THE ECONOMETRICS OF RATIONING MODELS*

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1. Introduction

Models of rationing behavior have a long tradition in the economics literature. In the last decade there has been a revival of interest in various aspects of the subject, both from theoretical and econometric points of view. Witness, for example, the now extensive literature on fix-price or temporary equilibrium models and on estimation and hypothesis testing in single and multimarket disequilibrium models.

In much of this work, rationing emerges as the result of some inherent rigidity, the source of which is not always made clear. There is, however, one strand of this literature in which rationing behavior emerges as an optimal strategy on the part of individual economic agents or policy makers. For example, Jaffee and Modigliani (1969) and Stiglitz and Weiss (1981) have studied the nature of optimal bank behavior which leads to credit rationing of loan customers. Similarly, Goldfeld, Jaffee and Quandt (1980) examined the circumstances under which it would be optimal for a policy maker such as a central bank to ration its "customers." The empirical work reported in that paper builds on earlier work in disequilibrium econometrics but, as the authors acknowledge, requires some strong, perhaps implausible, assumptions. The present paper seeks to put the econometrics of these types of rationing models on firmer grounds. As we shall see, this opens a wide range of interesting econometric issues which may well have more general applicability.

The outline of the paper is as follows. Section 2 provides some

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background on the econometrics of disequilibrium models. Section 3 summarizes the policy rationing setup that motivates the formal rationing models presented in section 4. The next section considers the econometric implementation of rationing models while some computational experience is described in section 6. The paper concludes with a brief summary and some suggestions for further research.

2. Some Background

As just suggested, many of the issues to be examined in this paper are closely related to the econometrics of disequilibrium models. A now standard version of this model has the following form:

$$D_t = \alpha_1 P_t + \beta_1' x_{1t} + u_{1t} \tag{2-1}$$

$$S_t = \alpha_2 P_t + \beta_2' x_{2t} + u_{2t} \tag{2-2}$$

$$P_t = P_{t-1} + \gamma(D_t - S_t) + u_{3t} \tag{2-3}$$

$$Q_t = \min(D_t, S_t) \tag{2-4}$$

where x_{1t}, x_{2t} are vectors pf exogenous variables, u_{it} are error terms, D_t (demand) and S_t (supply) are unobserved by the econometrician and P_t (price) and Q_t (quantity) are observed. Equations (2-1) and (2-2) are normally derived from choice theoretic considerations but the underpinnings of (2-3) are typically a bit more suspect. Equ. (2-4) is customarily justified on the basis that exchange is voluntary, although the applicability of this to aggregate data might be questioned. While there are many possible extensions of this basic model (see the survey of Quandt (1982)), for purposes of displaying the econometric issues this simple version will do.

Model (2-1) - (2-4) can be readily estimated by maximum likelihood techniques. The joint density of the u_{it} (typically assumed to be multinormal) gives rise to a joint pdf $g(D_t, S_t, P_t)$ from which the joint pdf of the observable variables is given by

$$h(Q_t,P_t) = \int_{Q_t}^{\infty} g(Q_t,S_t,P_t)dS_t + \int_{Q_t}^{\infty} g(D_t,Q_t,P_t)dD_t \qquad (2\text{-}5)$$

Given the form of g, (2-5) may be explicitly computed and the likelihood function is the product of terms such as (2-5).

A model with somewhat different underpinnings was analyzed by Goldfeld and Quandt (1975) who considered an agricultural commodity which may not be fully harvested under certain circumstances. Let q_t denote the crop, h_t the (ex ante) harvest amount, p_t the price and let x's refer to exogenous variables and u's to error terms. The basic equations of the model then are

$$q_t = \beta_1' x_{1t} + u_{1t} \qquad (2\text{-}6)$$

$$h_t = \beta_2' x_{2t} + \alpha_1 p_t + \alpha_2 q_t + u_{2t} \qquad (2\text{-}7)$$

$$p_t = \beta_3' x_{3t} + \alpha_3 y_t + u_{3t} \qquad (2\text{-}8)$$

$$y_t = \min(q_t,\ h_t) \qquad (2\text{-}9)$$

Equ. (2-6) states that the crop depends on exogenous variables only. Equ. (2-7) determines harvest intentions: these depend on exogenous variables (e.g., current wages), on the price and, perhaps, on the crop size itself. Equ. (2-8) is a conventional demand function containing y_t, the amount brought to market, which is determined by (2-9) to be the lesser of the amount of the crop and the amount intended to be harvested. In contrast to the disequilibrium model given above, price does clear the market. Nevertheless, the presence of the "min-condition" in both models gives rise to quite similar likelihood functions.

While there is a "min-condition" in the two models we have considered, it arises for rather different reasons. In the disequilibrium model, it arises because price rigidities lead to the rationing of one side of the market. In the agricultural example, the rationing element is imposed by "nature" given, of course, various exogenous variables which determine the initial planting and the desired harvest. We now consider a third type of model in which rationing in a market results from an optimal strategy on the part of a policy maker.

3. Policy Rationing: An Example

The idea behind this can most simply be illustrated by an example taken from Goldfeld, Jaffee, Quandt (1980). Consider a government financial authority which can lend to the private "banks" it supervises. The financial authority is assumed to raise its funds in the capital markets, paying a rate C_t, and in turn it charges a rate, R_t, for its loans. These loans, the demand for which is denoted by A_t^d, are then used to finance some sort of investment, say housing, H_t.[1]

The financial authority is assumed to have some target for housing, H_t^* and an overall loss function which has the form

$$L = (R_t - g(C_t))^2 + \delta_1(R_t - R_{t-1})^2 + \delta_2(H_t - H_t^*)^2 \qquad (3\text{-}1) .$$

The first term in (3-1) gives the utility loss when the loan rate deviates from a specified function, g( ), of the cost of funds and the second term accounts for bureaucratic inertia and possible other adjustment costs. The final term in (3-1) accounts for a utility loss when realized housing investment deviates from desired housing investment.

In the absence of rationing, (i.e., actual loans, A , are equal to

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[1]The institutional details are meant to capture the spirit of the U.S. Federal Home Loan Bank Board which extends loans, called advances, to private financial institutions known as savings and loan associations. For more background, see Goldfeld, Jaffee, Quandt (1980) and the references cited therein.

