V}_1)$, the game can also be described in the following way:

		Follower	
		s_2^*	\bar{s}_2
Leader	\hat{s}_1	a , 0	0 , b
	s_1^{mp}	/	- 1 , b - 1

which coincides with the normal form of the game analysed by Kreps-Wilson (1982a) where $a > 0$ if A.2 hold and $b < 1$ if the inducible region contains (s_1^*, s_2^*) .

The follower's uncertainty about the leader's payoff implies that, at each stage t , the follower assesses a probability p_t that the leader's loss function is such that $V_1^P \leq \bar{V}_1$, so that the leader

will carry out his threats. At stage $t + 1$, p_t will be revised on the base of the leader's decision at time t . The standard Bayes' rule is assumed to be used to compute p_{t+1} .

Furthermore, let δ be the initial probability that the leader finds it profitable to punish the follower when $s_2 \neq s_2^*$ at $t = 1$ (i.e. $p_1 = \delta$) and assume that both players remember the moves of the game, as the game progresses. Therefore, we are dealing with a game with imperfect information and perfect recall (see Kreps-Wilson, 1982a).

An equilibrium concept which is analogous to Selten's perfect equilibrium but which takes into account the uncertainty introduced into the model is the sequential equilibrium described by Kreps-Wilson (1982b).

Therefore, we want to determine the sequential equilibrium of the game (3.5). The function p_t is defined by the following four conditions:

- (i) If $s_{2t} = s_{2t}^*$, at stage t , then $p_{t+1} = p_t$.
- (ii) If $s_{2t} \neq s_{2t}^*$, $p_t > 0$ and the leader's reaction is $s_{1t} = s_{1t}^{mp}$.
Then $p_{t+1} = \max(b^{N-t}, p_t)$.
- (iii) If $s_{2t} \neq s_{2t}^*$ and either $s_{1t} = \bar{s}_{1t}$ or $p_t = 0$,
then $p_{t+1} = 0$.
- (iv) $p_1 = \delta$

Given this recursive definition of p_t , the following proposition can be proved:

Proposition 5: Suppose the game is characterized by assumptions A.1 and A.2 and the length of the game is finite. Then, the CLS strategy

$$s_1^{\text{cls}} = \begin{cases} s_1^* & \text{if } s_2 = s_2^* \\ s_1^{\text{mp}} & \text{otherwise} \end{cases}$$

where $(s_1^*, s_2^*) \in \mathbb{R}^{\text{mp}}$ can induce the follower to adopt $s_2 = s_2^*$ at any stage of the game if and only if $\delta > b$. Furthermore, if $\delta > b$ the sequence $\{s_{1t}^* = s_1^*, s_{2t}^* = s_2^*; t = 1, \dots, N\}$ is a sequential equilibrium of the game and the CLS strategy is credible.

Proof: From Kreps-Wilson (1982a), Proposition 1, we have that $p_t > b^{N-t}$ implies $s_{2t} = s_{2t}^*$. Furthermore, I.2 implies $b < 1$ and $\delta > b$ is necessary and sufficient for $p_t > b^{N-t}$ at any $t = 1, 2, \dots, N$. Therefore, $s_{2t} = s_{2t}^*$ for $t = 1, 2, \dots, N$. The leaders's consequent rational reaction is $s_{1t} = s_{1t}^*$ at any $t = 1, 2, \dots, N$ so that the sequence $\{s_{1t}^*, s_{2t}^*; t = 1, \dots, N\}$ constitutes a sequential equilibrium which, by definition, cannot be based on non-credible threats (see Kreps-Wilson, 1982b).

An immediate implication of Proposition 5 is the following:

Proposition 6: Suppose A.1 and A.2 hold. Then, if $\delta > b$ and the inducible region contains the team solution of the static game:

(i) the leader can achieve the absolute minimum of his multi-stage loss function;

(ii) the leader's CLS strategy is time-consistent.

Proof: Define $s_1^* = s_1^t$, $s_2^* = s_2^t$ and apply Proposition 5.

Then at any stage of the game the leader achieves the minimum of his loss function. The time consistency of the CLS strategy follows from Proposition 3.

