TEST CONSISTENCY WITH VARYING SAMPLING FREQUENCY

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ABSTRACT

This paper considers the consistency property of some test statistics based on a time series of data. While the usual consistency criterion is based on keeping the sampling interval fixed, we let the sampling interval take any path as the sample size increases to infinity. We consider tests of the null hypotheses of the random walk and randomness against positive autocorrelation (stationary or explosive). We show that tests of the unit root hypothesis based on the first–order correlation coefficient of the original data are consistent as long as the span of the data is increasing. Tests of the same hypothesis based on the first–order correlation coefficient of the first–differenced data are consistent against stationary alternatives only if the span is increasing at a rate greater than $T^{\frac{1}{2}}$. On the other hand tests of the randomness hypothesis based on the first–order correlation coefficient applied to the original data are consistent as long as the span is not increasing too fast. We provide Monte Carlo evidence on the power, in finite samples, of the tests studied allowing various combinations of span and sampling frequencies. It is found that the consistency properties summarize well the behavior of the power in finite samples. The power of tests for a unit root is more influenced by the span than the number of observations while tests of randomness are more powerful when a small sampling frequency is available.

Key Words: Hypothesis testing, unit root, randomness, near–integrated processes, time series.

1. INTRODUCTION

Foremost among the various criteria which have been proposed to assess the adequacy of a test statistic is the consistency property. It is generally considered a necessary condition which insures that a statistic would distinguish a fixed false null hypothesis from a disjoint true alternative if the sample size is sufficiently large.

More formally, consider a parameterized testing procedure. Let θ_0 be the null value of θ which we want to test and $P_T(\theta)$ the power function of the test considered based on a sample of size T and evaluated at a fixed alternative $\theta \neq \theta_0$. A test statistic is said to be consistent if its power function $P_T(\theta)$, evaluated at any *fixed* alternative $\theta \neq \theta_0$, converges to one as the sample size increases to infinity when the true value is θ (e.g., Rao (1973), p. 464). Consistency justifies our belief that a larger sample leads to a more powerful testing procedure. Of course how large "sufficiently large" is depends on the particular circumstances.

As stated, the consistency criterion appears straightforward and desirable but some ambiguity arises when considering a test statistic in the context of a time series of data. In this case the total number of observation depends upon both the total span (S) of the available data as well as the sampling frequency (h) via the relation $T = S/h$. An increase in the sample size T can be achieved by any of the following schemes: increasing S (with h fixed), decreasing h (with S fixed), increasing S and decreasing h, increasing S faster than increasing h or decreasing h faster than decreasing S.

The requirement of a *fixed* alternative in the definition of consistency has usually implied that h is treated as fixed since in this case the parametric discrete time representation of the process is unchanged. Consider, for example, an autoregressive process of order one with a parameter $\alpha_A = \overline{\alpha}$, say, when sampled annually. When sampled quarterly the autoregressive parameter becomes $\alpha_Q = \overline{\alpha}^{\frac{1}{4}}$. Therefore the autoregressive parameter is not fixed as the sampling frequency is changed. It is argued, accordingly, that to maintain the requirement of a fixed alternative in the definition of consistency one should only consider increasing the sample size without changing the sampling interval, by increasing the span one for one with the sample size.

This argument is, however, not without ambiguity. Under both the annual and quarterly schemes the stochastic processes are the same. We simply used a different framework to represent them. Hence these processes could be represented using a common fixed parameter. In general one can represent this fixed parameter by considering a "basic" sampling interval. This could, for example, be the sampling interval believed to represent the actual timing of the process. Usually, however, this choice is arbitrary since there is generally no basis suggesting such an interval. In order to avoid this type of arbitrariness one can view the basic process as one occurring in continuous time and consider the discrete time representation of the process in terms of the parameters of this continuous–time process and the actual sampling interval.

In this context there are no reasons to view the consistency criterion solely as a sequence of tests with the sampling interval fixed as the sample size increases. It is useful to consider a more general power function, say $P_{T,h(T)}(\theta)$, indexed by both the sample size T and the sampling frequency h (or equivalently by T and S(T)). The sampling frequency, h(T), is indexed by T to highlight the fact that as T increases h may vary. The criterion of consistency is the same as before, namely that $\lim_{T \to \infty} P_{T,h(T)}(\theta) = 1$ given any *fixed* alternative $\theta \neq \theta_0$ (θ_0 being the true value).

A number of interesting questions can be posed in this more general context. Do our test statistics have more power if we increase the sample size by decreasing the sampling interval (keeping a fixed span); for example, by going form annual to quarterly data ? If the test is consistent considering a sequence of sampling intervals that goes to zero as T tends to infinity then we should have some reasons to answer yes. Since in economics the sample size is often increased this way this aspect of the consistency criterion may provide useful insight. Or consider the following problem: is the power greater with 100 years of annual data or 20 years of monthly data ? If the test is not consistent as h decreases faster than T increases (i.e. S decreases) then we may have reasons to anticipate that the power would be greater with 100 years of annual data.

In this paper, we analyze in detail the general consistency properties of some statistics for testing i) the null hypothesis of a random walk and ii) the null hypothesis of randomness, against stationary or explosive first–order correlation. We study the behavior of the power function allowing any path for the sampling interval as the sample size increases to infinity.

Section 2 considers the null hypothesis of a random walk against the alternative hypothesis of either stationary positive autocorrelation or explosive correlation. We first consider the standard Dickey–Fuller statistics based on the normalized least–squares estimator in a first order autoregression. It is shown that this statistic is consistent against the specified alternative processes if and only if the span of the data increases as the sample size increases, i.e if the sampling interval decreases to zero at a rate slower than T. Hence the power function can be said to be influenced more strongly by the span of the data than by the number of observations per se. A simulation experiment illustrates the usefulness of the asymptotic results in assessing the behavior of the power function in finite samples. In this section , extensive use is made of some results originally developed by Phillips (1987,a,b).

We also consider in Section 2 , the same null and alternative hypotheses but tested by applying a test of randomness on the first–differences of the data. Here the power function converges to unity if and only if the span increases at least at a rate greater than $T^{1/2}$. This result is interesting because it suggests that, in some instances, it may be possible to increase the power by simply deleting observations, for example by using the same span but a longer sampling interval. Furthermore , it can be said that this procedure is dominated by the Dickey–Fuller procedure in the sense that the latter is consistent over a wider range of possible paths for the sampling interval as the sample size increases. Again we provide simulation evidence to support the usefulness of our asymptotic framework in finite samples.

In Section 3 , we consider the null hypothesis of randomness against the alternative hypothesis of positive serial correlation . The statistic investigated is simply the first–order autocorrelation coefficient. Here the behavior of the power function is rather different. The test statistic is consistent against stationary positive autocorrelation if and only if the span of the data does not increase too fast (in a sense to be made precise) relative to the number of observations. This behavior is unlike test for the random walk hypothesis. Here a smaller sampling interval is preferable. Again, simulation evidence support our asymptotic results.

Section 4 presents a discussion of some issues and proposes simplified consistency criteria to assess the behavior of power functions in finite samples. Section 5 offers some concluding comments. A mathematical appendix contains the proof of some theorems.

2. TESTING FOR A RANDOM WALK

Consider the simple Ornstein–Uhlenbeck diffusion process :

$$dy_t = \gamma y_t dt + \sigma\, dw_t \; ; -\infty < \gamma < \infty \;\; ; \;\; t > 0 \; ; y_0 = 0 \; ; \qquad (2.1)$$

where w_t is a unit Wiener process and γ, σ are constants. We consider testing for a random walk, hence our null hypothesis is:

$$H_0 : \; \gamma = 0 \;\; \leftrightarrow dy_t = \sigma\, dw_t \; , \; y_0 = 0 \; , t > 0 \; ;$$

and the alternative hypothesis is :

$$H_1 : \; \gamma = \bar{\gamma} \qquad -\infty < \bar{\gamma} < \infty \; , \bar{\gamma} \neq 0$$

The class of alternative hypotheses encompasses both stationary ($\gamma < 0$) and explosive ($\gamma > 0$) processes. In this context, it is straightforward to derive the following discrete time representations of the process (2.1) in terms of the sampling interval parameter h:

$$y_{th} = \alpha_h \, y_{(t-1)h} + v_{th} \; , y_0 = 0; \qquad (2.2)$$

where $\alpha_h = \exp(\gamma h)$ and $v_{th} \sim N(0, \sigma^2(\exp(2\gamma h) - 1)/2\gamma)$.