A_t^d), the authority chooses R_t so as to minimize (3-1) subject to[2]

$$A_t^d = \beta_1' Z_{1t} + \alpha_1 R_t + u_{1t} \tag{3-2}$$

$$H_t = \beta_2' Z_{2t} + \alpha_2 A_t + u_{2t} \tag{3-3}$$

If rationing is permitted, however, then the authority may simultaneously choose R_t and A_t, subject, of course, to the constraint

$$A_t \leq A_t^d \tag{3-4}$$

which assures that the banks cannot be forced to borrow more than they desire. It seems reasonable to presume that the authority might experience disutility from rationing so that we might, more generally, suppose they minimize the modified loss function

$$L' = L + \delta_3 (A_t^d - A_t)^2 \tag{3-5}$$.

The choice of R_t and A_t to minimize (3-5) or (3-1) subject to (3-2) - (3-4) requires some assumption about the way in which uncertainty is treated. In Goldfeld, Jaffee, Quandt (1980) it was implicitly assumed that the error terms in (3-2) and (3-3) were known by the authority but not by the outside econometrician. This readily yields an optimal policy for R_t and A_t, where the form of A_t depends on whether the model is in a rationing mode ($A_t < A_t^d$) or not ($A_t = A_t^d$). The same stochastic assumption also makes the estimation problem quite tractable. Nevertheless, despite its convenience, this treatment of the stochastic terms is rather unsatisfactory. At the very least, if one assumes policy makers observe structural disturbances one has to make the timing aspects of the problem more precise. More generally, it would seem more appropriate to regard the loss itself as a stochastic variable and have policy makers minimize expected loss. Both of these approaches are pursued in the next section.

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[2]For the moment, we ignore the complications presented by the stochastic elements in (3-2) and (3-3).

4. Rationing as an Optimal Policy

We shall now develop some alternative models of rationing by a policy maker. To keep things simple we assume a policy maker with a single instrument, x_t, the setting of which influences the "demand" for a variable, y_t^d, via a behavioral equation.[3] The policy maker is also able to ration demand, should it choose to do so. The loss function is given by

$$L = (x_t - x_t^*)^2 + v_1(y_t - y_t^*)^2 + v_2(y_t - y_t^d)^2 \tag{4-1}$$

where we posit

$$x_t^* = \alpha' z_{1t} + \varepsilon_{1t} \tag{4-2}$$

$$y_t^* = \beta' z_{2t} + \varepsilon_{2t} \tag{4-3}$$

$$y_t^d = \gamma_1 x_t + \gamma_2' z_{3t} + \varepsilon_{3t} \tag{4-4}$$

The "desired" values of x_t and y_t, x_t^* and y_t^*, are known exactly to the policy maker. Equs. (4-2) and (4-3) provide a model for those desired values where the stochastic terms reflect the inability of the outside econometrician to observe x_t^* and y_t^*. In contrast, the stochastic term ε_{3t} in (4-4) is unknown to the policy maker at the time it chooses x_t. This suggests choosing an optimal x_t by minimizing the expected loss, E(L). We shall, in fact, do this below. For reasons which will be apparent when we consider issues of estimation, however, we first consider another approach.

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[3]In analogy with the preceding section, x_t can be thought of as the loan rate while y_t^d corresponds to A_t^d. We make the simplifying assumption that the policy maker "cares" about y_t directly, thus omitting the connection between advances and housing of the previous example. No important generality is lost by this since one can think of the desired level of y as coming from equ. (3-3) with $H_t = H_t^*$. This setup does simplify the computations since there is only a single source of uncertainty facing the policy maker.

A simplified approach. More particularly, we posit that the policy maker first chooses x_t and an "anticipated" value of y_t under the assumption that $\varepsilon_{3t} = 0$. That is, x_t and y_t are chosen to minimize (4-1) subject to (4-2) - (4-4) and the condition

$$y_t \leq y_t^d \tag{4-5}$$

Since, at this stage, ε_{3t} is assumed zero, (4-5) assumes that the anticipated level of y_t does not exceed the expected level of borrowing, i.e., borrowers cannot be coerced, in an ex ante sense, to borrow more than they want. We further assume that after setting x_t, the policy maker learns the actual state of demand (i.e., in effect seeing the realization of ε_{3t}) and sets y_t in an optimal fashion, made precise below.[4]

The solution to the policy maker's first-stage problem is determined by forming the Lagrangean

$$\bar{L} = (x-x^*)^2 + v_1(y-y^*)^2 + v_2(y-\gamma_1 x-\gamma_2' z_3)^2 + \lambda(y-\gamma_1 x-\gamma_2' z_3) \tag{4-6}$$

where we have suppressed the time subscript. If we postulate that x and y are always positive, the following Kuhn-Tucker conditions are necessary and sufficient for the solution:

$$\frac{\partial \bar{L}}{\partial x} = 2(x-x^*) - 2v_2\gamma_1(y-\gamma_1 x-\gamma_2' z_3) - \lambda\gamma_1 = 0$$

$$\frac{\partial \bar{L}}{\partial y} = 2v_1(y-y^*) + 2v_2(y-\gamma_1 x-\gamma_2' z_3) + \lambda = 0$$

$$y \leq y^d \ ; \ (y^d-y)\lambda = 0$$

Straightforward algebra yields the following solutions:

$$x_I = x^* + \frac{v_1\gamma_1(y^*-\gamma_2' z_3-\gamma_1 x^*)}{1 + v_1\gamma_1^2} \quad \text{if} \quad y^* \geq \gamma_1 x^* + \gamma_2' z_3 \tag{4-7}$$

[4]It is possible to develop a model in which the policy maker is forced to commit itself to x_t and the "supply" of y_t before it observes ε_{3t}.

$$x_{II} = x^* + \frac{\gamma_1 v_1 v_2 (y^* - \gamma_2' z_3 - \gamma_1 x^*)}{v_1 + v_2 + \gamma_1^2 v_1 v_2} \quad \text{if} \quad y^* < \gamma_1 x^* + \gamma_2' z_3 \qquad (4\text{-}8)$$

Equ. (4-7) corresponds to the case in which there is no anticipated rationing ($\lambda \neq 0$) while (4-8) is associated with anticipated rationing ($\lambda = 0$).[5]

The role of v_2 in the analysis is worthy of note. First, from (4-7) and (4-8) we see that whether there is anticipated rationing or not is independent of v_2, although the quantitative amount of anticipated rationing most certainly does depend on v_2.[6] In particular, rationing may take place even if $v_2 = 0$. When $v_2 = 0$, x_{II} becomes particularly simple, namely $x_{II} = x^*$, and the corresponding y is given by $y_{II} = y^*$.

