IMPOSSIBILITY OF STRATEGY-PROOF MECHANISMS FOR ECONOMIES WITH PURE PUBLIC GOODS*

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1. INTRODUCTION

When a society consisting of several individuals has to select from a set of alternatives, it often relies on certain rules to make this choice. In economic theory such rules are called mechanisms (or voting schemes, or social choice functions). These mechanisms may be inherited from earlier generations, or they may be adopted via democratic processes. In order that a mechanism represent an optimal compromise (in any well-defined sense) of the conflicting interests of the members of a society, it must take into account individuals' preferences over the alternatives. However, these preferences are usually privately known and they have to be solicited for public use. Thus, those individuals who try to maximize utility have an opportunity to manipulate the final outcome by misrepresenting their preferences. As a result of such manipulations the actual outcomes may be far from satisfactory from the social point of view. Hence, in order to have a better understanding of social decision-making processes, it is important to know how severe the problem of manipulation is, and whether mechanisms immune to manipulation can be devised.

In the framework of social choice theory, Gibbard [1973] and Satterthwaite [1975] independently proved that, subject to a minor qualification, a mechanism is manipulable if it is nondictatorial. Since then many different proofs of this classical result have been provided by other authors. However, the original theorem is stated only for the case in which all possible preferences are admissible, and it leaves unanswered the question of whether similar results are true under various restrictions of the domain of admissible preferences.

For example, let us consider a canonical problem from public finance.[1] There are three public goods to be provided: education, tele-communication, and transportation. Since

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[1] This example was suggested by H. Moulin.

these are "goods", every individual has increasing preferences over them. In addition, individuals' preferences are also assumed to be continuous and quasi-concave. The feasible set is given by $A = \{ (x_1, x_2, x_3) \mid x_i \geq 0, \sum x_i \leq 1 \}$. Society has to choose an allocation from A. In this problem it is natural to consider mechanisms that satisfy the property of unanimity, *i.e.*, those mechanisms that choose x if all individuals regard x as the best alternative in A. The question is whether there exists a nonmanipulable and nondictatorial mechanism for this problem . Surprisingly, this seemingly straightforward question is not answered by any existing result.

In this paper I consider a general model of economies with pure public goods, in which individuals' preferences are continuous and quasi-concave. Although the existing literature on this subject contains some results which suggest that the Gibbard-Satterthwaite Theorem can be extended to such economies, it does not have a general and fully satisfactory resolution of the issue. The real difficulty in this area stems from the fact that most of the work on strategy-proofness relies heavily on pure logical induction, thus leaving little room to deal with the properties of continuity and quasi-concavity. For example, in Schmeidler and Sonnenschein's [1978] proof of the Gibbard-Satterthwaite Theorem, they assume that if one changes any individual's preference ranking by moving any pair of alternatives to the top of the original preference ranking, then the new preference is still admissible. This certainly violates the continuity of the preferences .

A recent paper by Barbera and Peleg [1988] presented a new and elegant proof of the Gibbard-Satterthwaite Theorem that is based on the so-called pivotal-voter approach developed earlier by Barbera [1983]. It is very direct and simple, invoking neither the Arrow Theorem nor any monotonicity argument. Yet it is so powerful that under its framework many other interesting issues can be addressed. The authors used it to prove that if the set of allocations is a metric space (not necessarily a subset of some finite-dimensional Euclidean space) and the space of admissible preferences contains all continuous utility functions, then any strategy-proof mechanism is dictatorial. The shortcoming of their work is that some double-peaked preferences are used in an essential way. Thus it failed to deal with quasi-concavity, which is a very important property for most economic problems.

However, a more refined use of the pivotal-voter approach enables me to establish impossibility results for public goods economies. In this paper it is shown that one simple dimension condition, analogous to the cardinality condition in the Gibbard-Satterthwaite Theorem, plays an important role in our models. It is also shown that although the space of all continuous, quasi-concave preferences is usually associated with economic environments, some smaller subspace of it (for example, the subspace of quadratic utility functions) is sufficient for a negative result to emerge. [2]

The paper is organized as follows. In Section 2, the main results are stated and compared to existing work in this area. In Section 3, a formal proof of our main results is presented. Finally, Section 4 contains an application of the results to public "goods" economies in which admissible preferences are further assumed to be increasing.

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[2] In the works of Maskin [1976], Kalai and Muller [1977], and Ritz [1985], it is established that a restricted domain admits a nondictatorial Arrovian welfare function if and only if it admits a nondictatorial, strategy-proof social choice procedure. However, it is important to recognize that the above relationship does not hold for Arrovian welfare functions and strategy-proof mechanisms (as they are usally defined). Otherwise our work would be vacuous, since it is already known that there exist no nondictatorial Arrovian social welfare functions in our models. The concept of social choice procedures is quite different from the concept of mechanisms as the former requires substantially stronger consistency conditions. Unfortunately, the lack of a common terminology has lead to some misconceptions about these results among many economists. One can find in Barbera, Sonnenschein, and Zhou [1988] an example of a domain that admits a class of nontrivial strategy-proof mechanisms but no nondictatorial Arrovian welfare functions (Footnote 3); and in Kalai, Muller, and Satterthwaite [1979] an example of a domain with the opposite set of characteristics (Example C in Section 1).