This solution exists and is unique in a mean–squared sense (see e.g. Bergstrom (1984)). Under the null hypothesis we have $\gamma = 0$ and $\alpha_h = 1$ for all h . It is interesting to remark that for any given fixed alternative $\gamma \neq 0$, α_h converges to 1 as h converges to zero. This implies that when testing H_0 , the discrete time parameter α_h converges to the null value as the sampling interval decreases. On the other hand, if h increases to infinity, α_h converges to 0 when $\gamma < 0$ and to infinity when $\gamma > 0$, hence it moves away from the null value of the parameter.

Of course, in practice, only a finite amount of data is available, say T . In the following we will denote by S the span of the data available, where S = hT . S denotes the length, in units of time, of the observed records of data . Therefore, in discrete time, the index t is in the range $t = 0, 1, ..., T = S / h$.

A wide variety of statistics have been proposed to test H_0 . We shall concentrate on two simple ones that are representative of broader classes of statistics . Consider first, the following regression estimated by ordinary least–squares:

$$y_{th} = \hat{\alpha}_h \, y_{(t-1)h} + \hat{u}_{th}$$

The statistic $T(\hat{\alpha}_h - 1)$ has been frequently used to test H_0 and the critical values under the null hypothesis have been tabulated by Dickey (1976) (see also Fuller (1976)). The same regression can be applied to the differenced series , say, $x_{th} = y_{th} - y_{(t-1)h}$ to test H_0 . In this case testing the null hypothesis is equivalent to testing $\alpha = 0$ since under this null hypothesis the sequence $\{x_{th}\}$ is i.i.d., while under the alternative it is correlated. This statistic is asymptotically equivalent to the usual first–order autocorrelation coefficient denoted by :

$$R_h = \Sigma_{t=2}^{T} x_{th} \, x_{(t-1)h} \, / \, \Sigma_1^T x_{th}^2 \, .$$

We shall be concerned with the asymptotic behavior of the normalized statistic $T^{1/2} R_h$.

These statistics have been chosen to represent the following class of tests: a) tests of a random walk based on the original series $\{y_{th}\}$; b) tests of a random walk based on a test of randomness applied to the differenced series $\{x_{th}\}$. Many other statistics in these classes are available . An extensive Monte Carlo experiment with a wide class of statistics is presented in Perron (1988) ; their behavior correspond to that of the statistics analyzed here .

2.1 Testing for a Random Walk Using $T(\hat{\alpha}_h - 1)$.

In order to analyze the limiting distribution of the statistics under the null hypothesis, we specify a triangular array of random variables $\left\{\{y_{nt}\}_{t=1}^{T_n}\right\}_{n=1}^{\infty}$. First consider the process under the null hypothesis of a random walk. For a given n, the sequence $\{y_{nt}\}_1^{T_n}$ is generated by (2.2) with $\gamma = 0$:

$$y_{nt} = y_{nt-1} + u_{nt} \; ; \; t = 1, ..., T_n \tag{2.3}$$

where the innovation sequence $\{u_{nt}\}_{t=1}^{T_n}$ is i.i.d. $N(0, \sigma^2 h_n)$. T_n and h_n are related as $T_n h_n = S_n$ with $T_n \in Z$. We require $T_n \to \infty$ as $n \to \infty$ but do not impose any prescribed path for h_n and S_n . The sequence $\left\{\{u_{nt}\}_1^{T_n}\right\}_1^{\infty}$ is called a triangular array of i.i.d. $N(0, \sigma^2 h_n)$ variates and $\left\{\{y_{nt}\}_1^{T_n}\right\}_1^{\infty}$ a triangular array of random walks.

Each row of the triangular array $\left\{\{y_{nt}\}_1^{T_n}\right\}_1^{\infty}$ is a sequence of random variables generated by a random walk in discrete time with a sampling interval h_n . In this framework, we can formally analyze the limiting behavior of the statistic $T(\hat{\alpha}_h - 1)$ as T increases but without imposing any conditions on h and S . For any given n , we let $T_n(\hat{\alpha}_n - 1)$ be the standardized least squares estimator in a first–order autoregression based on a sample of size T_n sampled at frequency h_n . The asymptotic distribution of the statistic is contained in the following lemma :

LEMMA 1: *If $\left\{\{y_{nt}\}_1^{T_n}\right\}_1^{\infty}$ is a triangular array of random walks for which the innovation sequences $\left\{\{u_{nt}\}_1^{T_n}\right\}_1^{\infty}$ is a triangular array of i.i.d. $N(0, \sigma^2 h_n)$ variates, then as $n \to \infty$:*

$$T_n(\hat{\alpha}_n - 1) \Rightarrow (1/2)(w(1)^2 - 1) \Big/ \int_0^1 w(r)^2 dr$$

where w(r) is a standard Wiener process, defined on C[0,1], and "⇒" denotes weak convergence in distribution.

This Lemma slightly generalizes results obtained by Phillips (1987a) which considered the limiting distribution of $T(\hat{\alpha}_n - 1)$ when h is fixed or when S is fixed . This Lemma implies that , when $y(0) = 0$, the limiting distribution of $T_n(\hat{\alpha}_n - 1)$ is invariant with respect to possible paths for h_n and S_n ; it is the same as long as T increases to infinity [1].

To analyze the limiting distribution of the statistic under the alternative hypothesis we shall again consider a triangular array of random variables $\left\{\{y_{nt}\}_{t=1}^{T_n}\right\}_{n=1}^{\infty}$. For a given n, the sequence $\{y_{nt}\}_{t=1}^{T_n}$ is generated by (see equation 2.2) :

$$y_{nt} = \exp(\gamma h_n) y_{nt-1} + u_{nt} \ , \ y_0 = 0 \ , \ t = 1, \ldots, T_n \ ; \tag{2.4}$$

where the innovation sequence $\{u_{nt}\}_{t=1}^{T_n}$ is i.i.d. normal with mean 0 and variance $\sigma^2\left[\exp(2\gamma h_n) - 1\right]/2\gamma$.

By analogy with the stochastic process described previously we refer to the sequence $\left\{\{y_{nt}\}_1^{T_n}\right\}_1^{\infty}$ as a triangular array of first–order autoregressive processes with $\left\{\{u_{nt}\}_1^{T_n}\right\}_1^{\infty}$ a triangular array of i.i.d. variates. This class of processes can accommodate stationary sequences ($\gamma < 0$) as well as explosive ones ($\gamma > 0$). We do not consider the case $\gamma = 0$ since it was analyzed in Lemma 1.

Our next Lemma concerns the limiting distribution of $T_n(\hat{\alpha}_n - 1)$ under (2.4) when $y(0) = 0$. Its proof is presented in the mathematical appendix .

LEMMA 2 : *If* $\left\{\{y_{nt}\}_1^{T_n}\right\}_1^{\infty}$ *is a triangular array of random variables defined by (2.4), then as* $n \to \infty$ *:*

a) If $S_n \to 0$ *as* $n \to \infty$ *:*

$$T_n(\hat{\alpha}_n - 1) \Rightarrow (1/2)\,(w(1)^2 - 1) \Big/ \int_0^1 w(r)^2 dr \quad ;$$

b) If $S_n = S$ *for all* n *:*

$$T_n(\hat{\alpha}_n - 1) \Rightarrow c + \left\{\int_0^1 J_c(r)^2 dr\right\}^{-1} \left\{\int_0^1 J_c(r)\, dw(r)\right\}$$

where $c = \gamma S$ *and* $J_c(r) = \int_0^r e^{(r-s)c}\, dw(s)$ *;*

c) If $S_n \to \infty$ *as* $n \to \infty$ *and* $\gamma < 0$ *:*

$$S_n^{-\frac{1}{2}} T_n(\hat{\alpha}_n - \alpha_n) \Rightarrow N(0, -2\gamma\sigma^2),$$

$$T_n(\hat{\alpha}_n - 1) \to -\infty \quad \text{and} \quad S_n^{-1} T_n(\hat{\alpha}_n - 1) \to \gamma \;;$$

d) If $S_n \to \infty$ *as* $n \to \infty$ *and* $\gamma > 0$ *:*

$$(2\gamma S_n)^{-1} \exp(\gamma S_n)\; T_n(\hat{\alpha}_n - \alpha_n) \Rightarrow \text{Cauchy},$$

$$T_n(\hat{\alpha}_n - 1) \to +\infty\,, \quad \text{and} \quad S_n^{-1} T_n(\hat{\alpha}_n - 1) \to \gamma\,.$$

Lemma 2 provides the limiting distribution of the standardized least–squares estimator under the alternative hypothesis, allowing the sampling interval to decrease as the sample size increases to infinity.