For any $\delta \leq b$ other sequential equilibria can be determined by using the leader's strategy and the follower's strategy described in Kreps-Wilson (1982a). (7)

However, these other sequential equilibria may be characterized by $s_{2t} \neq s_{2t}^*$ at some t , so that it is impossible for the leader to achieve the desired solution at any stage of the game. This implies that the CLS strategy loses its most appealing property, i.e. the absolute minimization of the leader's loss function.

Consequently, either the CLS strategy becomes time-inconsistent or it may not be the leader's optimal strategy. In contrast, when Proposition 5 holds, the CLS strategy defines the leader's optimal policy with respect to any other possible strategy, since it provides the absolute minimum of the leader's loss function.

Therefore the concept of CLS solution of the control problem is shown to be the best way of computing the leader's optimal policy if the uncertainty introduced into the game is large enough

to imply an initial probability of the leader's commitment to his threats greater than b . In other words, if the leader's reputation is good enough ($\delta > b$), then his announced policy will be credible, Pareto optimal and time-consistent.

Let us examine more carefully the condition $\delta > b$. It can be written as

$$(3.6) \quad \delta > \frac{V_2^* - \bar{V}_2}{V_2^P - \bar{V}_2}$$

so that it will be more easily satisfied when $V_2^* - \bar{V}_2$ is small and $V_2^P - \bar{V}_2$ is large, i.e. when the follower's relative loss from accepting the leader's desired strategy is small and the follower's relative loss when the leader's threats are actually carried out is large.

Finally, we want to emphasize that (3.6) is necessary for the leader to achieve with certainty his absolute minimum loss only if $N < \infty$. It is indeed possible to prove (see Kreps-Wilson, 1982a; Milgrom-Roberts, 1982) that if $N = \infty$, for any $\delta > 0$ the sequential equilibrium of the game (3.5) is determined by $\{(s_{1t}^*, s_{2t}^*); t = 1, \dots, N\}$.

Therefore, when the game is played an infinite number of times, the uncertainty which must be introduced into the game for the leader's CLS strategy to be credible can be very small. In contrast, if $N < \infty$, condition (3.6) must hold. These conclusions will be generalized to dynamic games in the next section.

4. DYNAMIC GAMES

The previous theoretical framework can easily be generalized to dynamic games. Therefore, the i -th player is supposed to minimize:

$$(4.1) \quad E[W_i(1, y_1, x_1, x_2)] = E\left[\sum_{t=1}^{T-1} g_t^i(y_t, x_{1t}, x_{2t}) + g_T^i(y_T)\right]$$

$$i = 1, 2$$

where the first argument of the function W_i indicates the first of the planning periods, subject to the dynamic system:

$$(4.2) \quad Y_{t+1} = f(y_t, x_{1t}, x_{2t}) + \varepsilon_t \quad t = 1, \dots, T$$

where $x_i^1 = [x_{i1}, \dots, x_{iT-1}]$, $i = 1, 2$, and ε_t is a vector of serially uncorrelated random variables.

Again we assume that the leader declares his strategy first, but he acts only after having known the follower's action (or the effects of this action). This assumption is particularly plausible when dynamic games are considered. The CLS strategy may indeed imply a punishment from time $t + 1$ on, any time the follower does not adopt the leader's desired strategy at time t .

The leader will therefore try to achieve the absolute minimum of his loss function by using the optimal strategy

$$(4.3) \quad s_{1t}^{cls} = \begin{cases} x_{1t}^* & \text{if } x_{2t-1} = x_{2t-1}^* \\ x_{1t}^{mp} & \text{if } x_{2t-1} \neq x_{2t-1}^* \end{cases}$$

where $\{x_{1t}^*, x_{2t}^*; t = 1, \dots, T-1\}$ is the team solution of the game and $\{x_{1t}^{mp}; t = 1, \dots, T-1\}$ is the punitive strategy determined by solving

$$(4.4) \quad \min_{x_{2t}} \max_{x_{1t}} E[W_2(t, y_t, x_1, x_2)] \quad t = 1, \dots, T$$

$$s.t. \quad y_{t+1} = f(y_t, x_{1t}, x_{2t}) + \varepsilon_t$$

The solution of this problem is a function $P(t, y_t)$ so that the inducible region can be defined as

$$(4.5) \quad IR^{mp} = \{(x_1, x_2) : E[W_2(t, y_t, x_1, x_2)] \leq B_t^{mp} \\ \text{for } t = 0, 1, \dots, T-2\}$$

where B_t^{mp} is defined as

$$(4.6) \quad B_t^{mp} = \min_{x_{2t}} E[g_t^2(y_t, x_{1t}, x_{2t}) + P(t+1, y_{t+1})]$$