2. THE MODEL AND THE STATEMENT OF THE THEOREMS

There are n agents in a society. They have to choose an allocation from a set of feasible allocations. A is the set of all conceivable allocations. It is assumed to be a convex subset in some finite-dimensional Euclidean space. For any set $B \subset A$, $Co(B)$ denotes the convex hull of B; $dim(B)$ denotes the dimension of B, which is the dimension of the smallest affine superset of B; and $\#(B)$ denotes the cardinality of B.

Each agent has a complete and transitive binary preference over A. Ω denotes the space of all the admissible preferences of the agents over A. Particularly, Ω_A denotes the space of all continuous and quasi-concave preferences over A; and Ω_0 denotes the space of the preferences which can be represented by a quadratic function $u(x) = -(x-a)'H(x-a)$, where H is a positive definite matrix, and a some point in A. Given a preference R in Ω and a set $B \subset A$, $Argmax(R; B)$ denotes the set of maximal points of R in B, and $argmax(R; B)$ denotes the same set if it contains a unique maximal point. Ω^n denotes the product space: $\Omega^n = \Omega \times \Omega \times ... \times \Omega$. The generic point $\boldsymbol{R} = (R_1, R_2, ..., R_n)$ in Ω^n is called a preference profile, where R_i is the i-th agent's preference over A. Sometimes $\boldsymbol{R} = (R_1, R_2, ..., R_n)$ is simply written as (R_i, R_{-i}).

A mechanism is a function $f: \Omega^n \to A$, which maps a preference profile to an allocation.[3] Since A may contain allocations that are not feasible, f is usually not onto A. The range of f is denoted by A_f. If each individual i announces a preference $R_i \in \Omega$, then $f(R_1, R_2, ..., R_n)$ is society's chosen allocation. However, each agent is free to report any preference he

[3] I will only discuss direct mechanisms in the paper. Nevertheless, the result can easily be generalized to arbitrary mechanisms by using the " revelation principle ".

wants to; in other words, he is not expected to report the truth unless it is to his best interest to do so. This consideration motivates the following definition.

Definition 1: A mechanism f is *strategy-proof* if for any profile $\boldsymbol{R} = (R_1, R_2, \ldots, R_n)$, any agent i, and any Q_i in Ω,

$$f(R_i, R_{-i})\; R_i\; f(Q_i, R_{-i}).$$

It follows directly from the definition that if a mechanism f is strategy-proof, then any agent i with any preference R_i is always willing to report the truth no matter what the other agents report. In other words, for any agent it is always a dominant strategy to report the truth. For a more detailed discussion of the concept of stratetegy-proofness and the related issues, readers are referred to existing literature, for example, Muller and Satterthwaite [1986]. Our analysis focuses on the structure of strategy-proof mechanisms. We first look at a special class of mechanisms.

Definition 2: A mechanism f is *strongly dictatorial* if there is an agent i (the *strong dictator*) such that for any profile $\boldsymbol{R} = (R_1, R_2, \ldots, R_n)$,

$$f(\boldsymbol{R}) \;=\; argmax\,(R_i; A_f).$$

It is trivial to verify that a strongly dictatorial mechanism is strategy-proof. However, the unique maximal point on the right-hand side does not necessarily exist unless we make some assumptions on either Ω or A_f. For a general discussion of mechanisms, we need a weaker condition simply to avoid non-existence problem.

Definition 3: A mechanism f is *weakly dictatorial*, or *dictatorial*, if there is an agent i (the *dictator*) such that for any profile $\boldsymbol{R} = (R_1, R_2, \ldots, R_n)$,

$$f(\boldsymbol{R}) \in Argmax\,(R_i; A_f).$$

A dictatorial mechanism is very degenerate since it mainly represents a single agent's interest. It can hardly be regarded as a satisfactory solution for a social decision problem. Given the above discussion, a natural question is: does there exist a strategy-proof mechanism that is also nondictatorial?

The Gibbard-Satterthwaite Theorem gives a negative answer to this question for the case in which the space of admissible preferences is unrestricted. It states that a strategy-proof mechanism f on an unrestricted space of admissible preferences is dictatorial whenever $\#A_f \geq 3$. This result reveals the essential difficulty in social decision-making when the relevant information is private. At the same time, many researchers have investigated various models in which either the space of admissible preferences or the set of outcomes has a certain structure that is imposed by the nature of the problem under consideration. The Groves' mechanism is perhaps the most notable example. It concerns the allocation of public goods. By introducing a transferable private good into the model, Groves characterizes an important class of non-dictatorial strategy-proof mechanisms.[4] It should be noticed that the analysis of the Groves' model strongly depends on the existence of the transferable private good. Hence it does not extend to the model of economies of pure public goods.

The case of economies with pure public goods has been discussed by many other authors. Satterthwaite and Sonnenschein [1981] proved the following: if the set of allocations A is a convex set in some finite-dimensional Euclidean space and F is a differentiable allocation mechanism on an admissible utility function space Ω, which is an open convex subset of $C^2(A)$, then the requirement that F be strategy-proof implies that there is a local dictator at each regular point of F. Border and Jordan [1983] also derived a negative result similar to the Gibbard-Satterthwaite Theorem for a specific case in which the allocation space is some E^k

[4] The main results of Groves' mechanisms can be found in Groves and Leob [1975], Green and Leffont [1979], Holmstrom [1979], and Hurwicz and Walker [1988].