Part (a) considers the case where both the sampling interval and the span decrease to zero as the sample size increases to infinity. By comparing this result with that of Lemma 1, we observe that the limiting distribution is the same under both the null and the alternative hypotheses. This result holds for any alternative $\gamma \neq 0$ whether negative or positive. Therefore, in this case, the power of a test of the random walk hypothesis ($\gamma_0 = 0$) converges to the size of the test as the sample size increases. The test statistic $T_n(\hat{\alpha}_n - 1)$ is therefore not consistent against a first–order correlation alternative if the span decreases as the sample size increases.

Part (b) considers the theoretically interesting case of a fixed span. Here the sampling interval h converges to zero at the same rate as the sample size increases to infinity. The limiting distribution derived in this case is equivalent to the so–called 'continuous records' asymptotics. Therefore the results obtained not only apply to a limiting sequence of test statistics as $h \to 0$ but also represents the exact distribution when a continuum of data is available (see Phillips (1987a)). In this case the limiting distributions under the null and the alternative hypotheses are different and so the limiting power of the tests does not converge to the size of the test as the sample size increases. However, $T_n(\hat{\alpha}_n - 1)$ is bounded in probability under both the null and the alternative hypotheses, hence the test is not consistent .

Let $\tilde{X}_c$ denote the random variable $\left\{\int_0^1 J_c(r)^2 dr\right\}^{-1}\left\{\int_0^1 J_c(r)\, dw(r)\right\}$. Then with $\gamma < 0$, the limiting power function of a size–β one–tailed test is given by :

$$\begin{aligned}\lim_{T \to \infty} P_{T,h(T)}(\gamma) &= \lim_{n \to \infty} \text{Prob}\left[T_n(\hat{\alpha}_n - 1) \le \delta_\beta\right] \\ &= \text{Prob}\left[\gamma S + \tilde{X}_{\gamma S} \le \delta_\beta\right] \\ &= \text{Prob}\left[\tilde{X}_{\gamma S} \le \delta_\beta - \gamma S\right]\end{aligned}$$

where δ_β is the β % point of the distribution of $(1/2)\,(w(1)^2 - 1)(\int_0^1 w(r)^2 dr)^{-1}$ which can be obtained from the work of Dickey (1976) (see the tabulated values in Fuller (1976) and Evans and Savin (1981)). The same principle holds when $\gamma > 0$ except that the inequality sign is reversed and δ_β is the $(1 - \beta)$ % point. A calculation of the exact power function of the test in this case can be obtained by deriving the exact cumulative distribution function of $\tilde{X}_{\gamma S}$ (see Perron (1987b)).

Parts (c) and (d) consider the case where the sampling interval converges to zero but the span increases to infinity. In that case, $T_n(\hat{\alpha}_n - 1)$ converges to $-\infty$ if the alternative is stationary ($\gamma < 0$) and to $+\infty$ if the alternative is explosive. The power of the test therefore converges to one in each case.

When the sampling interval of the data, h_n, increases with n we adopt a rather different approach to prove the consistency of the test statistic $T_n(\hat{\alpha}_n - 1)$. Consider first the case where $\gamma < 0$ (stationary alternatives) . Note that $T_n^{\frac{1}{2}}(\hat{\alpha}_n - \alpha)/(1 - \alpha^2)^{\frac{1}{2}}$ converges to a N(0, 1) as $n \to \infty$ for any fixed $\alpha < 1$. In our case $\alpha = \alpha_n = \exp(\gamma h_n)$ and $\alpha_n \to 0$ as $h_n \to \infty$. Since the convergence is uniform in a neighborhood of $\alpha = 0$, it also holds as $\alpha_n \to 0$. To prove consistency we proceed as follows:

$$\Pr_{\alpha_n}\left\{T_n(\hat{\alpha}_n - 1) < \delta_\beta\right\}$$

$$= \Pr_{\alpha_n}\left\{\frac{T_n^{1/2}(\hat{\alpha}_n - \alpha_n)}{(1 - \alpha_n^2)^{\frac{1}{2}}} < \frac{1}{(1 - \alpha_n^2)^{\frac{1}{2}}}\left[T_n^{\frac{1}{2}}(1 - \alpha_n) + T_n^{-\frac{1}{2}}\delta_\beta\right]\right\}$$

$$\equiv \Pr_{\alpha_n}\left\{\frac{T_n^{1/2}(\hat{\alpha}_n - \alpha_n)}{(1 - \alpha_n^2)^{\frac{1}{2}}} < d_n\right\}, \text{ say,}$$

$$\Rightarrow \Phi(\lim_{n\to\infty} d_n), \quad \text{as } n \to \infty .$$

But, $\lim_{n\to\infty} d_n = \lim_{n\to\infty} T_n^{1/2}(1 - \alpha_n) = \lim_{n\to\infty} T_n^{1/2}\left[1 - \exp(\gamma h_n)\right] = \infty$ (since $\gamma < 0$) and therefore

$$\Pr_{\alpha_n}\left\{T_n(\hat{\alpha}_n - 1) < \delta_\beta\right\} \to 1 , \text{ as } n \to \infty$$

and the test statistic is convergent. A similar result can be obtained when $\gamma > 0$ by using the fact that $\alpha^{T_n}(\hat{\alpha}_n - \alpha)/(\alpha^2 - 1)$ converges to a Cauchy random variable for any $\alpha > 1$ if $y(0) = 0$.

The results of Lemma 2 and the above discussion permits us to state the following result about the limiting power of the statistics $T_n(\hat{\alpha}_n - 1)$ as $n \to \infty$ when testing $\gamma = 0$ against an alternative $\gamma \neq 0$.

THEOREM 1: *The statistic $T_n(\hat{\alpha}_n - 1)$ for testing the null hypothesis $\gamma = 0$ is consistent against an alternative hypothesis $\gamma \neq 0$ if and only if the span of the data increases as the sample size increases ; that is :*

$$\lim_{n \to \infty} P_{T_n, S_n}(\gamma) = 1 \quad \text{iff} \quad S_n \to \infty \quad \text{as } n \to \infty ;$$

When the statistic is not consistent we have :

i) If $S_n \to 0$ *as* $n \to \infty$: $\lim_{n \to \infty} P_{T_n, S_n}(\gamma) = \beta$,

where β is the size of the test , and :

ii) If $S_n = S$ *for all* n :

$$a) \text{ if } \gamma < 0 : \lim_{n \to \infty} P_{T_n, S_n}(\gamma) = \text{Prob}\left[\tilde{X}_{\gamma S} < \delta_\beta - \gamma S\right]$$

where $\tilde{X}_c = \left\{\int_0^1 J_c(r)^2 dr\right\}^{-1} \left\{\int_0^1 J_c(r) dw(r)\right\}$ *with* $c = \gamma S$, *and* δ_β *is the* β*-percentage point of the distribution of* $(1/2)\left[w(1)^2 - 1\right]\left\{\int_0^1 w(r)^2 dr\right\}^{-1}$.

$$b) \text{ if } \gamma > 0 : \lim_{n \to \infty} P_{T_n, S_n}(\gamma) = \text{Prob}\left[\tilde{X}_{\gamma S} > \delta_{(1-\beta)} - \gamma S\right] .$$

Theorem 1 illustrates in a formal way the idea that the power of a test of the random walk hypothesis depends much more on the span of the data than on the number of observations per se. Indeed , given these theoretical results on can expect several features related to the behavior of the power function in finite samples : a) the increase in power should eventually decrease and become marginal as the number of observations is increased keeping a fixed span ; b) the power should eventually decrease if more observations are added while reducing the span of the data ; c) a longer span of data given a fixed number of observations should yield a higher power ; and d) it may be possible that the power function be higher with less observations when these fewer observations are spaced at longer intervals.