This case also permits a simple graphical interpretation of the solution which is given in Figure 1 where we have drawn the iso-loss ellipses along with two possible expected demand functions. In case I (see (4-7)) no rationing is planned whereas in case II (see (4-8)) the optimal strategy is to set $x_{II} = x^*$ and plan to ration borrowers to y^*.[7]

Of course, as suggested earlier, we do not force the policy maker to stick with its anticipated y. Rather, after announcing x, the policy maker learns y^d and chooses an optimal y based on this information. With x set, this amounts to minimizing $v_1(y-y^*)^2 + v_2(y-y^d)^2$ subject to $y \leq y^d$.

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[5]From (4-7) and (4-8), we see that when $y^* = \gamma_1 x^* + \gamma_2' z_3$, then $x_I = x_{II} = x^*$. To preserve space, we have not presented the two optimal solutions for y but, as the text suggests, when $\lambda \neq 0$ the anticipated y_I is $\gamma_1 x_I + \gamma_2' x_3$.

[6]It can be shown that anticipated rationing when $\lambda = 0$ is given by $v_1(\gamma_1 x^* + \gamma_2' z_3 - y^*)/(v_1 + v_2 + \gamma_1^2 v_2)$. By (4-8) this is positive, and decreasing in v_2.

[7]The same diagrammatic apparatus works when $v_2 \neq 0$ but the ellipses are centered differently.

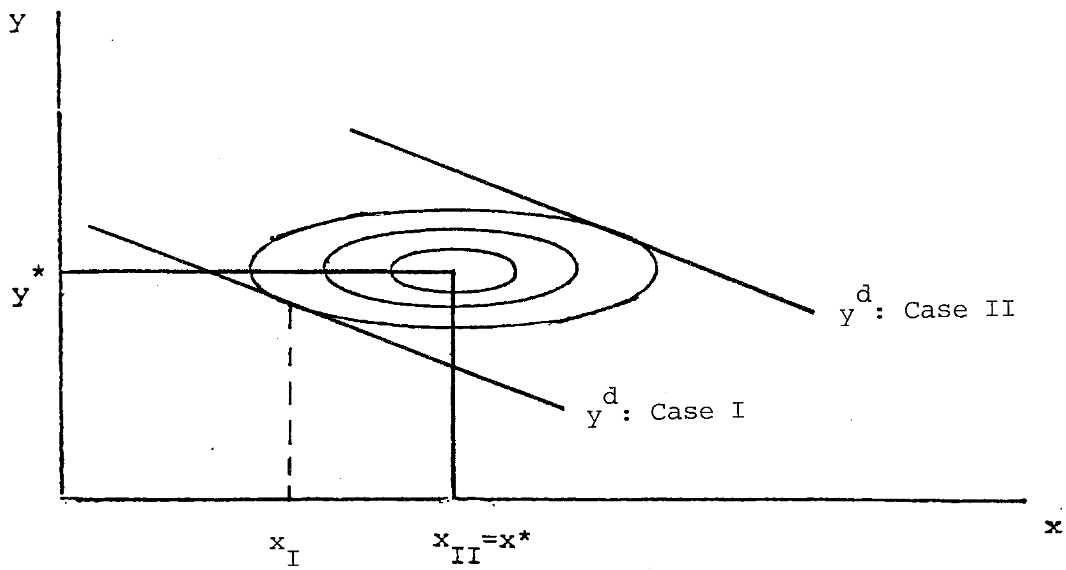

FIGURE 1.

Moreover, a bit of algebra reveals that this solution can be expressed compactly as[8]

$$y = \min \left\{ y^d, \frac{v_1 y^* + v_2 y^d}{v_1 + v_2} \right\} \tag{4-9}$$

Equ. (4-7), (4-8) and (4-9) completely characterize the optimal solution. We defer questions of estimation until the next section. Instead, we now develop the expected loss approach.

<u>The Expected Loss Approach</u>. In the case just considered, the policy process was characterized by a first stage in which x and an anticipated y were chosen by ignoring the uncertainty associated with y^d. To account for this uncertainty requires minimizing E(L) where, as before

$$L = (x-x^*)^2 + v_1(y-y^*)^2 + v_2(y-y^d)^2 \tag{4-1}$$

where y^d is given by equ. (4-4). If, as is common in such contexts, we assume that $\varepsilon_3 \sim N(0,\sigma_3^2)$, then $y^d \sim N(\gamma_1 x+\gamma_2' z_3,\sigma_3^2)$. For use in what follows we denote this pdf as $f(y^d)$, with the corresponding cdf given by $F(y^d)$.

A final question arises as to the proper way to regard y in (4-1). If, in parallel with the previous setup, we regard y as chosen in a second stage after y^d is revealed, then y is given by the min-condition, (4-9). Hence at the time x is set, the policy authority should properly regard y as a random variable but explicitly recognize that the distribution of this variable is determined by its own future behavior. The optimal strategy is then to

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[8]When $y = y^d$, the Lagrange multiplier which is non-negative is given by $2v_1(y^*-y)$, implying $y^* \geq y = y^d$. Then $y^* \geq y^d$ implies $(v_1y^*+v_2y^d)/(v_1+v_2) \geq (v_1y^d+v_2y^d)/(v_1+v_2) = y^d$. Hence (4-9) holds. For later use, we also note that when $y = y^d$ we have $y \leq y^*$ and when $y = (v_1y^*+v_2y^d)/(v_1+v_2) \leq y^d$ we have $y^* \leq y^d$ and hence $y \geq y^*$.

choose x to minimize E(L) when L is given by (4-1), y^d by (4-4) and y by (4-9).

The actual derivation of this optimal strategy is somewhat involved and we shall just sketch a few steps. Taking expectations of (4-1) yields the following objective function to be minimized.

$$E(L) = (x-x^*)^2 + v_1[E(y) - y^*]^2 + v_1 var(y) + v_2 E(y-y^d)^2 \qquad (4\text{-}10)$$