(or at least a direct product set in E^k) and the mechanism is onto the allocation space. While the deficiency of Border and Jordan's work is obvious, the difficulty with Satterthwaite and Sonnenschein's work is more subtle, yet serious. It will be discussed below.

We now present a general treatment of economies with pure public goods. We assume that the allocation set A is a convex set in some finite dimensional Euclidean space E^k and that the preference space is Ω_A, the space of all continuous and quasi-concave preferences over A. It turns out that negative results still prevail.

THEOREM 1: *Any strategy-proof mechanism f on $\Omega_A{}^n$ satisfying $dim\ (A_f) \geq 2$ is dictatorial*.

The dimension condition $dim\ (A_f) \geq 2$ in Theorem 1 is exactly the counterpart of the cardinality condition $\# A_f \geq 3$ in the Gibbard-Satterthwaite Theorem. These two conditions are very related in two ways. First, they are generally equivalent. The former implies the latter; and while the converse is not always true, if $\# A_f \geq 3$ and these points do not lie on the same straight line, then $dim\ (A_f) \geq 2$. Secondly, they play the same role in different contexts. When $\# A_f = 2$, majority voting provides a counter-example for the Gibbard-Satterthewaite Theorem. In our model, if $dim\ (A_f) = 1$, then a quasi-concave preference restricted on A_f simply becomes single-peaked. Hence, the mechanism that always chooses the median voter's most preferred outcome is strategy-proof and nondictatorial.

The above example also reveals the deficiency of Satterthwaite and Sonnenschein's work. An agent i is a local dictator at some preference profile if any small change of other agents' preferences does not change the local structure of the set of allocations he can achieve. Satterthwaite and Sonnenschein proved that strategy-proofness implies local dictatorship. They then claimed that this type of degeneracy is almost like that of global dictatorship and, therefore, that their result is parallel to the Gibbard-Satterthwaite Theorem. However, the above mentioned example makes their point hard to accept. In the example, at any regular point the median agent is the local dictator. Still, one is quite satisfied with it since it is the median voter's choice that best represents the compromise that the society is seeking . Thus there is really no strong feeling against local dictatorship. In fact, if we could find for cases $dim\,(A_f) \geq 2$ some strategy-proof mechanism with characteristics similar to the median-voter mechanism, it would be considered a positive result, regardless of whether it is locally dictatorial or not. Consequently, a real negative result is called for to demonstrate the impossibility of such mechanisms in higher dimensional cases. This was not observed in Satterthwaite and Sonnenschein's work.

Theorem 1 gives us a result that corresponds to the original Gibbard-Satterthwaite Theorem. It is directly applicable to location problems and other such problems. However, since Ω_A contains many preferences that have bliss points, it cannot be assumed for cases in which all public goods are real "goods" instead of "bads", like the example we put forward at the beginning of the paper. Hence, Ω_A is too large in many problems. In order to amend this, we introduce the following definition.

Definition 4 : An admissible preference space Ω is *abundant* on some set $B \subset A$ if it contains all quadratic preferences on B, *i.e.*, for any $v \in \Omega_0$ there exists some $u \in \Omega$, such that

$$u\,|_B \equiv v\,|_B\,.$$

The space of all preferences over A, Ω_A, or Ω_0, are all examples of abundant spaces on A. By definition, an abundant preference space Ω on B is also abundant on any subset of B. Generally, the size of an abundant space Ω on B will be reduced when the set B become smaller. Hence it is not a very strong requirement that a space Ω be abundant on some set $B \subset A$ for models of public goods economies. The next theorem shows that if f is strategy-proof and the admissible preference space Ω is abundant on the range of f, then f must be dictatorial.

THEOREM 2 : *Assume Ω is abundant on some convex set $B \subset A$. Any strategy-proof mechanism f on Ω^n satisfying $\dim(A_f) \geq 2$ and $Co(A_f) \subset B$ is dictatorial.*

Theorem 2 demonstrates that the space of all quadratic preferences contains enough preferences to make strategic manipulation inevitable. Of course, this is by no means necessary. From the proof we will see that many other spaces can also be used to serve the same purpose. Roughly speaking, the important point is that the preference space should be rich enough so that it is closed under any nonsingular transformation. This is also supported by the work of Border and Jordan [1983] in which they characterized a large class of nondictatorial strategy-proof mechanisms for the space of all separable quadratic preferences.[5]

It is easy to see that Theorem 1 is just a special case of Theorem 2. Theorem 1 is singled out because it has a very clear and simple form that matches the Gibbard-Satterthwaite Theorem. Theorem 2 is much sharper and its flexibility allows us to solve some problems with different restricted domains of preferences, especially those preferences that are also increasing. Such an application will be shown after the proof of Theorem 2.

3. PROOF OF THEOREM 2

Since we often deal with those preferences that have utility function representations in Theorem 2 , we will use utility functions instead of preferences in our discussion. We begin

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[5] In their paper, Border and Jordan were even able to establish negative results for preference spaces with arbitrarily small off-diagonal perturbations. However, it was done under the strong assumption that the mechanisms must be onto. For general mechanisms, this seems very unlikely to be true.

with two basic properties of any stategy-proof mechanism f on any domain.

