SUBGAMES AND THE REDUCED NORMAL FORM

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I establish that a reduced normal form (RNF) game N has a particular structure if and only if there is a game in the family of extensive form games having RNF N which has a subgame with an appropriate RNF. Examination of this structure suggests that what is important about subgames is not the particular form of the game tree, but rather a form of strategic independence between parts of the game. This strategic independence forms the basis for the definition of a "normal form subgame" which is closely related to the subgame in the extensive form but is immune to the criticism that it depends on seemingly irrelevant details of how a game is represented. This makes it possible to both motivate and implement such things as subgame perfection, subgame consistency, and backward induction without reference to the extensive form.

I

Introduction

This paper was motivated by the question "what do you do when twenty odd years of extensive form game theory have convinced you that there is something important about the subgame, but you are also convinced by the arguments of Thompson [1952] and Elmes and Reny [1987] that the reduced normal form (RNF) of a game is all that matters?" These two ideas are, after all, incompatible; for every extensive form game, there is another with the same RNF which has no subgames. As there are extensive form games with distinctively different subgames, but the same RNF, it is immediate that there will be no way of exactly recovering the "information" contained in the subgame structure of the original extensive form game. Thus, one may be able to look at a RNF and say "these are the extensive form games for which this is the RNF," but there is no way of

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singling out any particular one, and thus in specific, any particular set of subgames.

One way to approach this is to ask whether there is some more fundamental property of a subgame, something other than a set of conditions on the information sets and order structure of an extensive form tree, which still captures our fundamental intuition about why a subgame is important, but does not depend on "irrelevant details" of an extensive form game. Such a property would form the basis for a "normal form subgame," that is, a reasonable way of decomposing a RNF game into subproblems in much the same way as one decomposes an extensive form game into subgames.

To find such a property, I devote the majority of the paper to exploring the relation between the structure of a RNF game N and the structure of the elements of the family of extensive form games having RNF N, E(N), especially as regards the subgames of the elements of E(N). I establish that there will be an element of E(N) with a subgame having a particular RNF if and only if N has a certain structure.

This structure is characterized by a form of strategic independence between the restriction of the game to certain subsets of the players' strategy sets, and the remainder of the game. Even in the absence of any relation to the subgame, this independence would form the basis for an interesting subproblem in the RNF. However, the results of this paper provide a stronger motivation, by establishing a very close relation to the conventional subgame: such a subproblem can be found in the RNF if and only if a subproblem (subgame) with the same RNF can be found in some element of E(N). I thus term the structure a Normal Form Subgame.

It should be emphasized that this paper has two distinctly different aspects, the theorem relating extensive and normal form structures, and the interpretation I make of that relation. The validity of the theorem is independent of the validity of the opinion I advance about its interpretation. I would argue that much of its interest is as well: a reasonable first step in a better understanding of the relation between various extensive form and normal form solution concepts must surely be a better understanding of the relation between the extensive form and normal form structures on which they are defined.

The organization of the paper is a follows. Section II covers definitions and notation. Some of this material is standard and provided for the sake of completeness while much of the remainder is important only for a detailed understanding. The reader may wish to skip it on a first reading, referring back to it for specific terminology which is unfamiliar. Section III discusses the relation between subgames in extensive form games and their associated RNFs when the RNF is defined as containing no payoff equivalent pure strategies; I start with this case because the heart of the relation can be more clearly expressed for this RNF. I begin with a series of examples, and then go on to state and prove Theorem 1, the main theorem of the paper. Section IV begins with an example of why things become slightly more complicated when one extends the idea of the RNF to include removing strategies which are payoff equivalent to convex combinations of other strategies. I then present the extension to Theorem 1 which covers this case. In Section V, I discuss the implications of the theorem for understanding normal form games. I also present some very preliminary thoughts as to solution concepts defined in terms of this structure. Section VI briefly discusses other implications of equality restrictions in the normal form. I conclude in Section VII.

II

Terminology and Preliminaries

Borrowing from van Damme [1984], an extensive form game E is defined by: (1) a finite tree, (2) a partition of the non-terminal nodes of the tree between n players, denoted 1,...,n, and a nature player, denoted 0, (3) a partition of each player's set of nodes into information sets, (4) a specification of a player's options at each of her information sets, (5) a probability distribution over nature's choices at each of her information sets, and (6) a function G assigning to each terminal node of the tree an element of R^n representing payoffs to players 1,...,n from reaching that terminal node.

E is of perfect recall if nodes belonging to some player which follow from distinct information sets or moves for that player are never in the same information set, i.e., players never forget what they previously knew or did.

E is of perfect information if each information set has only one node.

An information set t precedes an information set s if there is a sequence of choices leading from some node in t to some node in s. s succeeds t it t precedes s.

If a subset of the nodes of E is such that (1) any time a node is in the set, so are all the nodes which succeed it, (2) if a node is in the set, so is every node in the same information set as that node, and (3) there is a unique node t in the set which precedes every other node in the set, then we say that the set of nodes along with the appropriately restricted player and information set partitions, probability distributions for moves by nature, and payoff function is the subgame beginning at t.

A pure strategy for player i is a mapping assigning to each of i's information sets a choice at that information set. Call the set of all pure strategies for player i S_i.

The normal form of E is $N = (H, \{S_0, \ldots, S_n\})$ where $H: \overset{n}{\underset{i=0}{X}} S_i \to R^n$ is the payoff function. $H(s_0, \ldots, s_n)$ equals G evaluated at the terminal node which would in fact be reached if $\{s_0, \ldots, s_n\}$ were followed in the extensive form[2]. Payoffs for mixtures of pure strategies are defined from H in the obvious way. We will often abuse notation by failing to differentiate between H and its restriction to various domains. We will also take as given the appropriate redefinition of H to deal with relabellings of its domain.

Two games $G^A = (\alpha, \{A_0, \ldots, A_n\})$ and $G^B = (\beta, \{B_0, \ldots, B_n\})$ are isomorphic if $\exists\ C_i: A_i \equiv B_i,\ i = 0, \ldots, n,$ such that $\forall\ (a_0, \ldots, a_n) \in \overset{n}{\underset{i=0}{X}} A_i,\ \alpha(a_0, \ldots, a_n) = \beta(C_0(a_0), \ldots, C_n(a_n))$. That is, two games are isomorphic if they differ only in the names given to the individual strategies of each player[3].

Following Kuhn [1953], term an information set for player i relevant relative to i's strategy s_i if that information set can be reached given s_i. Call two strategies s_i and t_i realization equivalent if they reach the same terminal node for every given specification of the strategies for the remaining players (i.e., if s_i and t_i differ only at irrelevant information sets).

Call two strategies s_i and t_i payoff equivalent (PE) if $H(s_0, \ldots, s_{i-1}, s_i, s_{i+1}, \ldots, s_n) = H(s_0, \ldots, s_{i-1}, t_i, s_{i+1}, \ldots, s_n)\ \forall\ (s_0, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$.

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[2]The reader will note that the treatment of nature is non-standard, i.e., that we have departed from the normal practice of "collapsing" the game using the probabilities attached to nature's moves. This is done for two reasons. First, it makes some of the analysis simpler. Second, note that the definition of an extensive form subgame depends only on (1) to (4) of the first paragraph, and that (1) to (4) treat nature symmetrically with other players. It thus seems reasonable that a normal form structure closely related to the extensive form will also have this symmetry. Comment 2 following Theorem 1 suggests an alternate approach.