To assess the significance of these theoretical results we conducted a simulation experiment similar to the one performed in Shiller and Perron (1985). We simulated the power of the statistics $T(\hat{\alpha}-1)$ for the null hypothesis of a random walk , $\gamma_0 = 0$) against a stationary alternative $\gamma = -0.2$. Each series was generated by the following process :

$$y_t = \exp(\gamma S/T)\, y_{t-1} + u_t \tag{2.5}$$

where $y_0 = 0$. We specify $u_t \sim N(0, 1)$, without loss of generality, since $\hat{\alpha}$ is invariant with respect to the variance of the innovation when $y_0 = 0$. We simulated 10,000 replications for each of the following values of S and T : 8, 16, 32, 64, 128, 256, 512. The critical values of $T(\hat{\alpha} - 1)$ under the null hypothesis were also simulated using 10,000 replications of a random walk model. The size of the tests was set at 5 %, and we considered only one–tailed tests against stationary alternatives.

Tables 1 presents the results . The values obtained clearly show the relevance of the asymptotic properties in assessing the behavior of the power function in finite samples . The infinity row in Table 1 is the limiting power of the test as the span increases to infinity keeping T fixed. As can be verified from equation (2.5), this is simply the power of a test of the random walk hypothesis against the alternative of randomness, since $\lim_{S \to \infty} \exp(\gamma S/T) = 0$.

In Table 1, the power values obtained for the statistic $T(\hat{\alpha} - 1)$ show that the power increases with the sample size keeping the span fixed but that it tends to level off as T increases. When both S and h decrease the power decreases . The power is always higher for a given sample size if the span is larger. Finally they are instances where fewer observations lead to higher power if the span is correspondingly higher. Consider for example 16 observations and a span of 64 units, the power is 0.74, while it is 0.155 with 64 observations and a span of 16 units. Other, more extreme, examples can be obtained from the Table.

2.2 Using a Test of Randomness on the First–Differenced Data.

We consider a sequence of statistics $\{R_n\}_1^\infty$ where,for a fixed n, R_n is the first–order correlation coefficient of the first–differences of the data :

$$R_n = \Sigma_{t=2}^{T_n} x_{nt}\, x_{n(t-1)} \,/\, \Sigma_{t=1}^{T_n} x_{nt}^2$$

where $x_{nt} = y_{nt} - y_{n(t-1)}$, and y_{nt} is given by (2.4) . In this section, we consider the case where $\gamma < 0$ and study the limiting power function of the statistic R_n which tests for a random walk against the alternative that the process $\{y_{tn}\}$ is stationary and positively correlated. Under these conditions, it is easy to verify that R_n is invariant with respect to the variance of the errors $\{u_{tn}\}$ when $y(0) = 0$. We may, without loss of any generality, set σ^2 in (2.4) such that $\{u_{nt}\} \sim$ i.i.d. N(0, 1). For simplicity of notation we simply write u_t instead of u_{nt}.

If $\{y_{nt}\}$ follows the stationary process (2.4) then the first differences $\{x_{nt}\}$ are an ARMA(1, 1) with a moving–average parameter on the unit circle, i.e.

$$x_{nt} = \alpha_n\, x_{nt-1} + u_t - u_{t-1} \,. \tag{2.6}$$

Now $\{x_{nt}\}$ has the following infinite moving average representation:

$$x_{nt} = \sum_{i=0}^{\infty} g_{ni}\, u_{t-i}$$

with $g_{n0} = 1$ and $g_{ni} = (\alpha_n - 1)\,\alpha_n^{\,i-1}$ for all $i \geq 1$. Since $\sum_{i=0}^{\infty} g_{ni} = 0$ and $\sum_{i=0}^{\infty} i\, g_{ni}^2 = (\alpha_n - 1) \,/\, (1 - \alpha_n^2)^2$, we have :

$$T_n^{\frac{1}{2}} \left[R_n - \rho_n(1)\right] \,/\, \sigma_n(R_n) \Rightarrow N(0,\, 1) \tag{2.7}$$

uniformly in α_n in the range [0,1] . Here $\rho_n(1)$ is the true first–order autocorrelation coefficient of the x_{nt}'s and $\sigma_n^2(R_n)$ is the variance of the statistic $T_n^{\frac{1}{2}} R_n$. A simple calculation yields :

$$\rho_n(1) = (\alpha_n - 1)/2 \,. \tag{2.8}$$

Now the variance of $T^{\frac{1}{2}}R_n$ is approximately given by (see e.g. Priestley (1981), p. 332) :

$$\mathrm{Var}(T_n^{\frac{1}{2}}R_n) \sim \sum_{m=-\infty}^{\infty} \Big[\rho_n^2(m) + \rho_n(m+1)\,\rho_n(m-1) + 2\rho_n^2(1)\rho_n^2(m) - 4\rho_n(1)\rho_n(m)\rho_n(m-1)\Big]$$

where $\rho_n(m)$ is the mth autocorrelation coefficient of the series $\{x_{nt}\}$. Since $\rho_n(m) = \alpha_n^{m-1}\rho_n(1)$, we obtain :

$$\mathrm{Var}(T_n^{\frac{1}{2}}R_n) \sim 1 - (\alpha_n - 1)^2\,(\alpha_n - 2)\,/\,\Big[2(1 + \alpha_n)\Big] \,. \tag{2.9}$$

The derivation of (2.8) and (2.9) was valid for $|\alpha_n| < 1$. Note, however, that when $\alpha_n = 1$, we have $T_n^{\frac{1}{2}}R_n \Rightarrow N(0, 1)$. Therefore the limiting distribution (2.7) is also valid with $\alpha_n = 1$ since (2.8) and (2.9) yield $\rho_n(1) = 0$ and $\mathrm{Var}(T^{\frac{1}{2}}R_n) \approx 1$, when $\alpha_n = 1$. Therefore (2.7) is valid uniformly in α_n over the range (0 , 1) including both end–points.

The power of the statistic $T_n^{\frac{1}{2}}R_n$ is given by $\Pr_{\alpha_n}\Big[T_n^{\frac{1}{2}}\,R_n < b\Big]$ where b is the lower β percentage point of the standardized normal distribution, with β the size of the test. We obtain the limiting power function as follows :

$$\lim_{n\to\infty} \Pr_{\alpha_n}\Big[T_n^{\frac{1}{2}}\,R_n < b\Big]$$

$$= \lim_{n\to\infty} \Pr_{\alpha_n}\left[\frac{T_n^{\frac{1}{2}}\big[R_n - \rho_n(1)\big]}{\sigma_n(R_n)} < \frac{\big[b - T_n^{\frac{1}{2}}\,\rho_n(1)\big]}{\sigma_n(R_n)}\right]$$

$$\equiv \lim_{n \to \infty} \Pr_{\alpha_n} \left[\frac{T_n^{\frac{1}{2}} \left[R_n - \rho_n(1) \right]}{\sigma_n(R_n)} < d_n \right] \text{, say ,}$$

$$\Rightarrow \Phi\,(\lim_{n \to \infty} d_n) \,.$$

in view of (2.7), with $\Phi(\bullet)$ the standardized normal distribution function. Consider first the case where $h_n = h$ for all n . It is clear that the limiting power function is one since $\rho_n(1) = (\alpha - 1)/2 \le 0$ for all n and $\sigma_n(R_n) = 1 - (\alpha - 1)^2 (\alpha - 2)/[2(1 + \alpha)]$ for all n with $\alpha = \exp(\gamma h)$.

Consider now the case where $h_n \to \infty$ as $n \to \infty$. In that case $\alpha_n \to 0$, $\rho_n(1) \to -(1/2)$ and $\sigma_n(R_n) \to \sqrt{2}$. Then $\lim_{n \to \infty} d_n = -\infty$ and again the limiting power function is one.

If, on the other hand, $h_n \to 0$ as $n \to \infty$, we have : $\alpha_n \to 1$, $\rho_n(1) \to 0$ and $\sigma_n(R_n) \to 1$. However,

$$\lim_{n \to \infty} T_n^{\frac{1}{2}} \rho_n(1) = \lim_{n \to \infty} T_n^{\frac{1}{2}} \left[\exp(\gamma h_n) - 1 \right] / 2$$

$$\to -\infty \text{ if } h_n = 0(T_n^a) \text{ for any } a > -1/2$$

$$\to 0 \text{ if } h_n = 0(T_n^a) \text{ for any } a < -1/2$$

$$\to \gamma/2 \text{ if } h_n = 0\,(T_n^{-1/2}) \,.$$

The implication of these results for the behavior of the power function as the sample size increases allowing any path for the sampling interval are summarized in the following theorem.