In other words, the follower will verify at any time t if his loss can be reduced by choosing $x_{2t} = x_{2t}^*$. If this is the case, the leader will use his punitive strategy from time $t + 1$ on. Therefore, the sequence $\{B_t^{mp}; t = 0, 1, \dots, T-2\}$ defines the inducible region for the dynamic game (see Tolwinski, 1983). It must be emphasized that in a deterministic setting the follower's decision at the last stage of the game cannot be

influenced by any threat, so that at the last stage of the game no policy can be induced.

A common assumption is to exclude any follower's action at the last stage of the game (Basar-Selbuz, 1979)(8) or to impose some restrictions on the leader's loss function (see Tolwinski, 1981). However, these assumptions affect the effectiveness and not the credibility of the CLS strategy. Indeed, they can be used to show that the leader's CLS strategy is effective even in the last stage of the game, so that the leader can achieve the absolute minimum of his loss function. However, Selten's argument can again be used to show that no threat will be carried out in the last period, so that in all the other stages of the game the follower will choose a strategy which differs from the leader's desired strategy. Furthermore, the credibility of a CLS strategy for dynamic games is related to the type of strategy (linear, nonlinear, continuous, etc.) which is adopted by the leader. Let us consider, for example, the solution of the CLS problem provided by Basar-Selbuz (1979) and Tolwinski (1981). The Basar-Selbuz CLS strategy is defined by:

$$(4.7) \quad x_{1t}^{\text{cls}} = x_{1t}^* + P_t(Y_t - \bar{Y}_t) \quad t = 1, \dots, T-1$$

where $\bar{Y}_t = f(Y_{t-1}, x_{1t-1}^*, x_{2t-1}^*)$ i.e. \bar{Y}_t is the state at time t if both decision-makers used the desired strategies at $t - 1$. The solution of the CLS problem is therefore a sequence $\{P_1, P_2, \dots, P_{T-1}\}$ such that $(x_1^*, x_2^*) \in \text{IR}^{\text{mp}}$. Basar-Selbuz (1979) provide the solution for general linear quadratic control

problems. However, (4.7) implies that if $x_{2t} \neq x_{2t}^*$ at any t , then in general we have $y_j \neq \bar{y}_j$ for $j > t$. Therefore the follower will be punished forever once a deviation, however unintentional, is observed, even if he returns to x_{2i}^* for $i > t$. This type of CLS strategy is not likely to be credible unless x_{1t}^{cls} belongs to the leader's rational reaction set for any x_{2t} and any t . (9)

Indeed, if this condition is not satisfied, when the payoff from establishing a reputation is high (the first stages of the game), the cost of carrying out the announced threats is also very high (the punitive strategy lasts for all future periods). In contrast, when this cost is low, the advantage of establishing a reputation is also very low (few periods remain for the leader to get his desired solution).

Let us consider now Tolwinski's solution. His CLS strategy is defined by

$$(4.8) \quad x_{1t}^{cls} = x_{1t}^* + h_t(y_t - \bar{y}_t) \quad t = 1, \dots, T-1$$

where h_t is a nonlinear function with $h_t(0) = 0$ and \bar{y}_t is defined as $\bar{y}_t = f(y_{t-1}, x_{1t-1}, x_{2t-1}^*)$. In this case, as long as $x_{2t-1} = x_{2t-1}^*$, $y_t = \bar{y}_t$ regardless of whether or not $x_{1t-1} = x_{1t-1}^*$. Thus, if the follower acted improperly for whatever reason at $t - 2$ but resumes the desired decision at $t - 1$, then the leader will only punish at $t - 1$ for one stage of the game.