If f is strategy-proof on some utility function space Ω^n, then for any pair of utility function profiles ($u_1, u_2, \ldots, u_n$) and ($v_1, v_2, \ldots, v_n$), the following inequalities hold :

$$u_1 (f (u_1, v_{-1})) \geq u_1 (f (v_1, v_{-1})) ,$$

$$u_2 (f (u_1, u_2, v_{N-\{1,2\}})) \geq u_2 (f (u_1, v_2, v_{N-\{1,2\}})) ,$$

$$\cdots\cdots\cdots\cdots$$

$$u_n (f (u_1, u_2, \ldots , u_n)) \geq u_n (f (u_1, u_2, \ldots, v_n)) .$$

From these inequalities, we can derive the following lemmas.

Lemma 1 : For any $u \in \Omega$, $Argmax (u ; range (f))$ is nonempty, and

$$f(u, u, \ldots , u) \in Argmax (u ; range (f)).$$

Proof : For any $a \in range (f)$, find some $(v_1, v_2, \ldots, v_n)$ such that $f(v_1, v_2, \ldots, v_n) = a$. Successive application of the above inequalities leads to $u(f(u, u, \ldots, u)) \geq u(a)$. **Q.E.D.**

Lemma 2 : Given a profile ($u_1, u_2, \ldots, u_n$), if there is $a \in range (f)$ such that, for all j, $a = argmax (u_j ; range (f))$, then

$$f (u_1, u_2, \ldots, u_n) = a .$$

Proof: Take any profile ($v_1, v_2, \ldots, v_n$) such that $f(v_1, v_2, \ldots, v_n) = a$. Apply the above inequalities successively. Since a is the unique maximal point for each u_i , not only the utility values but also the outcomes on both sides of each inequality must be equal. **Q.E.D.**

The properties stated in Lemma 1 and Lemma 2 represent two different expressions of conditional unanimity, *i.e.*, unanimity on the range of the mechanism. The first requires that when all agents have the same utility function, a mechanism choose an allocation that is at least as good as any allocation in its range; while the second requires that when all agents consider a specific allocation as the best allocation in its range, a mechanism choose this allocation.

Now we start to prove Theorem 2. Define a subspace Ω_f of Ω consisting of preferences that are more nicely behaved and thus more easily dealt with:

$$\Omega_f = \{ u \mid u \in \Omega, \text{ and } argmax\,(u\,; A_f) \text{ exists } \}.$$

Clearly f^*, the restriction of f on Ω_f^n, is still strategy-proof. Furthermore, the range of f^* remains the same as f. To see this, we take any $a \in A_f$. Since Ω is abundant on some $B \supset A_f$, there is a $u \in \Omega$ such that $u\,|_B \equiv -\|x - a\|^2\,|_B$. Obviously $a = argmax\,(u\,; A_f)$, hence $u \in \Omega_f$ and $a = f\,(u, u, ..., u)$ by Lemma 2. Thus $a \in A_{f^*}$. This shows $A_{f^*} = A_f$. In what follows, we first work with f^* on Ω_f^n. After finding a dictator for f^* on Ω_f^n, we can show that he is the dictator for f on Ω^n as well. For convenience of notation, we keep using f instead of f^*.

The key idea of the pivotal-voter approach is captured in the following concept. For each agent i, given his utility function u_i, define the *option set* for agents other than i as :

$$O_{-i}\,(u_i) = \{ a \in A \mid \text{there exists } u_{-i} \in \Omega_f^{n-1}, \text{ such that } a = f\,(u_i, u_{-i}) \}.$$

This is the set of allocations which agents other than i can achieve collectively when agent i's utility function is fixed at u_i. It is direct to observe that if an agent i is a dictator, then for any u_i, $O_{-i}\,(u_i) = argmax\,(u_i\,; A_f)$. The insight provided by the pivotal-voter approach is that the converse of the above observation is also true: if there is an agent i such that for any u_i, $O_{-i}\,(u_i) = argmax\,(u_i\,; A_f)$, then he must be the dictator. This is the strategy we adopt in our proof of Theorem 2. We divide our proof into several steps.

<u>Step 1</u>: $O_{-i}(u_i)$ is closed.

Proof: We first show that A_f is closed. For any point $a \in Bd(A_f)$, the boundary of A_f, $u(x) \equiv -\| x - a \|^2$ belongs to Ω, where $\|\cdot\|$ is the Euclidean distance.[6] By Lemma 1, $Argmax(u; A_f)$ is nonempty. However, it can be nothing but $\{a\}$ because $a \in Bd(A_f)$. Hence $a \in A_f$. This means that A_f is closed.

Now take any $a \in Bd(O_{-i}(u_i))$. Since A_f is closed, $a \in A_f$. Thus $u(x) \equiv -\| x - a \|^2$ belongs to Ω_f. Notice that when u_i is fixed, f is still strategy-proof on Ω_f^{n-1}. Lemma 1 then implies $a \in O_{-i}(u_i)$. This means that $O_{-i}(u_i)$ is closed. **Q.E.D.**

<u>Step 2</u>: For any u_i, $argmax(u_i; A_f) \in O_{-i}(u_i)$.