THEOREM 2 : *Consider* $\left\{\{y_{nt}\}_1^{T_n}\right\}_1^{\infty}$ *a triangular array of random variables defined by (2.4) with* $\gamma < 0$ *, and* $\{R_n\}_1^{\infty}$ *a sequence of test statistics, where* R_n *is the first–order correlation coefficient of the triangular array of first–differences of* y_{nt} . *Suppose that we test the null hypothesis that* $\gamma = 0$, *then* $T_n^{1/2}R_n$ *is a consistent test, as* $n \to \infty$, *under any alternative that* $\gamma < 0$ *if, and only if,* $h_n = 0(T_n^a)$ *for any* $a > -1/2$. *If* $h_n = 0(T_n^a)$ *for any* $a < -1/2$, *then the power of the test converges to its size. Finally, if* $h_n = 0(T_n^{-1/2})$, *the power of the test converges to* $\Phi\ (b - \gamma/2)$ *where* Φ *is the standardized normal distribution and b its associated* β *percentage point, such that* $\Phi\ (b) = \beta$, *the size of the test.*

Theorem 2 states that the statistics $T_n^{\frac{1}{2}}R_n$ is a consistent test of the random walk hypothesis against stationary positive autocorrelation if, and only if, the sampling interval decreases at a rate less than $T^{\frac{1}{2}}$ as T, the sample size, increases. Alternatively, it is consistent if the span of the data increases at a rate higher than $T^{\frac{1}{2}}$ as the sample size increases.

These results are to be contrasted with the ones obtained for the statistic $T(\hat{\alpha} - 1)$ which was consistent as long as the span was at least increasing, and this, at any rate in relation to the sample size. This implies that a test based on the original series $\{y_t\}$, such as $T(\hat{\alpha} - 1)$, is consistent over a much wider range of possible paths for the span or the sampling interval than is the case for a test based on the first–differenced series such as $T^{\frac{1}{2}}R$. According to a more general consistency criterion, the statistic $T(\hat{\alpha} - 1)$ can be said to dominate the statistic $T^{\frac{1}{2}}R$. A most interesting feature of this result is that more observations need not lead to a higher power if the span is kept fixed. Indeed as more observations are added the power will eventually decrease towards the size of the test. This implies that in some instances it may be possible to increase the power of the test by simply deleting some of the available data keeping the observations associated with a longer sampling interval.

We also conducted a simulation experiment to assess the relevance of these asymptotic results as a guide to the finite sample behavior. The framework is the same as in section 2.1 and the results are presented in Table 2. The results clearly show the relevance in finite samples of the fact that R has a power which converges to the size of the test as $h \to 0$. Indeed, for any given span, the power initially increases with the sample size until it reaches a maximum at a value of T between 16 and 64 (increasing with S) and then declines steadily as T is further increased.

3. TESTING FOR RANDOMNESS.

Consider the following continuous–time diffusion process :

$$dy_t = \gamma y_t dt + \sigma(c + |2\gamma|)^{1/2} dw_t \quad ; -\infty \leq \gamma < \infty \quad ; t > 0 \; ; y_0 = 0 \tag{3.1}$$

where w_t is again a unit Wiener process and γ , σ and c (> 0) are constants. The discrete–time representation of the process y_t sampled at an interval h is given by :

$$y_{th} = \alpha_h y_{(t-1)h} + v_{th} \quad ; (t= 1,2,...,T=S/h) \; ; y_0 = 0 \tag{3.2}$$

with $\alpha_h = \exp(\gamma h)$ and $v_{th} \sim N(0, \sigma^2(c + |2\gamma|)(\exp(2\gamma h) - 1)/2\gamma)$. As $\gamma \to -\infty$ (3.2) becomes :

$$y_{th} = e_{th} \qquad (t=1,2,...,T=S/h) \; ; \tag{3.3}$$

with $e_{th} \sim N(0, (c + 1)\sigma^2)$. Hence (3.3) represents a process where the series y_{th} is random. The parameter c is arbitrary and is simply introduced to avoid degeneracy of the process when $\gamma = 0$. In this framework, the null and alternative hypotheses are nested within the model (3.2) in such a way that we have the following specifications :

$$H_0 : \gamma = -\infty \quad ; \text{and} \quad H_1 : \gamma = \bar{\gamma} \quad , -\infty < \bar{\gamma} < \infty \; .$$

The class of alternative hypotheses permitted includes stationary first–order processes, a unit root process and explosive first–order difference equations.

To test the null hypothesis of randomness we consider the OLS estimator in a first–order autoregression which we denote as $\hat{\alpha} = \Sigma_1^T y_{th} y_{(t-1)h} \,/\, \Sigma_1^T y_{(t-1)h}^2$. It is , for our purposes , asymptotically equivalent to the usual first–order autocorrelation coefficient of the data ; using $\hat{\alpha}$ allows us to directly apply the results of Section 2.1. Furthermore, it is easy to see that $\hat{\alpha}$ is invariant with respect to the variance of the innovation process given that we specify a zero initial condition. Hence, we can, without loss of generality, choose σ^2 such that such that $v_{th} \sim N(0,1)$ for all h in (3.2).

We again consider a triangular array of random variables $\left\{\{y_{nt}\}_1^{T_n}\right\}_1^\infty$ defined under the null hypothesis by :

$$y_{nt} = v_{nt} \;; \qquad v_{nt} \sim N(0, 1) \tag{3.4}$$

and under the alternative hypothesis by :

$$y_{nt} = \exp(\gamma h)\, y_{nt} + v_{nt} \;; \qquad v_{nt} \sim N(0,1) \tag{3.5}$$

with $y(0) = 0$. Then we consider a triangular array of test statistics $\left\{\{T_n^{1/2}\hat{\alpha}_n\}_1^{T_n}\right\}_1^\infty$ defined for a fixed n by $T_n^{1/2}\,\hat{\alpha}_n = T_n^{-1/2}\Sigma_{t=1}^{T_n} y_{nt}\, y_{nt-1} \,/\, T_n^{-1}\Sigma_{t=1}^{T_n} y_{nt-1}^2$. Under the null hypothesis $T_n^{\frac{1}{2}}\,\hat{\alpha}_n \Rightarrow N(0, 1)$ as $n \to \infty$, no matter what is the path of h or S as n increases . Recall that we specify $T_n \to \infty$ as $n \to \infty$ with $T_n = S_n/h_n$.

Now we can use the results of Lemma 2 to characterize the behavior of $T_n^{\frac{1}{2}}\,\hat{\alpha}_n$ under the alternative hypothesis allowing h_n to decrease as n increases. This is possible given the invariance of the statistic with respect to the variance of the innovations.

First, we have that, if $S_n = S$ for all n or if $S_n \to 0$ as $n \to \infty$, $T_n(\hat{\alpha}_n - 1)$ is bounded in probability. Therefore $\hat{\alpha}_n \to 1$ and $T_n^{\frac{1}{2}}\hat{\alpha}_n \to \infty$ as $n \to \infty$. Since we consider only testing for randomness against positive autocorrelation, the relevant rejection region is bounded below by the upper $(1-\beta)$ % point of the normal distribution and the test statistic $T_n^{\frac{1}{2}}\,\hat{\alpha}_n$ is therefore consistent when either $S_n = S$ for all n or $S_n \to 0$ as $n \to \infty$.

When S_n increases to infinity and h_n decreases to zero we have, from Lemma 2 (parts (c) and (d)), that $S_n^{-1}T_n(\hat{\alpha}_n - 1) \rightarrow \gamma$ for both stationary and explosive alternatives. Since $S_n^{-1}T_n = h_n^{-1}$ which increases to infinity, we have $\hat{\alpha}_n \rightarrow 1$ and $T_n^{\frac{1}{2}}\hat{\alpha}_n \rightarrow \infty$ as $n \rightarrow \infty$. Therefore as long as the sampling interval decreases, $T_n^{\frac{1}{2}}\hat{\alpha}_n$ is a consistent test of the randomness hypothesis against positively correlated alternatives.