To evaluate (4-10) we need the mean and variance of y as well as $E(y-y^d)^2$. The first two of these require the pdf of y while the last term can be most directly calculated from the pdf of the variable $(y-y^d)$. From (4-9) and the algebra in footnote 8 we have that

$$y = \begin{cases} y^d & \text{if } y \leq y^* \\ \delta y^* + (1-\delta)y^d & \text{if } y \geq y^* \end{cases} \qquad (4\text{-}11)$$

where $\delta = v_1/(v_1+v_2)$. From (4.11) we see that the pdf of y, h(y) has the following form

$$h(y) = \begin{cases} f(y) & -\infty \leq y < y^* \\ f\left(\dfrac{y - \delta y^*}{1 - \delta}\right) \left|\dfrac{1}{1 - \delta}\right| & y^* \leq y \leq \infty \end{cases} \qquad (4\text{-}12)$$

where f( ) is the pdf of y^d.[9] From (4-12) and a bit of tedious manipulation one can derive E(y) and var(y). For example, when expressed in terms of the standard normal pdf ϕ (with corresponding cdf Φ) we have

$$E(y) = \delta y^*(1-\Phi(w)) + (\gamma_1 x+\gamma_2' z_3)\ (1-\delta+\delta\Phi(w)) - \delta\sigma_3\phi(w) \qquad (4\text{-}13)$$

$$w(x) = (y^*-\gamma_1 x-\gamma_2' z_3)/\sigma_3. \qquad (4\text{-}14)$$

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[9] When $v_2 = 0$ and $\delta = 1$, the form of (4-12) simplifies because $y = \min(y^d, y^*)$ and the pdf has a mass point at $y = y^*$. Thus, the second term in (4-12) reduces to $h(y) = 1 - F(y^*)$ when $y = y^*$ where F( ) is the cdf of y^d.

To evaluate $E(y-y^d)^2$ we note that

$$u = y - y^d = \begin{cases} 0 & \text{if } y \leqq y^* \\ \delta(y^* - y^d) & \text{if } y \geqq y^* \end{cases}$$

so that the pdf of w is given by

$$g(u) = \begin{cases} F(y^*) & \text{if } u = 0 \\ f\left(y^* - \frac{u}{\delta}\right) \frac{1}{\delta} & \text{if } u < 0. \end{cases} \tag{4-15}$$

Using the densities given in (4-12) and (4-15), we can compute $E(L|x)$ and differentiate this with respect to x to find the following first order condition:

$$H(x) \equiv (x-x^*) + \gamma_1 v_1(\gamma_1 x + \gamma_2' z_3 - y^*) \left[\frac{v_2}{v_1 + v_2} + \frac{v_1}{v_1 + v_2} \Phi(w) \right]$$

$$- \gamma_1 v_1 \sigma_3 \phi(w) \left[\frac{v_1}{v_1 + v_2} \right] = 0 \tag{4-16}$$

Since both $\phi(w)$ and $\Phi(w)$ depend on x in a nonlinear way,[10] it is not possible to give an explicit algebraic solution for x. One can, of course, solve (4-16) numerically. Moreover, we can shed considerable light on the properties of the solution to (4-16).

Properties of the Optimal Strategies. We can readily establish that

$$H'(x) = 1 + v_1\gamma_1^2[1+\delta(\Phi(w)-1)]. \tag{4-17}$$

Since $H'(x)$ is therefore strictly positive we know that there is a unique solution to $H(x) = 0$.[11] Let us denote this by $\hat{x}$. A question naturally arises as to the relationship of $\hat{x}$ to the solution, call it $\tilde{x}$, given by the

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[10] For example,

$$\phi(w) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left[\frac{y^* - \gamma_1 x - \gamma_2' z_3}{\sigma_3}\right]^2\right\}.$$

"certainty" approach summarized in (4-7) and (4-8). The nature of the relationship can most easily be seen by evaluating H(x), the left-hand side of (4-16), at x_I, x_{II} given by (4-7) and (4-8). A bit of algebra reveals

$$H(x_I) = -\gamma_1\sigma_3 v_1\delta[\phi(w_I) - (1-\Phi(w_I))w_I]$$
$$H(x_{II}) = -\gamma_1\sigma_3 v_1\delta[\phi(w_{II}) + \Phi(w_{II})w_{II}] \quad (4\text{-}18)$$

where from (4-14) $w_I = w(x_I)$ and $w_{II} = w(x_{II})$ are

$$w_I = (y^* - \gamma_1 x^* - \gamma_2' z_3)/\sigma_3(1+v_1\gamma_1^2)$$

$$w_{II} = (v_1+v_2)\ (y^* - \gamma_1 x^* - \gamma_2' z)/\sigma_3(v_1+v_2+\gamma_1^2\ v_1 v_2)$$

Intuitively, it should be the case that when σ_3^2 is small the solutions of $\hat{x}$ and $\tilde{x}$ should be quite close since $\tilde{x}$ should well approximate $\hat{x}$ when there is little uncertainty about demand. This intuition is confirmed by examining the limiting behavior of (4-18) as $\sigma_3^2 \to 0$. If $S = (y^* - \gamma_1 x^* - \gamma_2' z_3) > 0$ then $\Phi(w_I) \to 1$ and $H(x_I) \to 0$. Alternatively, if $S < 0$, $\Phi(w_{II}) \to 0$. Comparing this with (4-7) and (4-8), we see that $\lim_{\sigma_3 \to 0} \hat{x} = \tilde{x}$.

Equ. (4-18) can be used to shed further light on the relationship between $\hat{x}$ and $\tilde{x}$ for nontrivial values of σ_3^2. Using the standard properties of the normal distribution, both bracketed terms in (4-18) are positive. Hence, when $\gamma_1 < 0$, we see that $H(x_I)$ and $H(x_{II})$ are both positive. Since $H'(x) > 0$, we see that $\hat{x}$ must be less than both x_I and x_{II}. We have thus established that $\hat{x} < \tilde{x}$. Thus, a policy maker who minimizes expected loss will, by choosing a lower value for x, be more likely to

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[11] We are here ignoring the possibility that $H(0) > 0$, which would imply a boundary solution of $x = 0$. For certain values of the parameters, e.g., $\delta = 1$ and $\gamma_1 > 0$, it can be shown that $H(0) < 0$, but in general the boundary solution needs to be ruled out by assumption.

ration.[12] This result actually holds in more general sense, which we can see if we examine the comparative statics of the expected loss solution.

The relevant derivatives, obtained from straightforward calculations based on (4-16) are given in the Appendix. The qualitative results, under the assumption that $\gamma_1 < 0$, are summarized in Table 1, which also gives the qualitative effect of a change in each parameter on the quantity of expected rationing. The latter is defined as $E(y^d - y)$ and some details on these calculations are also given in the Appendix.