[3]One could extend this idea of isomorphism to allow renaming of the players, but as this adds nothing to the rest of the paper, we avoid the complication.

Realization equivalence implies payoff equivalence, but not conversely.

A <u>quasi reduced normal form</u> (QRNF) for E is $(H,\{Q_0,\ldots,Q_n\})$ where each Q_i is a subset of S_i such that (1) no two elements of Q_i are RE and (2) every element of $S_i\backslash Q_i$ is RE to some element of Q_i.

A <u>pure strategy reduced normal form</u> (PRNF) for E is $(H,\{P_0,\ldots,P_n\})$ where each P_i is a subset of S_i such that (1) no two elements of P_i are PE and (2) every element of $S_i\backslash P_i$ is PE to some element of P_i.

A <u>mixed strategy reduced normal form</u> (MRNF) for E is $(H,\{M_0,\ldots,M_n\})$ where each M_i is a subset of S_i such that (1) no element $m_i \in M_i$ is PE to any convex combination of the elements of $M_i\backslash m_i$, and (2) every element of $S_i\backslash M_i$ is PE to some convex combination of elements of M_i.

Any two PRNFs, MRNFs, or QRNFs for a given game are isomorphic. We will thus speak of <u>the</u> MRNF M(E), <u>the</u> PRNF P(E), and <u>the</u> QRNF Q(E) of E, taking the isomorphisms as understood. For generic extensive form games M(E), P(E), and Q(E) are the same. Further, M(E)=M(P(E))=M(P(Q(E))), or, what is the same thing, $M_i \subseteq P_i \subseteq Q_i$, i.e., one can find the MRNF of E in three steps, first checking that no two strategies always reach the same terminal node, then checking that no two of the remaining strategies always result in the same payoff, and finally checking that no remaining strategy can be replicated by a mixture of the remaining strategies. For convenience, we will refer to the strategies in Q(E) (and thus in P(E) and M(E)) simply by specifying what actions are taken at relevant nodes.

Q(E) can be found by an iterative process of removing a strategy s_i^1 from S_i which is RE to an element of $S_i\backslash s_i^1$, removing a strategy s_i^2 which is RE to an element of $S_i\backslash\{s_i^1,s_i^2\}$ etc., until no element $s_i^{t+1} \in S_i\backslash\{s_i^1,\ldots,s_i^t\}$ is RE to an element of $S_i\backslash\{s_i^1,\ldots,s_i^{t+1}\}$. A similar iterative process can be used to find

P(E) or M(E).

Call two games A and B <u>equivalent</u>, written A~B, if they have isomorphic MRNFs. Call two games A and B <u>P-equivalent</u>, written A≈B, if they have isomorphic PRNFs. We note for future use that isomorphism ⇒ equivalence ⇒ P-equivalence, and that isomorphism, equivalence, and P-equivalence are transitive. We are now in a position to discuss the main theorem of this paper.

III

Subgames and the Pure Strategy Reduced Normal Form

Assume an extensive form game E has a subgame E^s. Consider, for example E_1 of Figure 1a, where the particular subgame we shall be interested in is the one beginning at node 1_2, i.e., at player 1's second node. Assuming all of the labels are associated with distinct payoffs, P(E) is given by N_1 of Figure 1b. One could think of dividing a player's strategies in P(E) into two sets: those which take the unique action at each information set preceding E^s consistent with E^s being reached, and those which at some information set preceding E^s take an action which makes reaching E^s impossible. Call the first set ι_i, and the second set ω_i. For our example, strategies in ι_1 are ones which begin L_1, and strategies in ω_1 are those which begin R_1. Strategies in ι_2 begin l_1, and strategies in ω_2, r_1. We will concentrate on ι_1.

Given a choice of L_1, i.e., a choice in ι_1, there are only two relevant ways in which player 1's strategies can differ: in what they specify at 1_2, i.e., in the subgame, and in what they specify at 1_3, i.e., when the subgame can no longer be reached. Further, these choices are independent.

More generally, in Theorem 1, I show that a choice among the elements of

ι_i can always be represented as the product of two independent choices, one of what to do if the subgame is reached, and one of what to do when it is known that the subgame is no longer reachable. If we label the choices within the subgame by $1,\ldots,u_i$, and the choices for when it is known the subgame if no longer reachable by $1,\ldots,v_i$, then this implies that the elements of ι_i can be relabelled as $\{(l_i,k_i) | l_i \in \{1,\ldots,u_i\},\ k_i \in \{1,\ldots,v_i\}\}$.

In our example, $u_1 = 2$ and we can label L_2 as choice 1 within the subgame and R_2 as choice 2. Similarly $v_1 = 2$, and we can label $L_3 = 1$, and $R_3 = 2$. In the same manner, $u_2 = v_2 = 2$, $l_2 = 1$, $r_2 = 2$, $l_3 = 1$, $r_3 = 2$. The elements of ι_1 and ι_2 are labelled using this convention in Figure 1b. Thus for instance, (L_1,L_2,R_3) is relabelled $(1,2)$.

Now, if everyone is playing a strategy from ι_i, then E^s will indeed be reached, and it is irrelevant to the outcome what the players planned on doing in the event of the subgame not being reachable, i.e., k_i is irrelevant to the payoffs. Thus in Figure 1b, note that columns (1,1) and (1,2) do not differ when we restrict player 1 to strategies in ι_1.

Similarly, if anyone chooses a strategy in ω_i, then E^s is not reached, and it is irrelevant what people who played strategies from ι_i chose for within E^s, i.e., payoffs are independent of l_i. Thus, columns (1,1) and (2,1) are identical when we restrict player 1 to strategies in ω_1.

The above suggests some necessary conditions on a PRNF N for there to be an extensive form gave E with a subgame having a particular subgame E^s: (1) it must be possible to partition each player's strategy set into two sets, the first representing attempting to reach E^s, and the second representing making it impossible to reach E^s, (2) it must be possible to represent a decision among the elements of the first set as the product of two independent decisions, such that is everyone else is also playing from the first set, i.e., if E^s is reached, then only the first of the two decisions matters, while conversely, if anyone plays from their second set, then, among those who played from their first set, only the second of the two decisions matters. The obvious question is whether these conditions are also sufficient.

Figure 2a shows the simplest example of an N_2 satisfying these properties without being entirely degenerate. It should be clear that as player 2's decision is relevant if and only if player 1 chooses r_1, it makes no difference if player 2 knows player 1's decision before making his. So, we arrive at E_2 of Figure 2b.

In Figure 3a, I elaborate the example slightly. Now it does make a difference if player 2 knows player 1's move before making his. However, it is still true that if player 1 chooses r_3, then player 2's choice is irrelevant. Thus, if makes no difference if player 2 knows whether or not player 1 has chosen r_3 when he makes his move. Hence E_3 of Figure 3b.

In 4a, much the same story holds. Player 1's choice is clearly irrelevant if player 2 chooses c_3, and thus it makes no difference if player 1 knows whether or not this is the case when he makes his first move. Similarly, if player 2 is choosing between c_1 and c_2 , it makes no difference if he knows whether or not player 1 chose r_3. Thus E_4 of Figure 4b.