Consider now the case where both S_n and h_n increase with n. We proceed as in section 2.2 using a local power property argument. Consider first a test against stationary alternatives ($\gamma < 0$). Let b be the $(1 - \beta)$ percentage point of the standardized normal distribution. The power of the test is given by

$$\Pr_{\alpha_n}\left\{T_n^{\frac{1}{2}}\hat{\alpha}_n \geq b\right\}$$

$$= \Pr_{\alpha_n}\left[\frac{T_n^{\frac{1}{2}}(\hat{\alpha}_n - \alpha_n)}{(1 - \alpha_n^2)^{\frac{1}{2}}} \geq \frac{(b - T_n^{\frac{1}{2}}\alpha_n)}{(1 - \alpha_n^2)^{\frac{1}{2}}}\right]$$

$$\equiv \Pr_{\alpha_n}\left[\frac{T_n^{\frac{1}{2}}(\hat{\alpha}_n - \alpha_n)}{(1 - \alpha_n^2)^{\frac{1}{2}}} \geq d_n\right], \text{ say,}$$

$$\Rightarrow \Phi(\lim_{n \rightarrow \infty} d_n)$$

where Φ is the standardized normal distribution function. Now $\alpha_n = \exp(\gamma h_n)$ $\rightarrow 0$ and $T_n^{\frac{1}{2}}\alpha_n \rightarrow 0$ as $n \rightarrow \infty$ if $h_n = 0\,(\ln T^a)$ for any $a < -1/(2\gamma)$. This shows that, in this case, $\lim_{n \rightarrow \infty} d_n = b$ and

$$\lim_{n \rightarrow \infty} \Pr_{\alpha_n}\left[T_n^{\frac{1}{2}}\hat{\alpha}_n > b\right] = \beta$$

where β is the size of the test.

If $h_n = 0(\ln T^a)$ for any $a > -1/(2\gamma)$ then $\alpha_n \to 1$ and $T_n^{\frac{1}{2}} \alpha_n \to \infty$ as n $\to \infty$, hence $\lim_{n \to \infty} \Pr_{\alpha_n}\left[T_n^{\frac{1}{2}} \hat{\alpha}_n > b\right] = 1$. Finally if $h_n = 0(\ln T^a)$ for a $= -1/(2\gamma)$, $T_n^{\frac{1}{2}} \alpha_n \to 1$ and $\Pr_{\alpha_n}\left[T_n^{\frac{1}{2}} \hat{\alpha}_n > b\right] = \Phi(b-1)$.

When testing for randomness against an explosive alternative the result is rather different. Under the hypothesis that $\alpha_n > 1$ we have

$$\lim_{n \to \infty} \Pr_{\alpha_n}\left[\frac{\alpha_n^{T_n}(\hat{\alpha}_n - \alpha_n)}{(\alpha_n^2 - 1)} \geq y\right] = C(y), \text{ say,}$$

where C(y) is the probability that a Cauchy variate is greater than y. This result holds uniformly in α, for $\alpha > 1$, and we can therefore obtain the desired result:

$$\Pr_{\alpha_n}\left[T_n^{\frac{1}{2}} \hat{\alpha}_n > b\right]$$

$$= \Pr_{\alpha_n}\left[\frac{\alpha_n^{T_n}(\hat{\alpha}_n - \alpha_n)}{(\alpha_n^2 - 1)} \geq \frac{\alpha_n^{T_n}}{(\alpha_n^2 - 1)}\left\{T^{-\frac{1}{2}} b - \alpha_n\right\}\right]$$

$$\equiv \Pr_{\alpha_n}\left[\frac{\alpha_n^{T_n}(\hat{\alpha}_n - \alpha_n)}{(\alpha_n^2 - 1)} \geq d_n\right], \text{ say,}$$

$$\Rightarrow C(\lim_{n \to \infty} d_n).$$

But $\lim_{n \to \infty} d_n = \lim_{n \to \infty} \{-\alpha_n^{T_n+1}/(\alpha_n^2 - 1)\} = \lim_{n \to \infty} \{-\exp(\gamma S_n + \gamma h_n) / (\exp(2\gamma h_n) - 1)\} = -\infty$ since S_n increases faster than h_n. Therefore if $\gamma > 0$, $\lim_{n \to \infty} \Pr_{\alpha_n}\left[T_n^{\frac{1}{2}} \hat{\alpha}_n \geq b\right] = 1$ which shows the consistency of a test of randomness against explosive alternatives when both the span and the sampling interval increase.

It only remains to analyze the case where $\gamma = 0$. As shown in section 2.1, $T_n(\hat{\alpha}_n - 1)$ has a non degenerate limiting distribution regardless of the path of S_n or h_n . Therefore $\hat{\alpha}_n \rightarrow 1$ and $T_n^{\frac{1}{2}} \hat{\alpha}_n \rightarrow \infty$, which shows consistency. When the sampling interval h_n is fixed for all n , we obtain the usual consistency result for any alternative value of γ . We summarize our results on the consistency of a test of randomness against correlated alternatives in the following theorem.

THEOREM 3 : *a) The statistic* $T_n^{\frac{1}{2}} \hat{\alpha}_n$ *, for testing the null hypothesis of randomness, is consistent against the alternative hypothesis of stationary positive autocorrelation* $(-\infty < \gamma < 0)$*, if and only if the sampling interval is not increasing at a rate greater than* $ln(T^{-1/(2\gamma)})$ *as the sample size increases. If the sampling interval increases at a faster rate with the sample size, the power of the test converges to its size. That is, for* $\gamma < 0$ *:*

$$lim_{n \rightarrow \infty} P_{T_n, h_n}(\gamma) = 1 \quad \text{iff} \quad h_n = 0\,(lnT^a) \text{ for any } a < -1/(2\gamma)\,;$$

$$lim_{n \rightarrow \infty} P_{T_n, h_n}(\gamma) = \beta \quad \text{if} \quad h_n = 0\,(lnT^a) \text{ for any } a > -1/(2\gamma)\,;$$

and

$$lim_{n \rightarrow \infty} P_{T_n, h_n}(\gamma) = \Phi\,(b-1) \quad \text{if} \quad h_n = 0\,(lnT^a) \text{ for } a = -1/(2\gamma)\,;$$

where β *is the size of the test,* Φ *the standardized normal distribution and* $\Phi(b) = \beta$ *.*

b) The same statistic $T_n^{\frac{1}{2}} \hat{\alpha}_n$ *is consistent against the alternative hypothesis of explosive positive autocorrelation or against the random walk hypothesis* $(\gamma \geq 0)$*, regardless of the path taken by the sampling interval as the sample size increases.*

The practical implications of these results are as follows. In a test of randomness against stationary positive autocorrelation, the power function is greater if the sample size is increased by reducing the sampling interval rather than by increasing it. It may even be the case that a smaller sample size yields a greater power than a larger one if the former is associated with a small sampling frequency.

To assess the significance of these theoretical results we conducted a simulation experiment similar to those in Section 2. We generated 10,000 replication of (2.5) with $\gamma = -2.0$, hence analyzing the case where the alternative is a stationary process. The critical values under the null hypothesis were taken from the asymptotic standard N(0,1) distribution since it provides an adequate approximation even if the sample size is quite low.

The results in Table 3 are quite striking. Clearly, the power of the test converges to 1 as $h \to 0$, whenever S is fixed or decreases. Overall, it is apparent that the power increases whenever h decreases. On the other hand, the decrease in the power is quite dramatic when h increases. The infinity row corresponds to the theoretical case where an infinite span would be available with a fixed number of observations. As the span increases to infinity with a fixed T and γ (< 0), the process (2.5) is a random one, hence the power is equal to the size of the test. This is basically the intuitive reason why the power decreases as the sampling interval increases. In that case the discrete–time representation of the process under the alternative hypothesis gets closer and closer to the discrete–time representation of the process under the null hypothesis.