Table 1

P	$\frac{d\hat{x}}{dP}$	$\frac{dE(y^d-y)}{dP}$
x^*	+	−
y^*	−	−
v_1	?	?
v_2	+	−
σ_3	−	+

The results are generally as expected. A decrease in x^*, leads to a reduction in $\hat{x}$ but since $d\hat{x}/dx^* < 1$, also corresponds to an increase in expected rationing. An increase in the target y, y^*, is accompanied by both a reduced $\hat{x}$ and diminished rationing. Increasing v_2, which raises the disutility from rationing, leads to a higher $\hat{x}$ and reduced rationing. Indeed, in the limit as v_2 gets arbitrarily large, we have that $E(y^d - y)$ tends to zero as $\hat{x}$

[12]This conclusion does not depend on the sign γ_1. If $\gamma_1 > 0$ we would have $\hat{x} > \tilde{x}$ but demand would be higher at $\hat{x}$ and so rationing is still more likely. Since the assumption that $\gamma_1 < 0$ is more in keeping with our initial motivating example, we shall confine our attention to this case.

tends to x_I, the nonrationing certainty solution. An increase in v_1, the parameter affecting the disutility from deviating from y^*, in general has an uncertain sign. When $v_2 = 0$, raising v_1 unambiguously lowers $\hat{x}$ as would be expected. When $v_2 > 0$, however, the fact that lowering $\hat{x}$ raises $E(y^d)$ more than $E(y)$ serves to render ambiguous the overall effect. Finally, decreasing σ_3, the demand uncertainty, raises $\hat{x}$ and reduces expected rationing. This is in accord with the earlier comparison of $\hat{x}$ and $\tilde{x}$ as we have already seen that $\lim_{\sigma_3 \to 0} \hat{x} = \tilde{x}$.[13]

5. Econometric Implementation

The previous section developed two related alternative models of policy-maker behavior. The second of these is somewhat more appealing but, as we shall see, is rather less empirically tractable. The first model consists of equs. (4-2), (4-3), (4-4), (4-7), (4-8) and (4-9). For convenience these are collected below in a slightly different form:

$$x^* = \alpha' z_1 + \varepsilon_1 \tag{5-1}$$

$$y^* = \beta' z_2 + \varepsilon_2 \tag{5-2}$$

$$x = \frac{\alpha' z_1 + v_1\gamma_1\beta' z_2 - v_1\gamma_1\gamma_2' z_3}{1 + v_1\gamma_1^2} + \frac{\varepsilon_1 + v_1\gamma_1\varepsilon_2}{1 + v_1\gamma_1^2}$$

$$\text{if } S = \beta' z_2 - \gamma_1\alpha' z_1 - \gamma_2' z_3 + \varepsilon_2 - \gamma_1\varepsilon_1 \geq 0 \tag{5-3}$$

$$x = \frac{(v_1+v_2)\alpha' z_1 + \gamma_1 v_1 v_2\beta' z_2 - \gamma_1 v_1 v_2\gamma_2' z_3}{v_1 + v_2 + \gamma_1^2 v_1 v_2} + \frac{(v_1+v_2)\varepsilon_1 + \gamma_1 v_1 v_2\varepsilon_2}{v_1+v_2+\gamma_1^2 v_1 v_2}$$

$$\text{if } S < 0 \tag{5-4}$$

[13] It can be shown that for large σ_3 the expected loss solution is essentially linear in σ_3.

$$y^d = \gamma_1 x + \gamma_2' z_3 + \varepsilon_3 \qquad (5\text{-}5)$$

$$y = \min\left[y^d, \frac{v_1 y^* + v_2 y^d}{v_1 + v_2} \right] \qquad (5\text{-}6)$$

The only difference between this set of equations and those given earlier is that we have used (5-1) and (5-2) to rewrite (4-7) and (4-8) as (5-3) and (5-4). That is, we have explicitly introduced the stochastic terms and made apparent that the "switch condition" depends on the random variable S. The properties of S, in turn, depend on ε_1 and ε_2, which stem from our stochastic modeling of the behavior of the policy authority in the choice of its desired or target values for x and y.

We shall consider three alternative ways in which the parameters can be estimated, each of which applies the maximum likelihood method to some or all of the equations of the model. For this purpose, we shall assume that the ε_i are jointly normally distributed with variances σ_i^2.[14] The first approach consists of estimating the submodel consisting of (5-3) and (5-4). It can readily be seen that this submodel allows one to identify almost all the parameters of interest in the full model.[15] Equs. (5-3) and (5-4),

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[14]The most general derivation of the various likelihood functions would allow for a nonzero covariance between ε_1 and ε_2 but, given our previous discussion on timing aspects, would assume ε_3 independent of the other ε's. In the sampling experiments reported below, however, we have assumed ε_1 and ε_2 are independent.

[15]One obvious exception is σ_3^2. In addition, if z_2 and z_3 share any common variables, including an intercept, their coefficients cannot be separately identified.

constitute a version of a switching regression model to which one may apply the approach developed by Kiefer (1977). More particularly, (5-3) and (5-4) obviously constitute a model of the following form:

$$X_t = \begin{cases} B_1'Z_t + u_{1t} & \text{if } S_t > 0 \\ B_2'Z_t + u_{2t} & \text{otherwise} \end{cases}$$

$$S_t = B_3'Z_t + u_{3t}$$

It can be shown that the pdf of X_t is given by

$$g(X) = Pf_1 + (1-P)f_2 \qquad (5\text{-}7)$$

$$P = \Phi\left(\frac{B_3'Z}{\sigma_3}\right)$$

$$f_1 = \sigma_1^{-1}P^{-1}\Phi\left[\frac{B_3'Z/\sigma_3 + \rho_{13}r_1}{\sqrt{1-\rho_{13}^2}}\right]\phi(r_1)$$

$$f_2 = \sigma_2^{-1}(1-P)^{-1}\Phi\left[\frac{-B_3'Z/\sigma_3 - \rho_{23}r_2}{\sqrt{1-\rho_{23}^2}}\right]\phi(r_2)$$

$$r_j = (X-B_j'Z)/\sigma_j \qquad j = 1,2$$

and σ_j^2 and ρ_{ij} are variances and correlation coefficients for the u_j and where as before $\Phi(\ )$ and $\phi(\ )$ are the cdf and pdf of the standard normal, respectively. The product of terms like (5-7) then gives the relevant likelihood function.

While maximization of this likelihood function yields most of the parameters of interest, these estimates are not fully efficient, since they ignore y. Before considering estimation of the full model, it is worth noting that there is a second submodel which can be estimated. In particular, the submodel consisting of equations (5-2), (5-5) and (5-6) has a structure which is nearly identical to that of a simple disequilibrium model without a price equation. This is perhaps clearest when $v_2 = 0$ as the equations become $y^* = \beta' Z_2 + \varepsilon_2$, $y^d = \gamma_1 x + \gamma_2' z_3 + \varepsilon_3$ and the observable $y = \min(y^d, y^*)$. The likelihood function for this model is a special case of (2-5). Of course, use of this submodel does not permit estimation of all the parameters of interest nor does it make use of all the available information. It also appears to suffer from a somewhat more subtle defect in that the variable x is not independent of y^*. This can be seen most directly in equ. (4-7) and (4-8). This means that if x is regarded as exogenous in deriving the likelihood function, the resulting estimates may suffer from a type of endogenous policy bias. The quantitative importance of this, however, remains to be established.