Proof: By the definition of Ω_f, $argmax(u_i; A_f)$ is always well defined. Take any (v_i, v_{-i}) such that $f(v_i, v_{-i}) = argmax(u_i; A_f)$. f is strategy-proof implies

$$u_i(f(u_i, v_{-i})) \geq u_i(f(v_i, v_{-i})).$$

This means that $argmax(u_i; A_f) = f(u_i, v_{-i}) \in O_{-i}(u_i)$. **Q.E.D.**

In order to continue our discussion on these option sets, we need some notation. Let a, b be two points in some R^k, denote (a, b), $(a, b]$, $[a, b)$, and $[a, b]$ the segments, open or closed, connecting them. Given two sets S and T, we say that S is *star-shaped* (relative to T) with respect to a base point $b \in S$, if for any $c \in S$, $[c, b] \cap T \subset S$.

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[6] The precise statement here should be that there exists $u \in \Omega$ such that $u(x) |_{A_f} \equiv -\| x - a \|^2 |_{A_f}$. However, our use of language makes the argument a little simpler, without any effect on its validity. This same remark applies on several subsequent occasions.

Step 3 : $O_{-i}(u_i)$ is star-shaped (relative to A_f) with respect to $argmax(u_i; A_f)$.

Proof: Suppose the statement is false. Then there exist a and b such that $a \in O_{-i}(u_i)$, $b \in A_f \backslash O_{-i}(u_i)$, $b \in (a, argmax(u_i; A_f))$. Since $O_{-i}(u_i)$ is closed, we can further assume, without loss of generality, that there exists $p = \lambda(b - a)$, $\lambda > 0$, such that $(a, b + 2p] \cap O_{-i}(u_i) = \emptyset$. (See Figure 1.) Let Π denote the straight line passing a and b.

Choose $c = 1/2(a + b) + p$ and construct a sequence of utility functions $u^{(n)}$ in Ω_f :

$$u^{(n)}(x) = -(x - c)'H^{(n)}(x - c),$$

where the positive definite matrix $H^{(n)}$ is chosen so that any indifference curve of $u^{(n)}$ is an elliptic ball obtained by shrinking a standard ballby a factor of $1/n$ to Π in all directions orthogonal to Π. Consider the sequence of profiles $\{(u_i, u^{(n)}_{-i})\}$, where $u^{(n)}_j = u^{(n)}$ for all $j \neq i$, and the sequence of allocations $\{f(u_i, u^{(n)}_{-i})\}$. Since when u_i is fixed, f is a strategy-proof mechanism on Ω_f^{n-1} for agents other than i, $u^{(n)}(f(u_i, u^{(n)}_{-i})) \geq u^{(n)}(a)$ for all n by Lemma 1. This means $\{f(u_i, u^{(n)}_{-i})\}$ is a bounded sequence; therefore, it (or a subsequence of it) converges to some point d. By the construction of the sequence, d is on $[a, b + 2p]$. And since $O_{-i}(u_i)$ is closed, d is also in $O_{-i}(u_i)$. Thus $d = a$ because $(a, b + 2p] \cap O_{-i}(u_i) = \emptyset$.

On the other hand, let $v_i(x) \equiv -\|x - b\|^2$ and consider the sequence $\{f(v_i, u^{(n)}_{-i})\}$. For the same reason as above, $\{f(v_i, u^{(n)}_{-i})\}$ converges to some $e \in [a + 2p, b]$.

When $\{f(u_i, u^{(n)}_{-i})\}$ converges to a and $\{f(v_i, u^{(n)}_{-i})\}$ converges to e, the corresponding sequences $\{u_i(f(u_i, u^{(n)}_{-i}))\}$ and $\{u_i(f(v_i, u^{(n)}_{-i}))\}$ will converge to $u_i(a)$ and $u_i(e)$. Since f is strategy-proof, $u_i(f(u_i, u^{(n)}_{-i})) \geq u_i(f(v_i, u^{(n)}_{-i}))$ for all n. Therefore, $u_i(a) \geq u_i(e)$. But the quasi-concavity of u_i implies that

$$u_i(\, argmax\,(u_i\,;A_f)\,) > u_i(e) > u_i(a)\,.$$

We have a contradiction. Thus we have proved our claim. **Q.E.D.**

<u>Step 4</u>: For any pair u_i and v_i in Ω_f, if $argmax\,(u_i\,;A_f) = argmax\,(v_i\,;A_f)$, then

$$O_{-i}(u_i) = O_{-i}(v_i).$$

Proof: Suppose it is not true. Since both sets are star-shaped (relative to A_f) with respect to the same point $d = argmax\,(u_i\,;A_f) = argmax\,(v_i\,;A_f)$, when $O_{-i}(u_i) \neq O_{-i}(v_i)$, they must differ on some ray starting from d. We can assume, without loss of generality, that there exist a and b such that $a \in O_{-i}(v_i)$, $b \in O_{-i}(u_i)$, $b \in (a, argmax\,(u_i\,;A_f)\,]$, and $[a, b) \cap O_{-i}(u_i) = \emptyset$. (See Figure 2.) Denote Π the straight line passing a and b. As in step 3, we construct a sequence of utility functions $u^{(n)}$ in Ω_f :