4. SIMPLIFIED CONSISTENCY CRITERIA

In general deriving the limiting power function for any possible paths for the sampling interval as the sample size increases can be a complex task. Accordingly , it may be desirable to consider special cases of the general consistency criterion that are easier to derive and yet can still provide interesting information about issues concerning frequency versus number of observations. Consider the following cases :

Definitions :

a) *Small–h consistency* :

A test is small–h consistent if $\lim_{S \to \infty} [\lim_{h \to 0} P_{S,h}(\theta)] = 1$.

b) *Large–S consistency* :

A test is large–S consistent if $\lim_{T \to \infty} [\lim_{S \to \infty} P_{S,T}(\theta)] = 1$.

c) *Small–S consistency* :

A test is small–S consistent if $\lim_{S \to 0} [\lim_{h \to 0} P_{S,h}(\theta)] = 1$.

In each cases the limiting operations are taken sequentially. Determining whether a test statistic satisfies one or more of these criteria is relatively easier than analyzing the consistency properties for any possible paths for the sampling interval as the sample size increases. Yet these simple criteria summarize quite well the behavior of the power function in finite samples when considering various sampling intervals as the number of observations is changed. For example, if a test is small–h consistent and large–S consistent but not small–S consistent, we can infer that a larger span of data is preferable in terms of power, and that the power eventually decreases as the span decreases. On the other hand, if a test is small–S consistent but not large–S consistent, a smaller span is preferable. Let us consider the behavior of these simplified criteria with respect to the statistics and models considered in this paper.

Consider first the use of the statistic $T(\hat{\alpha}-1)$ when testing for a random walk. The limit of the power function as $S \to \infty$ when T is fixed is, in discrete time , the power function of a test of the random walk hypothesis against the alternative that the process is random (since $\alpha = \exp(\gamma S/T) \to 0$ as $S \to \infty$). This power function is increasing with T and converges to one as $T \to \infty$. Hence the statistic is large–S consistent. Consider now the other two criteria. Both involve first the limit of the power function as $h \to 0$ keeping S fixed. Now it can be shown that this power function converges to one as $S \to \infty$ and converges to the size of the test as $S \to 0$. Hence $T(\hat{\alpha} - 1)$ is small–h consistent but not small–S consistent. The simulation results presented in Section 2.2 attest to the usefulness of these criteria as guides to the behavior of the power function in finite samples.

Consider now using a test of randomness on the first–differences of the data to test for a random walk against stationary alternatives. Since the limit of the power function is the size of the test as $h \to 0$ keeping the span fixed , it is the case that here the test is neither small–h nor small–S consistent. Hence using these simple criteria , one would expect this test to have a power function declining as the sample size increases if the span does not increase sufficiently rapidly. For the same reason as before the test is however large–S consistent ; hence we can expect increasing power if the sampling interval increases with the number of observations. These implications are confirmed by our simulation experiment.

Finally , let us analyze the behavior of the first–order autocorrelation coefficient when testing for randomness. Consider first the case of non–stationary alternatives ($\gamma \geq 0$). The statistic satisfies all the criteria: large–S consistency, small–h consistency and small–S consistency. The test is small–h and small–S consistent because its power converges to one as the sampling interval tends to zero for any fixed span. Now the limit of the power function as the span increases keeping a fixed sample size, when $\gamma = 0$, is simply the power function (in finite samples) of a test of randomness against a random walk hypothesis. This power function converges to one as the sample size increases, so we have large–S consistency against an alternative that $\gamma = 0$. When $\gamma > 0$, the discrete time value of the autoregressive parameter diverges to infinity as the span increases. The test is therefore also large–S consistent in this case.

Consider now the case of stationary alternatives ($\gamma < 0$). The test is again small–S and small–h consistent since the power function converges to one as the sampling interval decreases (for any given fixed span). However, the test is not large–S consistent. As the span increases with a fixed sample size the discrete time value of the autoregressive parameter tends to zero (i.e. $\alpha = \exp(\gamma S/T) \rightarrow 0$ as $S \rightarrow \infty$ if $\gamma < 0$). Therefore the power is given by the size of the test for any sample size and the test is not large–S consistent.

5. CONCLUDING COMMENTS

This paper considered the consistency criterion for a test statistic in a more general framework than usual. Standard practice considers a test statistic consistent if its power against fixed alternatives converges to one as the sample size increases but constrains the sampling interval to remain unchanged as the sample size increases. While this criterion is useful and can be regarded to be a minimal property of a satisfactory test, it cannot shed light on some important features of the test when the sampling interval is changed at the same time as the sample size increases.

To answer some of these interesting practical questions we considered the consistency criterion under all possible paths of the sampling interval as the sample size increases. We analyzed in detail the properties of three commonly used test statistics: a) the normalized least–squares estimator in a first–order autoregression as a test of the random walk hypothesis; b) the first–order correlation coefficient of the first–differences of the data as a test of the random walk hypothesis ; and c) the first–order correlation coefficient as a test of randomness . Some of the results obtained are: 1) the normalized least–squares estimator is a consistent test of the random walk hypothesis as long as the span increases with the sample size (whether the alternative is stationary or explosive); 2) the first–order correlation coefficient of the first–differences of the data is a consistent test of the random walk hypothesis against stationary alternatives as long as the span of the data increases at a rate greater than $T^{\frac{1}{2}}$, where T is the sample size. In this sense the normalized least–squares estimator dominates the latter test statistic since it is consistent over a wider range of possible paths for the sampling interval. And 3) the first–order correlation coefficient is a consistent test of the hypothesis of randomness against stationary alternative as long as the sampling interval is not increasing too fast. If the alternative is explosive, then the test is consistent whatever the path for the span or the sampling interval.

We introduced three summary measures of the consistency properties as well: large–S, small–S and small–h consistency. These were found to adequately capture both the asymptotic and finite sample properties of the statistics considered. The normalized least–squares estimator is both large–S and small–h consistent but not small–S consistent. Therefore when testing the random walk hypothesis with this statistic a large–span of data is to be preferred, (in some cases) even if this entails less observations than would be possible at a smaller sampling frequency. The first–order correlation coefficient as a test of randomness is both small–h and small–S consistent but not large–S consistent (against stationary alternatives). In this case, a smaller sampling frequency is to be preferred, in some instances, even if this entails a smaller sample size. The first–order correlation coefficient of the first–differences is only large–S consistent but not small–h nor small–S consistent as a test of the random walk hypothesis. The result implies that a greater number of observations may lead to a test with smaller power if the span is not increased sufficiently.

The paper has considered only a few (though important in practice) test statistics in a specialized framework. The results, however, can be extended in several directions. Perron (1988) analyses the consistency properties of some thirteen different test statistics through Monte Carlo methods . What emerges from these sets of additional results is that the consistency properties of $T(\hat{\alpha} - 1)$ derived in Section 2 also apply to any test of the random walk hypothesis using the original series (as opposed to a test of randomness on the first–differences). With respect to tests of randomness on the first–differenced data , the simulations suggest that the result obtained in Sections 2.2 and 3 also apply to a wide class of statistics.

Indeed, it is not difficult to generalize the results of Sections 2.2 and 3 to any test statistic which has a non–degenerate limiting distribution under a local alternative ; i.e., for which $\lim_{T \to \infty} Pr_{\alpha} [T^{1/2}\{ X_n - \mu(\alpha)\}/\sigma(\alpha) \leq b] = \Phi(b)$ uniformly over $0 \leq \alpha \leq 1$. Since tests of randomness usually satisfy this property, the results of Sections 2.3 and 3 apply to most tests of randomness and tests of the random walk hypothesis using differenced data. For these reasons, the results of the present study indicate that, under the framework and hypotheses considered, tests of the random walk hypothesis should be based on a statistic using the undifferenced series since they dominate, in terms of a more general consistency criterion, those based upon the differenced series. Of course, this result is contingent upon the specific class of alternative hypotheses considered.

Finally, there is ample scope for further research by analyzing more complex models based on a time series of data; for example, models with lagged dependent variables and exogenous regressors as well as models with time averaged data. The major practical problem to be addressed in this context is to see whether it is worth, in terms of power and efficiency, increasing the sample size if this entails a fixed or, even more realistically, a reduced span. For example, is it better to have 100 annual observations, 150 postwar quarterly observations, or, say, 250 monthly observations. Some answers to these questions can be given using the approach suggested in this paper.