We finally turn to the full model (5-1) - (5-6). The derivation of the relevant likelihood function is quite similar to the corresponding derivation for the agricultural market considered in Goldfeld and Quandt (1975) and we shall simply sketch the steps. Let us denote the nonstochastic part on the right-hand sides of (5-3) and (5-4) by A_1 and A_2, respectively. We consider the model in the following form:

$$x_1 = A_1 + \frac{\varepsilon_1 + v_1\gamma_1\varepsilon_2}{1 + v_1\gamma_1^2} \tag{5-8}$$

$$x_2 = A_2 + \frac{(v_1+v_2)\varepsilon_1 + \gamma_1 v_1 v_1 \varepsilon_2}{v_1 + v_2 + \gamma_1^2 v_1 v_2} \tag{5-9}$$

$$y_1 = \gamma_1 x_1 + \gamma_2' z_3 + \varepsilon_3 \tag{5-10}$$

$$y_2 = \gamma_1 x_2 + \gamma_2' z_3 + \varepsilon_3 \tag{5-11}$$

$$y^* = \beta' z_2 + \varepsilon_2 \tag{5-12}$$

We know from (5-3) that when $S \geqq 0$, we observe $x = x_1$, and the corresponding $y^d = y_1$ where y_1 is defined in (5-10). Some algebra also reveals that when $S \geqq 0$ we have $x_1 \leqq x_2$.[16] The full model (5-1) - (5-6) consists of four regimes depending on whether S is positive or not and on whether y^d is less than or greater than $(v_1 y^* + v_2 y^d)/(v_1+v_2)$. Using footnote 8 says these four regimes correspond to $(S \geqq 0, y^d \geqq y^*)$, $(S \geqq 0, y^d < y^*)$, $(S < 0, y^d \geqq y^*)$, and $(S < 0, y^d < y^*)$. In terms of our reformulated version of the model, the four regimes can be expressed as

$$(x,y) = \begin{cases} (x_1, y_1) & \text{when } x_1 \leqq x_2,\ y_1 < \delta y^* + (1-\delta)y_1 \\ (x_1, \delta y^* + (1-\delta)y_1) & \text{when } x_1 \leqq x_2,\ y_1 \geqq \delta y^* + (1-\delta)y_1 \\ (x_2, y_2) & \text{when } x_2 < x_1,\ y_2 < \delta y^* + (1-\delta)y_2 \\ (x_2, \delta y^* + (1-\delta)y_2) & \text{when } x_2 < x_1,\ y_2 \geqq \delta y^* + (1-\delta)y_2 \end{cases}$$

The likelihood function then has four pieces which are obtained by first writing down the joint densities $f(x_1, y_1, \eta_1)$, $g(x_2, y_2, \eta_2)$ where $\eta_i = \delta y^* + (1-\delta)y_i$ and successively integrating out y_1 and η_1 from the first and y_2 and η_2 from the second. That is,

$$\begin{aligned} h(x,y) = & \int_y^\infty f(x,y,\eta_1)d\eta_1 + \int_y^\infty f(x,y_1,y)dy_1 \\ & + \int_y^\infty g(x,y,\eta_2)d\eta_2 + \int_y^\infty g(x,y_2,y)dy_2 \end{aligned} \qquad (5\text{-}13)$$

[16] It can be shown that $x_1 - x_2 = \gamma_1 KS$ where $K > 0$. Thus the statement in the text assumes $\gamma_1 < 0$. The derivation of the likelihood function goes through when γ_1 is positive with the obvious changes in the relevant inequalities. It should also be observed that $y_1 - y_2 = \gamma_1(x_1 - x_2)$ so $\gamma_1 < 0$ means $x_1 < x_2$ implies $y_1 > y_2$.

The similarity between the form of (5-13) and the corresponding equation (A-13) in Goldfeld and Quandt (1975) is readily apparent. The actual expression for (5-13) is sufficiently messy, however, that we omit it.

We now turn to the problem of estimating the expected loss model. As in the previous case, we can directly estimate the submodel (5-2), (5-5) and (5-6) although this would have the same sort of difficulties as before. The more interesting problem is to estimate an equation for x where x is implicitly given by

$$(x-x^*) + \gamma_1 v_1(\gamma_1+\gamma_2' z_3-y^*)\ [1 - \delta + \delta\Phi(w)] - \gamma_1 v_1 \sigma_3 \delta\phi(w) = 0 \qquad (5\text{-}14)$$

and where $x^* = \alpha z_1 + \varepsilon_1$, $y^* = \beta z_2 + \varepsilon_2$, and $w = (y^*-\gamma_1 x-\gamma_2' z_3)/\sigma_3$

Estimation of (5-14) presents several problems, the least of which stems from the fact that (5-14) is only an implicit nonlinear equation. A more serious problem stems from the error structure of (5-14). While the x^* term introduces an additive error, y^* enters (5-14) in various nonlinear ways and the overall implied error structure is extremely complicated. It would thus appear that in the general case, the expected loss model is not easily estimable. In the special case when ε_2 is small, we can use (5-14) to derive the approximate pdf of x and do maximum likelihood estimation. More specifically if $B = (\alpha,\beta,\gamma_1,\gamma_2,v_1,v_2,\sigma_3^2)$ is the vector of parameters to be estimated and we let $H_i = H(x_i|B)$ then the condensed log-likelihood function, save for a constant is

$$\sum_{i=1}^{N} \log\ [1+v_1\gamma_1^{\,2}(1-\delta+\delta\Phi(w_i))] - \frac{N}{2} \log \left(\sum_{i=1}^{N} H_i^2 \right) \qquad (5\text{-}15)$$

This approach will be precise for $\varepsilon_2 \equiv 0$ but the quality of the approximation for other cases needs to be examined. This is done below.