$$u^{(n)}(x) = -(x-a)'H^{(n)}(x-a)\,,$$

where $H^{(n)}$ is also similarly defined. Consider the sequence of allocations $\{ f(u_i, u^{(n)}_{-i}) \}$ where $u^{(n)}_j = u^{(n)}$ for all $j \neq i$. By a similar argument as in step 3, we can show that $\{ f(u_i, u^{(n)}_{-i}) \}$ converges to b. Since v_i is quasi-concave,

$$v_i(\, argmax\,(v_i\,;A_f)) > v_i(b) > v_i(a)\,.$$

Therefore, there is a finite n_o such that $v_i(f(u_i, u^{(no)}_{-i})) > v_i(a) = v_i(f(v_i, u^{(no)}_{-i}))$. This contradicts that f is strategy-proof. **Q.E.D.**

<u>Step 5</u>: Either (i) $O_{-i}(u_i) = argmax\,(u_i\,;A_f)$, for all $u_i \in \Omega_f$, or

(ii) $O_{-i}(u_i) = A_f$, for all $u_i \in \Omega_f$.

Proof: We first show that for any given $u_i \in \Omega_f$, either $O_{-i}(u_i) = \text{argmax}\,(u_i\,;A_f)$, or $O_{-i}(u_i) = A_f$. If this is not true, then there exist a and b such that $a \notin O_{-i}(u_i)$,

$b \in O_{-i}(u_i)$, and $b \neq argmax\ (u_i; A_f)$. (See Figure 3.) The condition that $dim\ (A_f) \geq 2$ further guarantees that we can find a pair like this also satifying that a, b, and $argmax\ (u_i; A_f)$ are in general position, *i.e.*, they do not lie on the same straight line. Let Π_1 denote the straight line passing a and $argmax\ (u_i; A_f)$, and Π_2 the straight line passing a and b.

Take $u(x) \equiv -\| x - a \|^2$ and shrink the indifference curve of it to Π_2 in all directions orthogonal to Π_2. We can get a utility function $v \in \Omega_f$ such that $a = argmax\ (v; A_f)$, and any point on $[\, a, argmax\ (u_i; A_f)\,]$ does not belong to $Argmax\ (v; O_{-i}(u_i))$. Since $Argmax\ (v; O_{-i}(u_i))$ is compact, we can find $v_i \in \Omega_f$, by applying the same procedure to $-\| x - argmax\ (u_i; A_f) \|^2$, such that $argmax\ (v_i; A_f) = argmax\ (u_i; A_f)$, and $v_i(a) > v_i(c)$ for any c in $Argmax\ (v; O_{-i}(u_i))$. By Lemma 2, for profile (v_i, v_{-i}), where $v_j = v$ for all $j \neq i$, $f(v_i, v_{-i}) \in Argmax\ (v; O_{-i}(v_i)) = Argmax\ (v; O_{-i}(u_i))$, since $O_{-i}(u_i) = O_{-i}(v_i)$ by step 4 . But when agent i with utility function v_i falsely announces that he has utility function $-\| x - a \|^2$, the allocation would be a by Lemma 2, which is better to him than $f(v_i, v_{-i})$. This contradicts that f is strategy-proof.

Now we show that if for some $u_i \in \Omega_f$, $O_{-i}(u_i) = argmax\ (u_i; A_f)$, then for all $v_i \in \Omega_f$, $O_{-i}(v_i) = argmax\ (v_i; A_f)$. Suppose that it is not true. By what we have just shown, there must exist some $u_i \in \Omega_f$ such that $O_{-i}(u_i) = A_f$. Notice $argmax\ (v_i; A_f) \neq argmax\ (u_i; A_f)$ by step 4 . Add to them some $c \in A_f$ such that these three points are in general position. Then an argument similar to that in the above paragraph will lead to a contradiction.

Q.E.D.

<u>Step 6</u> : There is an agent i such that for all $u_i \in \Omega_f$, $O_{-i}(u_i) = argmax\ (u_i; A_f)$.

Proof : We proceed by induction for n, the number of the agents . The case $n = 2$ is simple. If the statement is not true, then for any (u_1, u_2), $O_{-1}(u_1) = A_f$, and $O_{-2}(u_2) = A_f$. f

is strategy-proof implies $f(u_1,u_2) = argmax(u_1; A_f)$, and $f(u_1, u_2) = argmax(u_2; A_f)$. But it is impossible for any profile (u_1, u_2), where $argmax(u_1; A_f) \neq argmax(u_2; A_f)$.