N_5 of Figure 5a is more difficult. As in the two previous examples, the conditions seem to be satisfied for the block with entries a,b,c, and d to be a simultaneous move subgame. However, now there is no simple structure such as that of N_3 where first player 1 could find out whether player 2 chose to play c_3 or not, and then conditional on playing $\{c_1,c_2\}$ player 2 would find out if player 1 chose r_3 or not. Now it is necessary to condition both players' receipt of the information as to whether their opponent chose within his first two strategies on whether they themselves choose their first two strategies. **Not only is the subgame simultaneous, but now the decision to enter it must be as well.**

An extensive form game structure which does this is illustrated in Figure 5b. In the first stage, players simultaneously choose whether to make the subgame reachable or not. In the second, if both players attempted to reach the subgame, then it is reached and both are told this. If either chose not to attempt to reach the subgame, then it is not reached.

There is not currently an information structure defined at the terminal nodes e,f, and g of Figure 5b. However, the importance of this example is that with the right information structure, arbitrarily complex extension can be 'grafted on' at each of these nodes. This will allow us to generate extensive form representation with the relevant subgame for any PRNF satisfying the structural requirements.

To see the right information structure, consider N_6 in Figure 6. Here, if player 2 plays to reach E^s, i.e., plays anything but c_5, but player 1 plays to make E^s unreachable, i.e., plays r_3, player 2 still has a decision to make. Further, conditional on playing $\{c_1,...,c_4\}$ in the first stage, he can be told in the second stage whether he is facing r_3 or $\{r_1,r_2\}$. Hence E_6 of Figure 6b.

In N_7 of Figure 7a, on the other hand, player 2 has an option if he chooses to 'veto' the subgame, i.e., to restrict himself to $\{c_3,c_4\}$. Thus, it makes a big difference if player 2 is told whether or not player 1 chose r_3, and we arrive at the information structure of Figure 7b. Player 2 is not told, if he chooses to restrict himself to $\{c_3,c_4\}$, what player 1 chose.

The general principle then is that players who chose to restrict themselves to strategies consistent with reaching the subgame in the first stage can be told in the second stage whether or not the subgame is reached. Players who 'vetoed' the subgame in the first stage are given no extra information.

Finally, consider a RNF such as N_8 of Figure 8a. Figure 8b shows an extensive form game with RNF N_8 having a subgame with RNF given by the 2 by 2 submatrix in the top left corner of N_8, while Figure 8c does the same for the 2 by 2 submatrix in the bottom right of N_8. However, by considering the implied sized if the players' strategy spaces, it can be seen that no one extensive form game having RNF N_8 can have subgames with both these RNFs. This demonstrates that in general the most we can ask is for a relation between the RNF and all the elements of E(N). There will not in general be any one extensive form game which captures all the subgames of the elements of E(N), even up to having the same RNF.

Using these examples as a guide, Theorem 1 can now be stated and proved.

Theorem 1:

Let $N = (H,\{S_0,\ldots,S_n\})$ and $M = (H,\{R_0,\ldots,R_n\})$ be PRNF games, with $R_i = \{r_1,\ldots,r_{ui}\}$. Then, $\exists$ an extensive form game E with P(E) = N having a subgame E^s with $P(E^s) = M$ if and only if each S_i can be partitioned as $\{\iota_i, \omega_i\}$ and the elements of ι_i relabelled as $\{(l_i,k_i) | (l_i,k_i) \in \{1,\ldots,u_i\}\chi\{1,\ldots,v_i\}\}$ (where $v_i = \text{size}(\iota_i)/u_i$) such that

(1) $H((l_0,k_0),\ldots,(l_n,k_n)) = H((l_0,k_0'),\ldots,(l_n,k_n')) = F(r_{l0},\ldots,r_{ln})$ $\forall\ 1 \leq l_i \leq u_i,\ 1 \leq k_i,k_i' \leq v_i$.

and

(2) if w.l.o.g. players $0,\ldots,m$ play $\tau_i \in \omega_i$, while players $m+1,\ldots,n$ play $(l_i,k_i) \in \iota_i$, then $H(\tau_0,\ldots,\tau_m,(l_{m+1},k_{m+1}),\ldots,(l_n,k_n)) = H(\tau_0,\ldots,\tau_m,(l_{m+1}',k_{m+1}),\ldots,(l_n',k_n))$ $\forall\ 1 \leq l_i,l_i' \leq u_i,\ 1 \leq k_i \leq v_i$.

Comment 1:

There is no loss of generality in assuming all players participate in the subgame: players who do not participate in the subgame are given a singleton strategy set (u_i=1). Similarly, it should be noted that the partition $\{\iota_i, \omega_i\}$ may have $\iota_i = S_i$ and $\omega_i = \emptyset$, i.e., some players may have no choice as to whether or not a given subgame is reached.

Comment 2:

If one uses the PRNF with the nature player "collapsed out," then the theorem becomes slightly more complicated. There will exist an extensive form game with a particular subgame if and only if there exists a group of conformable games such that (i) the original game is a convex combination of these games, and (ii) if the set of conformable games is interpreted as a game with a nature player then Theorem 1 is satisfied. The weights in the convex combination can be interpreted as the weights given by nature to the various "layers" of the game.

Proof:

To prove the "if" part of the theorem, assume we have ι_i and ω_i and a renaming of the elements of ι_i such that (1) and (2) hold. Consider the following 2 stage extensive form game with perfect recall (illustrated in Figure 9 for 2 players). Stage 1: all players simultaneously choose to restrict themselves either to ι_i or ω_i. Stage 2: all players simultaneously choose a strategy. If all players chose ι_i in the first stage, then all players are told this, and given a choice among $\{(l_i,1) | l_i \in \{1,...,u_i\}\}$. If not, then those players who chose ι_i in the first stage are told that not everyone chose ι_i, and given a choice among $\{(1,k_i) | k_i \in \{1,...,v_i\}\}$. Players who chose ω_i in the first period are given no extra information and choose among ω_i.

I define the payoffs of this game by specifying the payoff function for the QRNF. The strategies in the QRNF of E are $Q_i = \{(\iota_i,$ choice among $\{(1_i,1) | 1_i \in \{1,\ldots,u_i\}\}$ if everyone else also chooses ι_i, choice among $\{(1,k_i) | k_i \in \{1,\ldots,v_i\}\}$ otherwise$)\} \cup \{(\omega_i,$ choice $\tau_i \in \omega_i)\}$. This is written more compactly as $Q_i = \{((1_i,1),(1,k_i))\} \cup \{\tau_i\}$. The payoff function G of E is given by H taking as arguments the choices made in the second stage, i.e.,

$$G(((1_0,1),(1,k_0)),\ldots,((1_n,1),(1,k_n)) = H((1_0,1),\ldots,(1_n,1))$$

while if w.l.o.g. players $0,\ldots,m$ chose ω_i in the first stage then

$$G(\tau_0,\ldots,\tau_m,((1_{m+1},1),(1,k_{m+1}),\ldots,(1_n,1),(1,k_n)))$$
$$= H(\tau_0,\ldots,\tau_m,(1,k_{m+1}),\ldots,(1,k_n)).$$

I claim E defined as this game with E^s defined as the second stage which follows from everyone choosing ι_i in the first stage satisfies the theorem. There are three things to prove (a) that P(E) = N, (b) that E^s is indeed a subgame, and (c) that $P(E^s) = M$.