Even with this simplification, there remains a problem of identification. In particular, inspection of (5-14) and (5-15) reveals that if $(\alpha,\beta,\gamma_1,\gamma_2,v_1,v_2,\sigma_3^2)$ maximize (5-15) then so will $(\alpha,\beta/\lambda,\gamma_1/\lambda,\gamma_2/\lambda,\lambda^2 v_1,\lambda^2 v_2,\sigma_3^2/\lambda^2)$.[17] In other words, with the exception of α, using (5-15) only permits estimation of the parameters relative to each other.

These difficulties perhaps suggest the wisdom of trying another approximate way to estimate the expected loss model, namely by using the model given by (5-3) and (5-4). We have already seen that (5-14) collapses to this model as σ_3^2 tends to zero. It remains to be seen how this approximation works more generally.

6. Some Computational Experience

We now describe some sampling experiments aimed at providing computational experience with rationing models. To minimize computational costs we have chosen a bare-bones specification with an intercept and one exogenous variable in each stochastic relationship. More specifically, the parameters appear as follows:

$$L = (x-x^*)^2 + v_1(y-y^*)^2 + v_2(y-y^d)^2$$

$$x^* = \alpha_0 + \alpha_1 z_1 + \varepsilon_1$$

$$y^* = \beta_0 + \beta_1 z_2 + \varepsilon_2$$

$$y^d = \gamma_{20} + \gamma_{21} z_3 + \gamma_1 x + \varepsilon_3$$

There are nine basic parameters $(v_1,v_2,\alpha_0,\alpha_1,\beta_0,\beta_1,\gamma_{20},\gamma_{21},\gamma_1)$ and, given the assumption that the ε_i are independently normally distributed, three variances σ_i^2, i = 1,2,3.

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[17] This statement assumes there is no intercept in γ_2.

The previous section outlined four methods for estimating the rationing model. We shall refer to these as Exploss (eqn. 5-14), Switch (eqns. (5-3), (5-4)), Min (eqns. (5-2), (5-5) and (5-6)) and Full (eqns. (5-1)-(5-6)). While the Full method estimates all parameters, the remaining methods can only estimate some subset. For Switch the estimable parameters are $(v_1, v_2, \alpha_0, \alpha_1, \beta_1, \gamma_{21}, \gamma_1, \sigma_1^2, \sigma_2^2)$ while for Min they are $(\beta_0, \beta_1, \gamma_{20}, \gamma_{21}, \gamma_1, \sigma_2^2, \sigma_3^2)$. In addition, Min provides an estimate of $\delta = v_1/(v_1 + v_2)$. As noted earlier, the Exploss method only provides estimates of relative parameters. We have implemented this by constraining γ_1 to its true value in the sampling experiments.[18] Other things equal, this would tend to favor Exploss as compared with the remaining methods. Fortunately, it is easy to correct for this tendency by transforming the results of the sampling experiments to yield estimates corresponding to alternative values for γ_1.

Aside from the estimating methods, the other potentially important dimension of the analysis is the underlying model. We have two possibilities: the policy maker minimizes expected loss or minimizes loss ignoring uncertainty. We shall generate data using both assumptions, and the presumption is that the Exploss method should work better with the expected loss data.

The other details of the experiments are as follows. The z_i's were generated from the uniform distribution with ranges (0,100), (0,100) and (200,400), or with ranges multiplied by 10. The parameter values were $v_1 = 4$, $v_2 = 2$, $\alpha_0 = 70$, $\alpha_1 = 1$, $\beta_1 = .5$, $\gamma_{20} = 60$, $\gamma_{21} = -.25$, $\gamma_1 = .5$. For the small z range, $\beta_0 = 20$ while $\beta_0 = -655$ for the large z range. These values were chosen so that the mean of the switch variable $S = (y^* - \gamma_1 x^* - \gamma_2' z_3)$

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[18] Preliminary computational tests revealed no reason to favor the choice of one constraint to another. Moreover, γ_1 seems to be a parameter for which one might have some a priori information.

was zero. For all experiments σ_1^2 = 250 while the values of σ_2^2 and σ_3^2 were varied as indicated below. Finally, all experiments consisted of 50 replications and, except for one case, we dealt with sample sizes of size 40.

An overview of the sampling experiments is provided in Table 2. There are two preliminary experiments, cases I and II, and six basic cases, III-VIII. The latter come in pairs and in the first of each pair, the data are generated by ignoring demand uncertainty while the second generates data by assuming expected loss minimization. The range of the z's and σ_3^2 vary across the pairs. All four estimation methods are used in the basic cases.

Table 2

Design of Experiments

Case	σ_2^2	σ_3^2	Z-Range	Data Generation
I	0	125	small	expected loss sample size = 80
II	0	125	small	expected loss
III	250	125	small	certainty
IV	250	125	small	expected loss
V	250	2000	small	certainty
VI	250	2000	small	expected loss
VII	250	500	big	certainty
VIII	250	500	big	expected loss

The two preliminary experiments deal only with the Exploss method, clearly the one about which there are the most questions. In both these cases σ_2^2 = 0, a priori the most favorable circumstance for the Exploss method. Table 3 reports the mean absolute deviations (MADs) as a percentage of the absolute value of the true parameter. For Case I (sample size = 80)

Table 3

Preliminary Exploss Results

	Case I % MAD	Case II % MAD	% WIN I vs. II	% WIN II vs. IV
v_1	107	190	59	52
v_2	74	234	59	87
α_0	12	15	59	41
α_1	15	20	59	54
β_1	16	33	57	54
γ_{21}	14	24	61	67

the %MADs are about 15%, except for v_1 and v_2 where they are much larger. For Case II (sample size = 40), the MADs are roughly twice as large, a factor consistent with the relative sample sizes. For v_1 and v_2 there is some indication that the MADs are affected by outliers. This phenomenon is even more strongly evident in a number of other cases where $\sigma_2^2 \neq 0$.

As a consequence, in analyzing the results we shall mainly rely on a non-parametric statistic, the fraction of times one estimator is closer to the truth than another. This statistic appears in the third column of Table 3 comparing the Exploss method for the two sample sizes. Averaged across parameters, Exploss for Case I wins 59 percent of the time. While, strictly speaking, the behavior across parameters is not independent, a rough measure of the standard error of this percentage is .029, so one can reject the hypothesis that the Exploss method performs equally well as sample size changes.

As noted earlier, strictly speaking, our Exploss method only applies when when $\sigma_2^2 = 0$. We can see the consequences of using the method when $\sigma_2^2 \neq 0$ by comparing Cases II and IV. This is done in the last column of Table 3. The average percentage win statistic is again 59, suggesting that Exploss performs significantly better when $\sigma_2^2 = 0$. Unfortunately, this assumption is unlikely to be met in practice so in comparing the Exploss method with the other three, we utilize nonzero values of σ_2^2.