Therefore, Tolwinski's strategy is more likely to be credible since at any $t < T$ the leader can compensate his punishment loss with the payoffs he can obtain, in all future periods, from

establishing the credibility of his threats. (10)

Furthermore, by using Tolwinski's strategy, the analysis of the previous section can be repeated simply by adding a time index to the losses V_i^* , \bar{V}_i , V_i^P . Therefore, if the conditions given by Kreps-Wilson (1982a) are satisfied for any t , where b^{N-t} is

substituted by $\prod_{i=t+1}^T b_i$, and b_i is defined as $b_i = (V_{2i}^* -$

$\bar{V}_{2i}) / (V_{2i}^P - \bar{V}_{2i})$, then the CLS strategy is credible and attains the absolute minimum of the leader's loss function. However, the deterministic structure of the game cannot be maintained. The conclusions derived from Proposition 5 can be applied to dynamic games only if some uncertainty about the leader's payoff is introduced into the model. How this uncertainty affects the solution of stochastic dynamic games is a matter to be investigated. The general solutions of the CLS problem provided by Basar-Selbuz (1979) and Tolwinski (1981) can be applied only to linear quadratic deterministic dynamic games and few attempts to solve stochastic dynamic games have appeared in the literature (see Ho-Luh-Muralidharan, 1981; Chang-Ho, 1981 and Chow, 1981). Summing up, we can conclude that three major ingredients are necessary to determine a credible and effective CLS strategy:

(i) The team solution (x_1^*, x_2^*) must belong to the inducible region, otherwise the CLS strategy is time-inconsistent.

(ii) Either the leader has a reputation such that the probability of his commitment to his announced threats is high or, if it is low, the time-horizon is infinite.

(iii) The punishment for any follower's deviation from the

desired strategy must last a finite number of periods, and the loss for the leader must be finite.

6. CONCLUSION

This paper has tried to achieve several goals: first, a new interesting solution of the control problem has been presented and its main features have been discussed. This solution, called Closed-Loop Stackelberg solution, is based on an optimal announcement strategy so that a credibility problem arises.

Therefore, this paper has also shown under which conditions the optimal announcement is credible. Static, repeated and dynamic games have been considered.

However, several extensions of the results contained in this paper should be provided. For instance, a general CLS solution for stochastic games has not been provided (see Chang-Ho, 1981, for a first attempt) and the new problems arising when multi-level games are considered have not been examined (see Luh-Chang-Ning, 1984). Furthermore, more effective CLS strategies can be determined when two or more followers are introduced into the game, so that the leader can exploit their interaction in order to achieve his team solution (see Chang-Ho, 1983). Finally, several problems related to the information structure of the two players have not been considered. If, for example, the follower's strategy is not observable by the leader, who must therefore induce the follower to reveal his actual decision, then the CLS strategy becomes more complex and a two-sided credibility problem must be solved (see Ho-Luh-Olsder, 1982).

FOOTNOTES

(1) See, for example, Basaz-Selbuz (1979), Tolwinski (1981, 1983), Chang-Ho (1983), Chang-Luh (1984), Basar (1979), Ho-Luh-Muralidharan (1981), and Luh-Chang-Ning (1984). It must be stressed that the CLS problem for dynamic games does not have a simple solution. See Simaan-Cruz (1973).

(2) This is not a new idea in the economic literature. See, for example, the issue of the Review of Economic Studies (1979) devoted to the "incentive comptability" problem and the book by Green-Laffont (1979).

(3) For the sake of simplicity, we assume that the minimum problem (2.2) has a unique solution.

(4) This cheating solution, which can be considered a particular, time-inconsistent version of a CLS strategy, has been studied by Hamalainen (1981).

(5) Luh-Chang-Chang (1984) define a policy as time-inconsistent when it does not satisfy the principle of optimality along the equilibrium desired path (s_1^t, s_2^t) and define a policy as not credible when it does not satisfy the principle of optimality off the optimal path. These definitions are consistent with our analysis.

(6) A similar assumption is used by Kreps-Wilson (1982a) in order to provide a solution of the Chain-Store Paradox.

(7) The multiplicity of sequential equilibria that can be determined may be considered a limit of this solution concept, if no other criterium is provided that enables us to choose between different equilibria.

(8) Basar-Selbuz (1979) also provide the CLS solution without assuming the follower does not act at the last stage of the game, but, in this case, the team solution is not attained.

(9) This condition is equivalent to the conditions required by Luh-Chang-Chang (1984) for a CLS strategy to be credible. See also footnote 4.

(10) However, Tolwinski's solution is highly nonlinear as will be shown later on.

(11) n is the dimension of the state vector y_t .

(12) However, an explicit discussion of the existence of a non-empty inducible region is not provided by Basar-Selbuz (1979) and Tolwinski (1981).

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