To prove (a), we start by showing N = Q(E) (i.e., that N and Q(E) are isomorphic). Define an isomorphism $\rho: Q_i \equiv S_i$ by $\rho((1_i,1),(1,k_i)) = (1_i,k_i)$ and $\rho(\tau_i) = \tau_i$. Note that

$$G(((1_0,1),(1,k_0)),\ldots,((1_n,1),(1,k_n))$$
$$= H((1_0,1),\ldots,(1_n,1))$$
$$= H((1_0,k_0),\ldots,(1_n,k_n))$$
$$= H(\rho_0((1_0,1),(1,k_0)),\ldots,\rho_n((1_n,1),(1,k_n))$$

and

$$G(\tau_0,\ldots,\tau_m,((1_{m+1},1),(1,k_{m+1}),\ldots,(1_n,1),(1,k_n)))$$
$$= H(\tau_0,\ldots,\tau_m,(1,k_{m+1}),\ldots,(1,k_n)).$$
$$= H(\tau_0,\ldots,\tau_m,(1_{m+1},k_{m+1}),\ldots,(1_n,k_n))$$

$$= H(\rho_0(\tau_0),\ldots,\rho_m(\tau_m),\rho_{m+1}((1_{m+1},1),(1,k_{m+1})),\ldots,\rho_n((1_n,1),(1,k_n))$$

where the first equality in each sequence is by definition of G, the second by (1) and (2) respectively, and the third by definition of ρ_i. Thus, N = Q(E) and so P(E) = P(Q(E)) = P(N). However, N is already a PRNF, so P(N) = N, and thus P(E) = N.

To show (b), let t be the first node of the second stage which results when everyone chooses ι_i in the first stage. The player who is called upon to move at t (i.e., the player who moves first in the extensive form representation of the second stage of the game) knows the first stage choice of all the players, and thus knows he is at t. Similarly, those players who follow in the second stage also know the first stage moves of all players, and thus they know they are within the set of nodes which follow from t. This set of nodes, along with the payoff structure inherited from E is thus a subgame.

To see (c), note that E^s has strategy sets $\{(1_0,1),\ldots,(1_n,1)\}$ and payoff function H and so by (1) $E^s \approx M$. Thus $P(E^s) = P(M) = M$.

To prove the "only if" part of the theorem, assume there exists E such that P(E) = N with a subgame E^s such that $P(E^s) = M$. Partition the information sets of player i in E into three sets: those which precede E^s, p_i, those within E^s, w_i, and those which are neither in E^s nor precede it, o_i. Partition S_i into ι_i', consisting of those strategies which make it possible to reach E^s, i.e., those which take the unique action at each information set preceding E^s which would lead towards E^s if the player were at the node in this information set which preceded E^s, and ω_i' consisting of those strategies which make it impossible to reach E^s. Strategies in ι_i' are those which take the unique action at each information set in p_i which is consistent with reaching E^s, strategies in ω_i' are those which take some other action at some information set in p_i.

Because p_i, w_i, and o_i are a partition of i's information sets, we can generate S_i, the set of all mappings from information sets to choices at those information sets as $S_{pi} \chi S_{wi} \chi S_{oi}$, where S_{pi} (respectively S_{wi}, S_{oi}) is the set of all mappings from information sets in p_i (respectively w_i, o_i) to choices at those information sets.

Let s_i^* be the unique element of S_{pi} which specifies taking the action at each information set which is consistent with reaching E^s. Then,

$$\iota_i' = s_i^* \chi S_{wi} \chi S_{oi}$$

and

$$\omega_i' = S_{pi} \backslash s_i^* \chi S_{wi} \chi S_{oi}.$$

Consider deriving P(E), i.e., reducing each S_i to a subset of itself such that no two remaining strategies are PE. Recall that this can be done by iteratively removing strategies which are PE to some strategy which remains. Begin by removing elements of ω_i' until no element of ω_i' is PE to any remaining strategy. Call what remains of ω_i' after all such removals ω_i. Any two remaining strategies which are PE must both be in ι_i'.

Assume $s_i^* \chi s_{wi} \chi s_{oi}$ PE $s_i^* \chi s_{wi}' \chi s_{oi}'$ where at least one of $s_{wi} \neq s_{wi}'$ or $s_{oi} \neq s_{oi}'$ holds. Take the case $s_{wi} \neq s_{wi}'$ (the argument is completely analogous if $s_{oi} \neq s_{oi}'$). Then,

$$s_i^* \chi s_{wi} \chi s_{oi} \text{ PE } s_i^* \chi s_{wi}' \chi s_{oi}'$$

$$\Rightarrow s_i^* \chi s_{wi} \chi s_{oi} \text{ PE } s_i^* \chi s_{wi}' \chi s_{oi} \ \forall\ s_{oi}$$

$$\Rightarrow \text{can reduce } \iota_i' \text{ to } s_i^* \chi S_{wi} \backslash s_{wi}' \chi S_{oi}.$$

Let $s_i^* \chi Q_{wi} \chi Q_{oi}$ be what remains of ι_i' after all such removals. I claim $(H,\{Q_{w0},\ldots,Q_{wn}\}) \approx M$. To see this, let q and t $\in Q_{wi}$. Then, if E^s is reached, $H(q_0,\ldots,q_{i-1},s_i^* \chi q \chi q_{oi},q_{i+1},\ldots,q_n) = H(q_0,\ldots,q_{i-1},s_i^* \chi t \chi q_{oi},q_{i+1},\ldots,q_n)$

if and only if q and t are PE in E^s, while if E^s is not reached,

$H(q_0,\ldots,q_{i-1},s_i^* \chi q \chi q_{oi},q_{i+1},\ldots,q_n) = H(q_0,\ldots,q_{i-1},s_i^* \chi t \chi q_{oi},q_{i+1},\ldots,q_n)$ independent of q and t because these strategies are identical on every information set outside E^s. Thus,

$$q \text{ PE } t \text{ in } E^s \Leftrightarrow s_i^* \chi q \chi q_{oi} \text{ PE } s_i^* \chi t \chi q_{oi},$$

so it is clear that Q_{wi} is a relabelling of R_i. That $H(s_0^* \chi r_0 \chi q_{oi},\ldots,s_n^* \chi r_n \chi q_{on}) = F(r_0,\ldots,r_n)$ is also clear. If we take $v_i = \text{size}(\iota_i)/u_i$, then we can clearly relabel Q_{oi} as $\{q_{oi}^1,\ldots,q_{oi}^{vi}\}$. This suggests a natural relabelling of ι_i as $\{(l_i,k_i)|(l_i,k_i) \in \{1,\ldots,u_i\}\chi\{1,\ldots,v_i\}\}$ where $s_i^* \chi r_{li} \chi q_{oi}^{ki}$ is renamed (l_i,k_i). It is immediate that ι_i and ω_i with this relabelling satisfy (1) and (2).

IV

Subgames and the Mixed Strategy

Reduced Normal Form

Consider the extensive form game of Figure 10a. Its PRNF, given by Figure 10b, has the now familiar structure. However, the MRNF of this game is illustrated in Figure 10c, and it can be seen that the structure we have found so far has been damaged. What is different about this example? Essentially, the "only if" part of Theorem 1 used that when two pure strategies are PE, one can remove either of them. Thus, if a row in ι_i and a row in ω_i were PE, one could remove the row in ω_i. When one is using payoff equivalence to mixed strategies, this no longer holds. x may be PE to some convex combination of y and z without y being PE to any convex combination of x and z or z being PE to any convex combination of x and y. Thus, if a strategy in ι_i is PE to some convex combination of strategies not all in ι_i, there may be no choice but to remove that strategy. An argument similar to that used in Theorem 1 shows that if a strategy in ι_i is PE to a convex combination of elements all of which are in ι_i, it will be possible to remove an entire class of strategies, i.e., to reduce either u_i or v_i by 1.