Now we assume the statement is true for $n = k$. Let us consider the case $n = k + 1$. If the statement is false, then by step 5, for any profile $(u_1, u_2, ..., u_{k+1})$, $O_{-i}(u_i) = A_f$ for all agents. Therefore when we fix any agent's utility function, f is still a mechanism for the other k agents which satisfies the conditions of the theorem. If we first fix some $v_1 \in \Omega_f$, then by the induction hypothesis, we can find a dictator $i \neq 1$. Hence for any $u_{-1} \in \Omega_f^{n-1}$,

$$f(v_1, u_2, ..., u_{k+1}) = argmax(u_i; A_f).$$

But if we fix some v_i such that $argmax(v_i; A_f) \neq argmax(v_1; A_f)$, again by the induction hypothesis, we can find another dictator $j \neq i$ so that for any $u_{-i} \in \Omega_f^{n-1}$,

$$f(u_1,u_2, ..., v_i, ..., u_{k+1}) = argmax(u_j; A_f).$$

If we choose a particular u_{-i} such that $v_k = v_k$ for $k \neq i$, then the above two equations lead to a contradiction. Thus we complete the induction. **Q.E.D.**

Up to now, we have found a dictator i for the restriction of f on Ω_f^n. The final thing to do is to demonstrate that he must be a dictator for f on Ω^n as well. To see that, consider the profiles in which the dictator's preference belongs to Ω_f, while the other agents might have preferences outside it. If there were such a profile that could lead to an allocation different from the dictator's best, then it is easy to find a manipulation by some agents other than the dictator. Thus as long as the dictator announces some prefence in Ω_f, the mechnism f has to choose the dictator's best alternative. Therefore the dictator can always enforce any allocation $a \in A_f$ by claiming that he has the utility function $-\|x - a\|^2$. This concludes the proof of Theorem 2.

4. AN APPLICATION

In the problem put forward at the beginning of the paper, admissible preferences are also increasing. This property is very common in many problems associated with pure public goods. In this section we apply our result to a general model of such problems.

We assume $A^* = E^k_+$, the nonnegative orthant of E^k, $k \geq 3$. Ω^* is the space of all continuous, strictly quasi-concave, and strictly increasing preferences on E^k_+. This model is standard and has been considered by many authors.[6] Since we are considering allocation mechanisms that are related to decision-making instead of just ranking alternatives, some feasibility constraint should be imposed. It not only makes the model more realistic, it also keeps the problem well-defined (as we will see that the range of any strategy-proof mechanism must be properly bounded). For simplicity, we assume that it is given by

$$A^{**} = \{ x \in E^k_+ \mid \Sigma\, p_i x_i \leq I \},$$

where p_i 's and I are all positive numbers with p_i representing the price of public good i and I the total budget. Therefore, A^{**} is a convex set in E^k_+, and its boundary $Bd\,(A^{**})$ is given by $\{ x \in E^k_+ \mid \Sigma\, p_i x_i = I \}$. A mechanism is again a function $f: (\Omega^*)^n \to A^{**}$, which chooses a feasible allocation for every preference profile in $(\Omega^*)^n$. Finally, we impose another condition that is intuitively appealing.

<u>Definition 4</u>: A mechanism f is *unanimous* if for any $u \in \Omega^*$,

$$f\,(u, u, \ldots, u) = argmax\,(u\,; A^{**}).$$

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[6] For example, Kalai, Muller, and Satterthwaite [1979] considered this model and proved the nonexistence of Arrovian social welfare functions.

Unanimity, as we defined above, is a very natural and mild requirement. But our next theorem demonstrates that a unanimous mechanism violates incentive compatibility unless it is dictatorial. It is also worth mentioning that we do not need to add any dimensional condition on the range of f once we require unanimity.

THEOREM 3 : *Any unanimous mechanism* $f: (\Omega^*)^n \to A^{**}$ *is strategy-proof if and only if it is strongly dictatorial.*

Proof: It is trivial that f is strongly dictatorial implies that f is strategy-proof. Now assume that f is a strategy-proof and unanimous. If it can be shown that Ω^* is abundant on the range of f, then f is dictatorial according to Theorem 2. Thus f is strongly dictatorial since $Argmax\,(u\,; A^{**})$ is always singleton when f is strictly quasi-concave.

We first show that the range of f is $Bd\,(A^{**})$. Given $a \in Bd\,(A^{**})$, it is easy to find some $u \in \Omega^*$ such that $a = argmax\,(u\,; A^{**})$. Thus, that f is unanimous implies that a belongs to the range of f. On the other hand, suppose a belongs to the range of f. Let us find a preference profile $(u_1, u_2, ..., u_n)$ such that $a = f(u_1, u_2, ..., u_n)$. We normalize $u_1, u_2, ..., u_n$ so that $u_1(a) = u_i(a)$ for all $i \neq 1$, and then construct a utility function u by

$$u(x) = Min_{1 \leq j \leq n} \{ u_j(x) \} + Min_{1 \leq i \leq k} \{ (x_i+1) / (a_i+1) \}.$$

Since $u_i \in \Omega^*$ for all i, $Min_j \{ u_j(x) \} \in \Omega^*$. It is obvious that $Min_i \{ (x_i+1) / (a_i+1) \}$ is continuous, weakly quasi-concave, and weakly increasing. Therefore, $u \in \Omega^*$. And we further claim that if $u(x) \geq u(a)$ for some $x \neq a$, then $u_i(x) > u_i(a)$ for all i. To see this, we consider the following two cases : (i) $x_i \geq a_i$ for all i ; or (ii) $x_i < a_i$ for some i. In the first case, since u_i is strictly increasing, $u_i(x) > u_i(a)$ for all i. In the second case, $Min_i \{ (x_i+1) / (a_i+1) \} < 1 = Min_i \{ (a_i+1) / (a_i+1) \}$. Because $u(x) \geq u(a)$, it must be true that $Min_j \{ u_j(x) \} > Min_j \{ u_j(a) \}$. This also leads to $u_i(x) > u_i(a)$ for all i. We now replace u_1 in $(u_1, u_2, ..., u_n)$ by u. That f is strategy-proof implies