Table 4 reports the results of the average percentage win statistics for a bivariate comparison of methods across the various experiments. As far as the Exploss method is concerned, the comparison with the Switch method, which also uses the x but not the y data, is perhaps most relevant. There is a slight tendency for the Exploss method to do better with the expected loss data generation but, with the exception of the last two cases, the Exploss method does not do all that well. Moreover, the results for the last two cases give a misleading impression of the success of the Exploss method. In particular, it will be recalled that the Exploss method constrains γ_1 to its true value. If, alternatively, in Case VIII we had constrained γ_1 to 10% above or below its true value, the average percentage win statistics would be 40.7 and 42.3 respectively.[19]

What this suggests is that the Exploss method, even where the data are appropriately generated, does not have much to recommend it over the Switch method. The comparison with the Full method is even less favorable, as the second row of Table 4 indicates. Once again, the last two columns give a mis-

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[19]The deterioration of Exploss is most severe in Cases VII and VIII. In Case VI, for example, moving γ_1 up or down by 20% leaves the average percentage win statistic virtually unchanged. The reason is that the absolute performance for all methods is both better and less disparate for Case VIII. See Table 5.

Table 4

Average Percentage Win Statistics*

	III	IV	V	VI	VII	VIII
Exploss vs Switch	47.0	45.3	48.3	55.3	67.0	70.7
Full vs Exploss	59.0	66.3	53.7	52.0	52.3	47.0
Full vs Switch	61.7	58.6	51.7	58.9	71.0	68.3
Full vs Min	66.7	70.3	59.7	60.7	46.7	41.7

Table 5

Relative MADs

	Case IV				Case VIII			
	Full	Exploss	Switch	Min	Full	Exploss	Switch	Min
v_1	1.0	15.9	20.9	–	1.0	1.3	2.3	–
v_2	1.0	59.2	16.4	–	1.0	1.1	1.4	–
α_0	1.5	1.2	1.0	–	1.3	1.0	1.4	–
α_1	1.0	14.0	1.7	–	1.2	1.0	1.8	–
β_0	1.0	–	–	2.5	1.2	–	–	1.0
β_1	1.0	2.1	1.3	4.9	1.0	1.4	2.3	1.3
γ_{20}	1.0	–	–	1.0	1.4	–	–	1.0
γ_{21}	1.0	3.5	2.7	2.6	1.0	1.3	2.4	1.1
γ_1	1.0	–	2.7	2.4	1.0	–	3.2	1.3
δ	1.0	2.9	2.1	1.4	1.1	1.1	1.0	1.1

*Entries give the average percentage of wins for the first-named method in each row.

leading impression of the Exploss method. For example, for Case VIII constraining γ_1 to 10% above or below its true value, pushes the percentage win for Full to over 70%.

More generally, as Table 4 shows, the Full method, hardly surprisingly, emerges as the most reliable estimating technique, whatever the underlying data generation scheme.[20] This can also be seen in Table 5 which reports the MADs for two cases, normalized so that the lowest entry in each row is unity. The advantages of the Full method are particularly striking in Case IV but are also evident in Case VIII where the methods perform more comparably.[21]

7. Conclusions

This paper presents two types of rationing models that differ in their treatment of the underlying uncertainty. From a theoretical perspective the expected loss approach has the more appeal. Indeed, a number of interesting extensions of this model can be readily suggested. For one, it would be possible to extend the model to allow for more than one source of uncertainty. Such a situation was implicit in the setup described by equs. (3-2) and

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[20] The comparison of the Full vs Switch methods is based on seven parameters, the six in Table 3 plus γ_1, while Full vs Min is based on β_0, β_1, γ_{20}, γ_{21}, γ_1, and δ. The last parameter is only directly estimated by the Min method. As the Min method has relatively few parameters in common with Exploss and Switch, these comparisons are not presented.

[21] The discrepancy between the relative performance of the Full and Min methods in Case VIII in Tables 4 and 5 largely stems from the percentage wins for the intercepts. In general, the Full method does least well for the intercept terms. It should also be noted that there is some mild evidence of a "policy bias" with the simple Min model. More particularly, for about two-thirds of all the parameter estimates the Min model had larger biases than the Full model.

(3-3). Another source of multiple uncertainty would arise if we allowed for the possibility that the policy maker is uncertain both with respect to the strength of demand and in regards to the elasticity of demand with respect to its instrument (i.e., γ_1). Preliminary analysis of such a model reveals a number of new wrinkles that we plan to report on in a subsequent paper. Still another related extension would be to adopt an explicit multiperiod setting where, perhaps, anticipations or rationing might be important.[22]

From an econometric point of view, the Full method, even when it is only approximately valid, appears to be the most useful of the four estimating methods. Nevertheless, it remains a challenge to provide a more satisfactory way to estimate directly the expected loss model.

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[22] Several potential areas of application of the extended expected loss model can be noted. One example stems from the work of Abel (1985) who analyzes inventory behavior in the face of stockouts, a setup that leads to min conditions. It also appears fruitful to apply the expected loss model to the multiperiod interest-rate setting behavior analyzed in Goldfeld and Jaffee (1970).

Appendix

The relevant expressions for the comparative statics of the expected loss model referred to in the text are given below.

$$\frac{dx}{dx^*} = 1/H'(x) \qquad\qquad \frac{dx}{dy^*} = -v_1\gamma_1(1-\delta+\delta\Phi)/H'(x)$$

$$\frac{dx}{dv_1} = -\gamma_1\sigma_3[\delta(2-\delta)(w(1-\Phi)-\phi)-w]/H'(x)$$

$$\frac{dx}{dv_2} = -\gamma_1\delta^2\sigma_3[w(\Phi-1)+\phi]/H'(x) \qquad\qquad \frac{dx}{d\sigma_3} = v_1\gamma_1\delta\phi/H'(x)$$

To calculate the quantity of expected rationing, $E(y^d-y)$, one can proceed directly from the pdf of y^d-y or combine (4-13) and (4-16). The result is:

$$R = E(y^d-y) = \left(\frac{1+v_1\gamma_1^2}{\gamma_1 v_1}\right)x + \gamma_2' z_3 - y^* - x^*/\gamma_1 v_1$$

From this one can directly calculate the relevant derivatives. For example:

$$\frac{dR}{dx^*} = \gamma_1\delta(1-\Phi)/H'(x) \qquad\qquad \frac{dR}{dy^*} = -\delta(1-\Phi)/H'(x)$$

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