Using the above as motivation, Theorem 2 is stated without proof.

Theorem 2:

Let $N = (H,\{S_0,\ldots,S_n\})$ and $M = (F,\{R_0,\ldots,R_n\}$ be MRNF games, with $R_i = \{r_1,\ldots,r_{ui}\}$. Then, $\exists$ an extensive form game E with $M(E) = N$ having a subgame E^s with $M(E^s) = M$ if and only if $\exists$ ι_i, ι_i' and ω_i and a relabelling of each ι_i as $\{(l_i,k_i)|(l_i,k_i) \in \{1,\ldots,u_i\}\chi\{1,\ldots,v_i\}\}$ (where $v_i = \text{size}(\iota_i)/u_i$) such that

(1) ι_i' and ω_i partition S_i

(2) $\iota_i = \iota_i'$ plus some finite number of distinct non-pure strategies which put positive weight on some element of ω_i

(3) $H((l_0,k_0),\ldots,(l_n,k_n)) = H((l_0,k_0'),\ldots,(l_n,k_n')) = F(r_{10},\ldots,r_{1n})$ $\forall\ 1 \leq l_i \leq u_i,\ 1 \leq k_i,k_i' \leq v_i$.

and

(4) if w.l.o.g. players $0,\ldots,m$ play $\tau_i \in \omega_i$, while players $m+1,\ldots,n$ play $(li,k_i) \in \iota_i$, then $H(\tau_0,\ldots,\tau_m,(l_{m+1},k_{m+1}),\ldots,(l_n,k_n)) = H(\tau_0,\ldots,\tau_m,(l_{m+1}',k_{m+1}),\ldots,(l_n',k_n))$ $\forall\ 1 \leq l_i,l_i' \leq u_i,\ 1 \leq k_i \leq v_i$.

Comment 1:

ι_i' and ι_i differ on a subset of those extensive form games for which M(E) and P(E) are not the same. For generic payoffs on the terminal nodes, M(E) and P(E) are the same.

Comment 2:

Comment 2 following Theorem 1 (concerning games with a nature player) again applies.

Proof:

A simple but notationally tedious corollary to Theorem 1.

V

The Normal Form Subgame

This section briefly considers the interpretation and applications of the structure introduced by Theorems 1 and 2. I begin by defining a normal form subgame (NFS) using the results above as a guide. I then discuss the interpretation and motivation of the NFS. Next, I discuss some applications of the NFS in understanding games and their solutions. More extensive interpretation and application of the normal form subgame and the derivation, interpretation, and application of other normal form structures is the subject of current joint research by George Mailath, Larry Samuelson, and myself.

Assume RNF's M and N have the relation given by Theorem 2. Then, I will call M a normal form subgame (NFS) of N. Theorem 2 can thus be restated: a RNF game N will have a NFS M if and only if some extensive form game with MRNF N has a subgame with MRNF M.

In the introduction to the paper, I asked whether there might be a more fundamental property of the subgame, something which captures our intuition as to why the subgame is important, but does not depend on "irrelevant" details of the extensive form. I would argue that the strategic independence captured by the definition of the NFS is a good candidate for this more fundamental property of a subgame.

The argument is that when a player is making a decision which only matters under certain circumstances, then in making that decision, the player should reason as if those circumstances held. So, consider that some player is making some decision about his strategy choice, for example, the weighting to give one of his pure strategies, or the relative weighting to give one group of his strategies compared to another, and ask two questions: (a) for what pure strategy choices by the remaining players does this decision matter? and (b) what is relevant about the remaining player's choices among these strategies for this decision? Now, for each of the remaining players, conduct the same experiment for the choices identified in the last step. That is, ask what strategies for the other players make these choices relevant, and what choices by the remaining players over these strategies are relevant. One could imagine iterating this procedure, at each stage asking what strategy sets, and choices over those strategy sets, are relevant to the choices identified at the last stage.

An important property shared by extensive and normal form subgames is that they are fixed points of this reasoning. Assume that M is a NFS, and consider the problem of players choosing the l_i. This decision matters only when all the players are choosing from ι_i, and the only choice that matters is the choice among the l_i. Put differently, to make my choice of how my strategy will project onto M I need only consider your choice of how your strategy will project onto M, and vice versa. There is thus a strong strategic independence between the subgame and the game as a whole.

There are at least two interesting applications of the NFS to understanding games and their solutions. The first of these is to the question of subgame consistency. Loosely, a solution concept is said to be subgame consistent if the restrictions it requires of the solution as a whole are also satisfied when the solution is restricted to the subgame. As an example, subgame perfection can be thought of as the minimally restrictive strengthening of Nash equilibrium which satisfies subgame consistency in a given extensive form. Subgame consistency is one possible desiderata for solution concepts (see for instance Hillas, 1989). A major application of the idea of subgame consistency and other "backwards induction-like" properties has been as a desiderata for stability. As the entire concept of stability presupposes that the RNF contains all that is relevant about a game, it is somewhat problematic that both the definitions and the motivations for these various backwards induction properties of stability rely on the extensive form.

The introduction of a definition for a subgame depending only on the normal form is thus useful in two ways. First, the discussion of the preceding paragraphs as to the strategic independence of the NFS suggests that the projection of our strategies onto a NFS should be in some way reasonable, i.e., that a reasonable strategy for N should also satisfy some reasonableness criteria when projected onto M. This provides an argument for subgame consistency (and backwards induction in general) which is phrased purely in terms of the RNF. Second, solution concepts defined in terms of the normal form, such as stability or properness, can now be judged on their subgame consistency without reference to any particular extensive form.

A second application of the NFS is in defining solution concepts. As an example, one could think about defining normal form subgame perfect equilibria. This sort of application, both of the NFS and related structures, is a major focus of Mailath et al.

Two interesting problems, both of which are mirrored in extensive form analysis, arise when taking about normal form subgame consistency or about solution concepts using the NFS.

First, as normal form strategies are currently defined, it is not always meaningful to speak of a strategy's projection onto a given NFS: one or more of the players may not be putting any weight on any strategy in ι_i. This is in contrast to the extensive form, where we specify not only what is chosen at relevant information sets, but what is chosen at irrelevant ones as well.

A normal form analog to this specification of strategies "out of equilibrium" might involve a sort of normal form sequentiality in which strategies are not only specified by the weight they put on any given pure strategy, but also by the relative weights they put on the elements of subsets of the pure strategies, even though the total weight given to the entire subset may be zero. Specifying strategies in this way would remove any problems with what one meant by the projection of a strategy profile onto the NFS. We hope to develop and extend this idea in Mailath et al.

Second, it is entirely possible that some NFS being reached may be inconsistent with rationality common knowledge. Thus, in the experiment of the last paragraph, asking a player to make his decision on the hypothesis that it matters may also require that the player hypothesize that his opponent is irrational (or that rationality common knowledge has been violated at some higher level). What is would mean for a solution to have a "reasonable" projection onto such a NFS is thus unclear. This observation is interesting in light of Reny's (1988) contention that the difficulties of common knowledge of rationality do not occur in simultaneous (i.e., normal form) games.