$$u (f (u , u_2, ..., u_n)) \geq u (f (u_1, u_2, ..., u_n)) = u(a) , \quad \text{and}$$

$$u_1(a) = u_1(f (u_1, u_2, ..., u_n)) \geq u_1 (f (u , u_2, ..., u_n)) .$$

These two inequalities imply $f (u , u_2, ..., u_n) = a$. Doing this repeatedly for all i leads to $f (u , u, ..., u) = a$. Since f is unanimous, $a = argmax (u ; A^{**})$. This is possible only when $a \in Bd (A^{**})$. Therefore, the range of f is $Bd (A^{**})$.

Secondly we show that Ω^* is abundant on $Bd (A^{**})$. Given any quadratic function $u(x) = - (x - a)'H (x - a)$, we define $v_s(x)$ (s is a scalar) by :

$$v_s(x) = - (x - a - sH^{-1}p)'H (x - a - sH^{-1}p) .$$

We can write $v_s(x)$ as

$$\begin{aligned} v_s(x) &= - (x - a - sH^{-1}p)'H (x - a - sH^{-1}p) \\ &= u(x) - 2s (x - a)'p + s^2 p 'H^{-1}p . \end{aligned}$$

Since the second and the third terms are constants on $Bd (A^{**}) = \{ x \in E^k_+ \mid \Sigma p_i x_i = I \}$, $u(x)$ and $v_s(x)$ represent the same preference on $Bd (A^{**})$. The gradient of $v_s(x)$ is proportional to $s p - H (x - a)$. If we choose an s which is large enough, then all the first order derivatives of $v_s(x)$ are positive on any bounded set in E^k_+, especially on A^{**}. Therefore, v_s is also increasing on any bounded set in E^k_+. Then it is not difficult to construct some $w \in \Omega^*$ such that w coincides with v_s, thus u on A^{**}. Therfore, Ω^* is abundant on $Bd (A^{**})$. **Q.E.D.**

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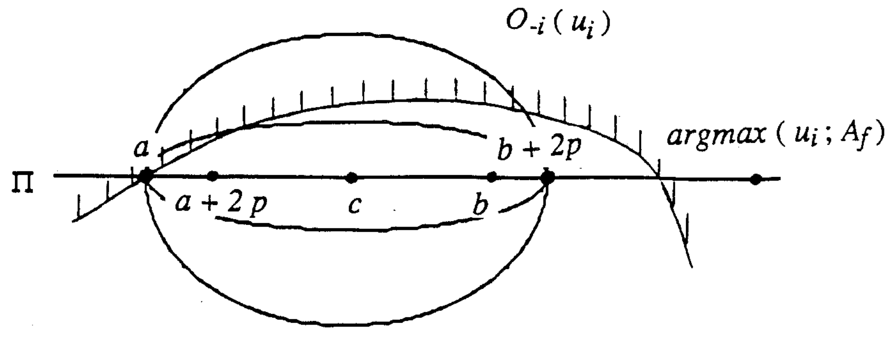

Figure 1

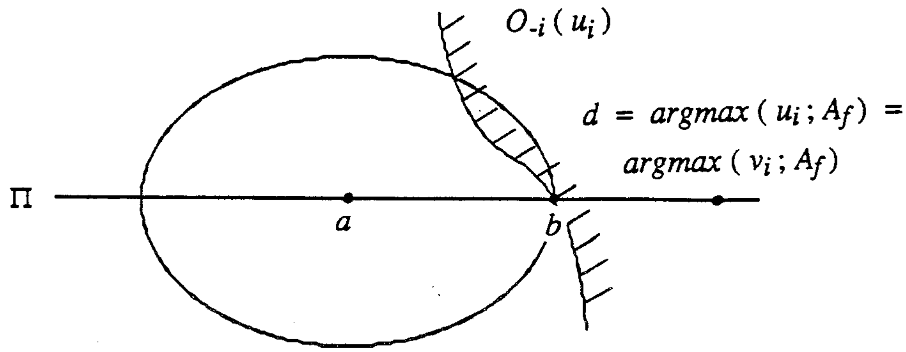

Figure 2

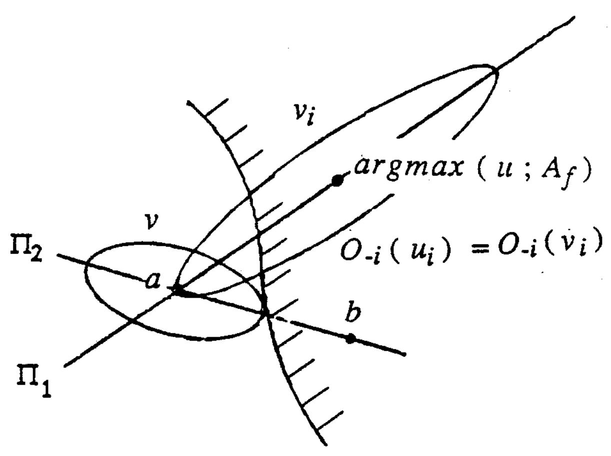

Figure 3