FOOTNOTES

1. This results do not hold if $y(0) \neq 0$ and the reader is referred to Perron (1987a) for details. In this case the asymptotic distribution is the same if the span S is strictly increasing as the sample size increases . Here the effect of y(0) becomes negligible since the observations do not become "too close" together. Such is not the case when considering the distribution of $T_n(\hat{\alpha}_n - 1)$ with a continuum of data, i.e. when S is constant or decreases with n . It is interesting to note, in particular that if S decreases to zero as n increases and $y(0) \neq 0$, $T_n(\hat{\alpha}_n - 1)$ becomes degenerate at 0.

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MATHEMATICAL APPENDIX

The proofs of Lemmas 1 and 2 are simple modifications of developments in Phillips (1987a,b). To prove Lemma 1, we first note that $\hat{\alpha}_n$ is invariant with respect to the variance of the innovations $\{u_{nt}\}$ in (2.3) given that we specified $y_0 = 0$. Hence, without loss of generality, we can set $u_{nt} \sim N(0,1)$ for all n. $\{y_{nt}\}$ is then not a function of h_n and the usual result carries over, see Phillips (1987a).

To prove Lemma 2, we again note the invariance of $\hat{\alpha}_n$ with respect to the variance of the innovations . Hence, without loss of generality , we set $u_{nt} \sim N(0,1)$ for all n in (2.4). Consider first the case where $S_n = S$ for all n and write (2.4) as :

$$\begin{aligned} y_{nt} &= \exp(\gamma S/T_n)\, y_{nt-1} + u_{nt} \\ &= \exp(c/T_n)\, y_{nt-1} + u_{nt} \end{aligned} \qquad (A.1)$$

where $c = \gamma S$. (A.1) corresponds to the near–integrated process analyzed in Phillips (1987b,1988) and others. Since $T \to \infty$ as $n \to \infty$, we can apply Phillips (1987b) result for the least–squares estimator :

$$T_n(\hat{\alpha}_n - \alpha_n) \Rightarrow \{\textstyle\int_0^1 J_c(r)^2 dr\}^{-1}\{\int_0^1 J_c(r)dw(r)\} \qquad (A.2)$$

Given that :

$$T_n(\hat{\alpha}_n - 1) = T_n(\hat{\alpha}_n - \alpha_n) + T_n(\alpha_n - 1) \qquad (A.3)$$

and $T_n(\alpha_n - 1) = T_n(\exp(c/T_n) - 1) \to c$ as $T_n \to \infty$, we have :

$$T_n(\hat{\alpha}_n - 1) \Rightarrow c + \{\textstyle\int_0^1 J_c(r)^2 dr\}^{-1}\{\int_0^1 J_c(r)dw(r)\}\ .$$

This proves part (b). To prove part (a), note that as $S_n \to 0$ we have $c \to 0$ and, hence, $J_c(r) \to w(r)$. Therefore , if $S_n \to 0$ as $n \to \infty$, we have :

$$T_n(\hat{\alpha}_n - 1) \Rightarrow \{\int_0^1 w(r)^2 dr\}^{-1} \{\int_0^1 w(r)dw(r)\} .$$

This proves part (a) using Ito's formula. To prove parts (c) and (d), we first remark that if $S_n \to \infty$ as $n \to \infty$, then , in (A.2), $c \to -\infty$ if $\gamma < 0$ and $c \to +\infty$ if $\gamma > 0$. To provide the proof, we need the following lemma proved in Phillips (1987b).

LEMMA A.1 : Define $J_c(r) = \int_0^r \exp((r-s)c)dw(s)$ with w(s) a unit Wiener process defined on C[0,1], then :

i) as $c \to -\infty$:

a) $(-2c) \int_0^1 J_c(r)^2 dr \underset{p}{\to} 1$;

b) $(-2c)^{1/2} \int_0^1 J_c(r)dw(r) \Rightarrow N(0,1)$;

ii) as $c \to +\infty$:

c) $(2c)^2 \exp(-2c) \int_0^1 J_c(r)^2 dr \Rightarrow \eta^2$;

d) $(2c)\exp(-c) \int_0^1 J_c(r)dw(r) \Rightarrow \eta\xi$;

where η and ξ are independent N(0,1) variates.

Consider first the limit of (A.2) as $c \to -\infty$. Using Lemma A.1, we have :

$$(-2c)^{-1/2} T_n(\hat{\alpha}_n - \alpha_n) \Rightarrow N(0,1) .$$

Noting that $c = \gamma S_n$, we deduce :

$$S_n^{-1/2} T_n(\hat{\alpha}_n - \alpha_n) \Rightarrow N(0, -2\gamma\sigma^2) .$$

Now since $h_n = T_n/S_n$ is decreasing as $n \to \infty$, we have :

$$S_n^{-1}T_n\,(\exp(\gamma S_n/T_n) - 1) \to \gamma \text{ as } S_n^{-1}T_n \to \infty . \tag{A.4}$$

Then , using (A.3) and (A.4) we deduce that $S_n^{-1}T_n(\hat{\alpha}_n - 1) \to \gamma$ and, given that $\gamma < 0$, $T_n(\hat{\alpha}_n - 1) \to -\infty$, as $n \to \infty$. To analyze the case where $\gamma > 0$, first use Lemma A.1 to note that :

$$(2\gamma S_n)^{-1}\exp(2\gamma S_n)T_n(\hat{\alpha}_n - \alpha_n) \Rightarrow \text{Cauchy} \tag{A.5}$$

upon replacing c by γS_n . Using (A.3) through (A.5) and the fact that $S_n^{-1}(2\gamma S_n)\exp(-2\gamma S_n) \to 0$ as $S_n \to \infty$, we have $S_n^{-1}T_n(\hat{\alpha}_n - 1) \to \gamma$ and $T_n(\hat{\alpha}_n - 1) \to +\infty$, given that $\gamma > 0$.

TABLE 1

Power of a one–tailed test with the statistic $T(\hat{\alpha}-1)$

$H_0 : \gamma = 0 \; ; \; H_1 : \gamma = -0.2$

Number of observations : T = S/h

S \ T	8	16	32	64	128	256	512
8	0.101	0.095	0.101	0.105	0.107	0.108	0.111
16	0.169	0.171	0.182	0.174	0.183	0.187	0.197
32	0.320	0.366	0.398	0.416	0.436	0.423	0.439
64	0.555	0.731	0.825	0.869	0.899	0.896	0.891
128	0.700	0.947	0.996	1.000	1.000	1.000	1.000
256	0.724	0.988	1.000	1.000	1.000	1.000	1.000
512	0.728	0.991	1.000	1.000	1.000	1.000	1.000
INF	0.737	0.991	1.000	1.000	1.000	1.000	1.000

TABLE 2

Power of a one–tailed test with the statistic $T^{1/2}R$

$H_0 : \gamma = 0 \; ; \; H_1 : \gamma = -0.2$

Number of observations : T = S/h

S \ T	8	16	32	64	128	256	512
8	0.035	0.055	0.058	0.058	0.058	0.059	0.061
16	0.059	0.072	0.070	0.071	0.068	0.061	0.057
32	0.090	0.116	0.109	0.096	0.080	0.068	0.062
64	0.169	0.231	0.204	0.160	0.127	0.104	0.086
128	0.231	0.429	0.447	0.364	0.260	0.186	0.139
256	0.254	0.588	0.758	0.728	0.595	0.420	0.274
512	0.261	0.630	0.905	0.970	0.953	0.858	0.656
INF	0.254	0.631	0.919	0.998	1.000	1.000	1.000

TABLE 3

Power of a one–tailed test with the statistic $T^{1/2}\hat{\alpha}$

$H_0 : \gamma = -\infty \; ; \; H_1 : \gamma = -2.0$

Number of observations : T = S/h

	S \ T	8	16	32	64	128	256	512
	8	0.083	0.392	0.956	1.000	1.000	1.000	1.000
	16	0.050	0.115	0.651	0.999	1.000	1.000	1.000
S	32	0.047	0.057	0.185	0.898	1.000	1.000	1.000
P	64	0.050	0.047	0.064	0.286	0.994	1.000	1.000
A	128	0.049	0.048	0.047	0.065	0.455	1.000	1.000
N	256	0.051	0.045	0.049	0.052	0.074	0.703	1.000
	512	0.048	0.046	0.048	0.051	0.048	0.092	0.921
	1024	0.049	0.048	0.044	0.046	0.048	0.047	0.110