VI

Some Other Implications

This paper suggests that equality restrictions on payoffs are important. In fact, I would go so far as to suggest that both the normal and the extensive forms are incomplete representations of games, and that this is the true source of much of the debate concerning their relative merits. The missing component of both descriptions is an explicit description of the equality restrictions that the modeler is willing to assume exist across the payoffs to different strategies. The extensive form already incorporates one such restriction which is commonly made and which the normal form fails to capture: the payoffs associated with strategy profiles do not depend on "out of equilibrium" behavior. However, changing the particular extensive form used changes the restriction implied. By incorporating these restrictions explicitly, the problem is resolved.

A similar point applies to the subgame. I have argued that the subgame can in fact be more properly viewed as a set of restrictions on the payoff function. The extensive form implicitly captures these restrictions for conventional subgames. However, seemingly innocuous changes to the extensive form have the effect of creating and destroying subgames. By imposing these restrictions explicitly, we again lose our dependence on a specific extensive form.

Recognizing the importance of these restrictions also has implication for metrics on games. A common metric used for normal form games is the euclidean: if the payoffs in two normal form games of the same dimensions do not differ very much, then the two games are said to be close to each other. However, the two games may satisfy very different equality restrictions on their payoffs. Arbitrarily small changes as measured by the euclidean metric are very large changes in terms of the underlying structure of the game.

VII

Conclusions

An important "structural feature" of extensive form games is the subgame. This paper demonstrates that there is a close relative to the subgame depending only on the RNF. Thus, at least as concerns subgames, the structural information of extensive form games is captured by the RNF. This lends support to the contention that the RNF contains all the strategically relevant information of a game.

The existence of a normal form subgame, and therefore normal form analogs to backward induction and subgame consistency, removes any inconsistency in asking that these should be important properties of normal form solution concepts.

The normal form subgame also suggests some interesting new normal form solution concepts, the most obvious of which is normal form subgame perfection. Less obviously, the question of the projection of a solution onto a given normal form subgame leads us to the idea that the description of normal form equilibrium strategies is incomplete. This paper suggests that a natural analog to specifying behavior at irrelevant nodes is to specify the relative probabilities within subsets of a player's strategies, even though the total weight given that subset may be zero.

Finally, this paper emphasizes the more general point that normal form games do have structure, and that one aspect of this structure is equality restrictions in the normal form. Thus, rather than being viewed as non-genericities, equalities in the normal form should be taken as suggestive of underlying structure in the strategic problem being modelled.

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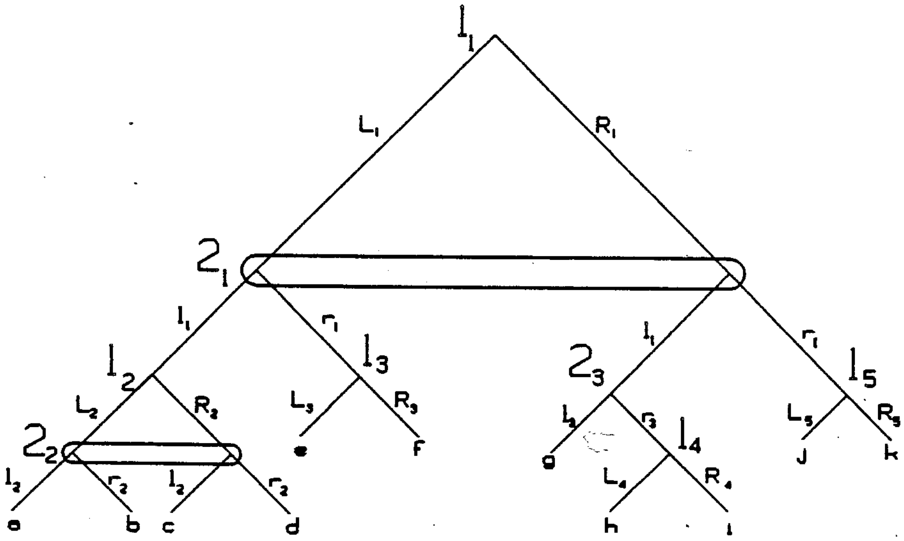


E_1

FIGURE 1a

				ℓ_2				ω_2
				(1.1)	(2.1)	(1.2)	(2.2)	
				1_1				
				$l_2 l_3$	$r_2 l_3$	$l_2 r_3$	$r_2 r_3$	r_1
ℓ_1	(1.1)	L_1	$L_2 L_3$	a	b	a	b	e
	(2.1)		$R_2 L_3$	c	d	c	d	e
	(1.2)		$L_2 R_3$	a	b	a	b	f
	(2.2)		$R_2 R_3$	c	d	c	d	f
ω_1		R_1	$L_4 L_5$	g	g	h	h	j
			$R_4 L_5$	g	g	i	i	j
			$L_4 R_5$	g	g	h	h	k
			$R_4 R_5$	g	g	i	i	k

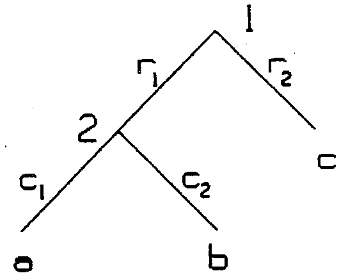
N_1

FIGURE 1b

$$N_2 = \begin{array}{c} \\ r_1 \\ r_2 \end{array} \begin{array}{cc} c_1 & c_2 \\ \left[\begin{matrix} a & b \\ c & c \end{matrix} \right. & \left. \right] \end{array}$$

$$\begin{array}{c|cc} & c_1 & c_2 \\ \hline r_1 & a & b \\ r_2 & c & c \end{array}$$

FIGURE 2a

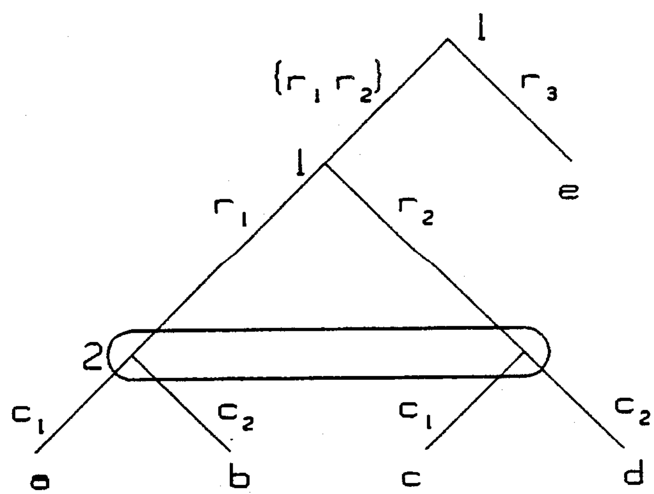


E_2

FIGURE 2b

$$N_3: \quad \begin{array}{c|cc} & c_1 & c_2 \\ \hline r_1 & a & b \\ r_2 & c & d \\ r_3 & e & e \end{array}$$

FIGURE 3a



E_3

FIGURE 3b

	c_1	c_2	c_3
r_1	a	b	f
r_2	c	d	f
r_3	e	e	f

N_4

FIGURE 4a

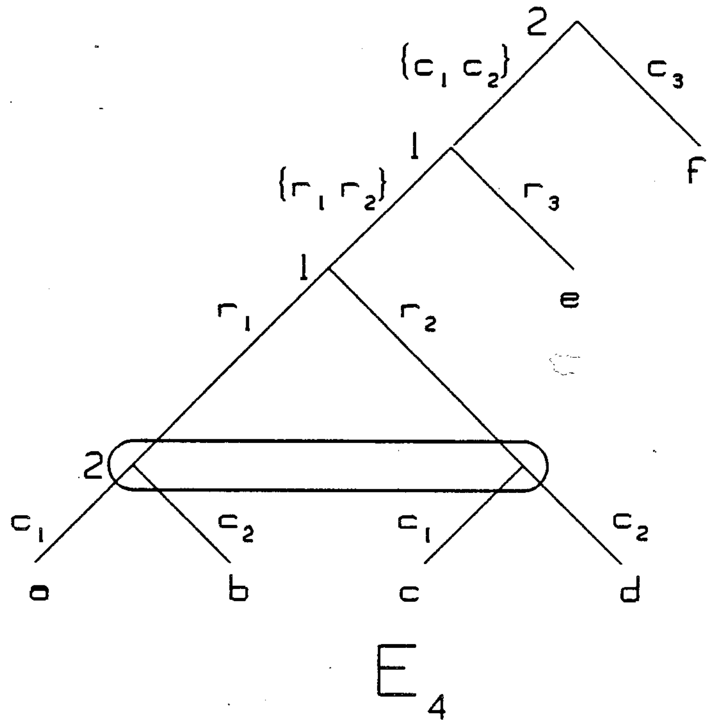


FIGURE 4b

	c_1	c_2	c_3
r_1	a	b	f
r_2	c	d	f
r_3	e	e	g

N_5

FIGURE 5a

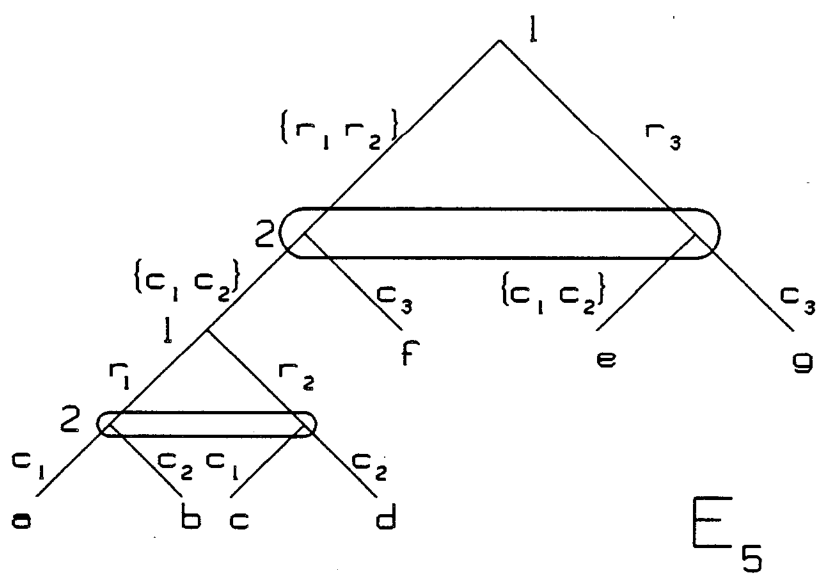


FIGURE 5b

	c_1	c_2	c_3	c_4	c_5
r_1	a	b	a	b	f
r_2	c	d	c	d	f
r_3	e	e	h	h	g

N_6

FIGURE 6a

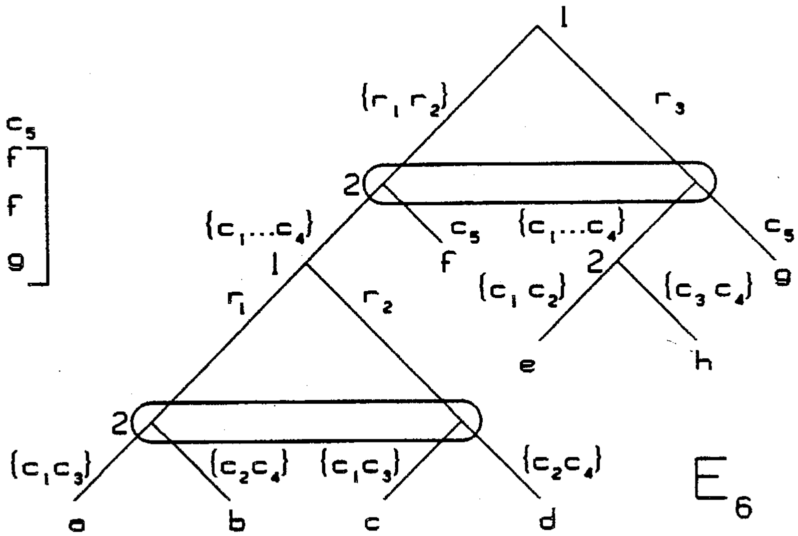


FIGURE 6b

	c_1	c_2	c_3	c_4
r_1	a	b	f	h
r_2	c	d	f	h
r_3	e	e	g	i

N_7

FIGURE 7a

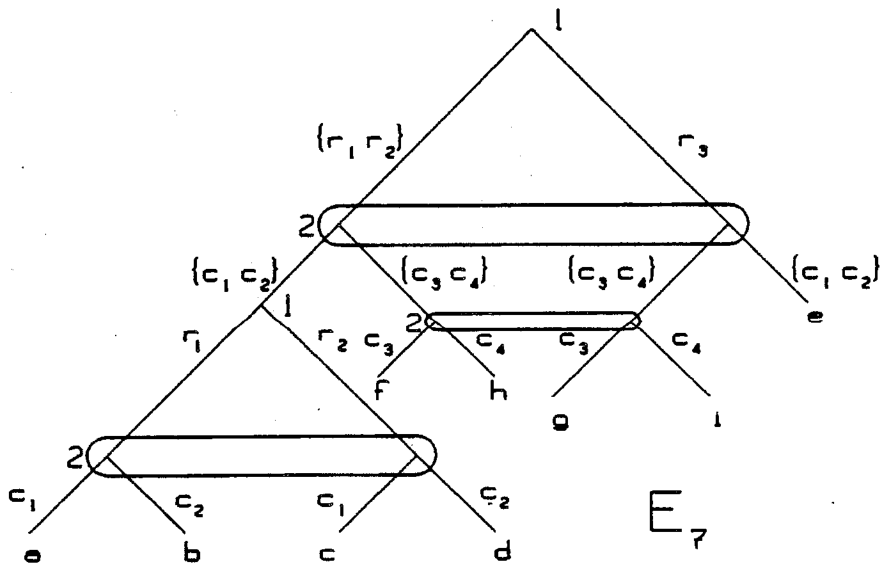


FIGURE 7b

	c_1	c_2	c_3
r_1	a	d	d
r_2	e	b	d
r_3	e	e	c

N_8

FIGURE 8a

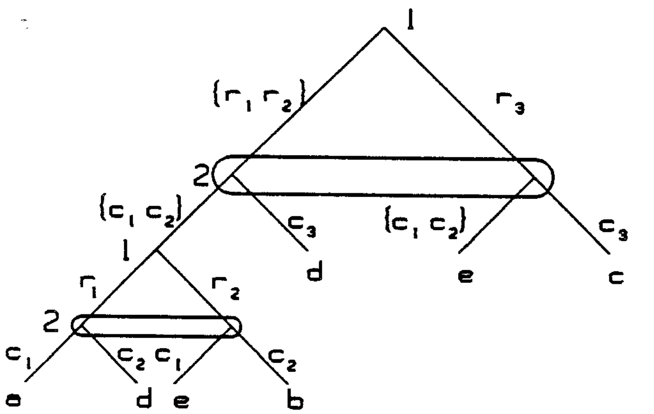


FIGURE 8b

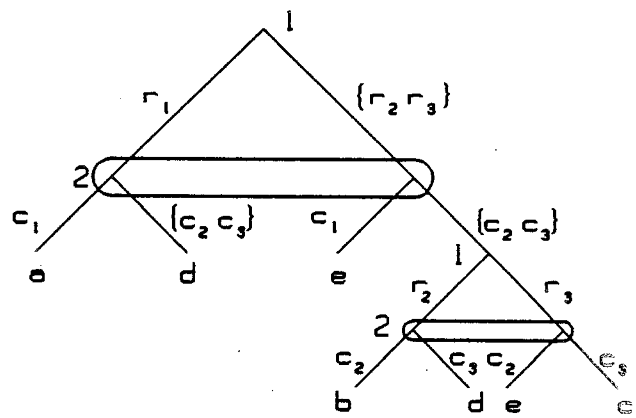


FIGURE 8c

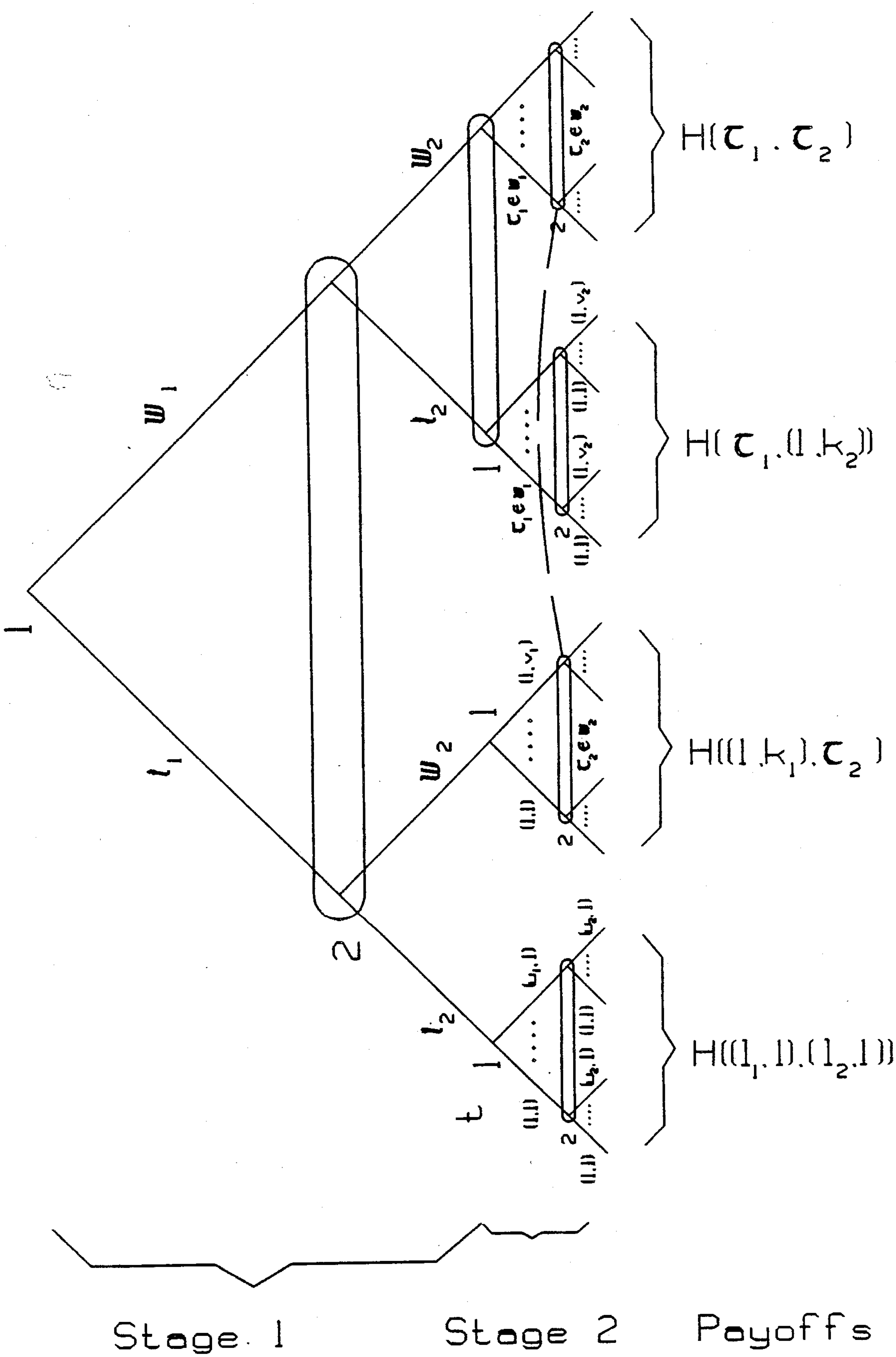


Figure 9

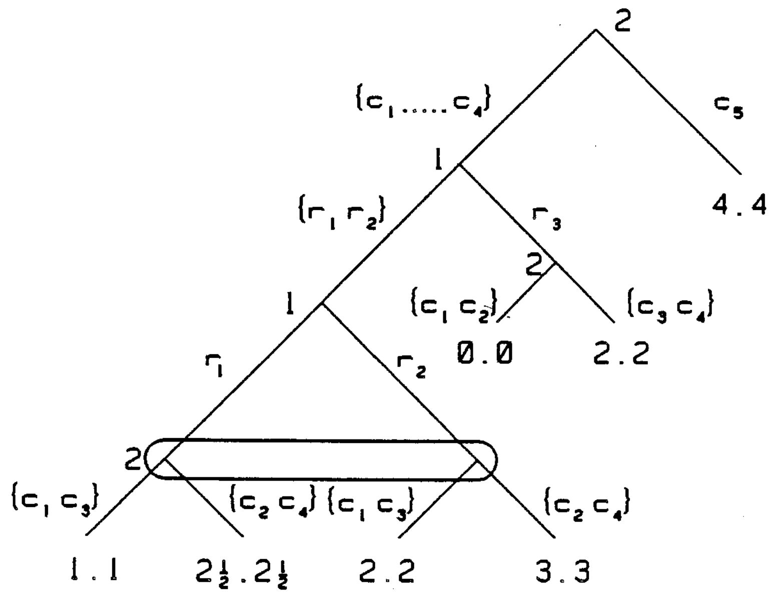


FIGURE 10a

	c_1	c_2	c_3	c_4	c_5
r_1	1.1	2½.2½	1.1	2½.2½	4.4
r_2	2.2	3.3	2.2	3.3	4.4
r_3	0.0	0.0	2.2	2.2	4.4

$c_4 = \frac{1}{2}c_1 + \frac{1}{2}c_5$

Figure 10b

	c_1	c_2	c_3	c_5
r_1	1.1	2½.2½	1.1	4.4
r_2	2.2	3.3	2.2	4.4
r_3	0.0	0.0	2.2	4.4

Figure 10c