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ON UTILITY

I. THE VALIDITY OF THE EXPECTED UTILITY HYPOTHESIS

II. ON BERNOULLIAN UTILITY FOR GOODS AND MONEY

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THE VALIDITY OF THE EXPECTED UTILITY HYPOTHESIS

1. Introduction. There is to be considered an agent who on any occasion is capable of certain acts. Any act has an outcome, which is supposed to be in a finite set X of objects $X_1, \ldots, X_n$, which represent the mutually exclusive and exhaustive possibilities for the outcome of any act on any occasion.

If the relation of act to outcome is certain, the choice of act is transferred to the choice of outcome. In this case a model of behaviour is represented by a preference order (irreflexive, transitive binary relation) between the objects in X, which determines a best object among any that are attainable on any occasion, and hence an act which is possible and which has that object as outcome.

The realism of such a model is not a proper issue. It is inseparable from, and is indeed identical with, the very notion of preference; which may or may not be a rewarding notion, but which is one uniquely conspicuous in every attempt to give an account of choice.

In many circumstances of behaviour, the outcome of an act is not certain. Instead, every object X_r which could be an outcome has a certain probability α_r $(\alpha_r \geq 0, \Sigma \alpha_r = 1)$. Then certainty is the special case in which all probabilities but one are zero. In these more general circumstances, choice between acts is transferred not to choice between the objects which are outcomes but to choice between probability distributions over those objects.

Symbolically, $(X ; \alpha)$ can denote a probability distribution $\alpha = \{\alpha_r\}$ over the objects $X = \{X_r\}$. Since the set X is fixed, this is specified by the distribution α alone, which ranges in the space Δ of all possible such distributions. In particular, there are the special distributions $\delta_s = \{\delta_{rs}\}$ where $\delta_{ss} = 1$ and $\delta_{rs} = 0$ $(r \neq s)$, assigning probability 1 to X_s and 0 to every X_r $(r \neq s)$. There is made the identification

$$(X ; \delta_r) = X_r ,$$

so that X_r is connected with the distribution δ_r.

A model of behaviour in circumstances of uncertainty is now represented by a preference order between the distributions Δ, which determines a best distribution among any that are attainable on any occasion, and hence an act which is possible and which has that distribution as outcome. But since certainty appears as a special instance of uncertainty, the preference order between objects X which gives a model for behaviour with certainty, induces a corresponding order between the special distributions δ_r which must be contained in this preference order between distributions Δ.

The expected utility hypothesis for uncertain choice is the hypothesis that every object X_r has a certain utility $U(X_r)$ determining

$$U(\alpha) = \Sigma\, U_r \alpha_r$$

as the expected utility for an act which gives outcome X with probability distribution α; and that preference between distributions corresponds to relative magnitude of expected utility, so choice is determined by the maximum of possible expected utility. The utilities $U = \{U_r\}$, where $U(\delta_r) = U(X_r) = U_r$, represent the same choice behaviour on this model when they are multiplied by some positive number, or some number is added to them. Therefore, since addition of a suitable number will make no change, it can be assumed, without loss of generality, that any utility vector U is such that $\Sigma\, U_r = 0$. To any utility vector U there corresponds a relation P_U between probability distributions $\alpha, \beta \in \Delta$ defined by

$$\alpha \, P_U \, \beta \equiv U(\alpha) > U(\beta) \; .$$

The expected utility hypothesis is thus a special form of the distribution preference hypothesis in which the preference relation between distributions is required to have the special form P_U .

This form of hypothesis is the familiar basis for the "Theory of Games and Economic Behavior" of von Neumann and Morgenstern.[1] It made an earlier appearance in work of Bernoulli[2] and Ramsey[3], and has assumed a new importance in theories such as those of Wald[4] and Savage[5], on foundations of statistical method. It has applications to analysis in many different spheres of behaviour, as shown, for example, by the investigations of Luce[6] and Suppes[7]; and, by Borch[8], it has been made the keystone of a theory of insurance. The question which is going to have attention here is the admissibility, and constructability of the hypothesis, directly on the basis of observation.

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[1] John von Neumann and Oskar Morgenstern. Theory of Games and Economic Behavior (Princeton, 1944, 2nd Ed. 1947).

[2] Daniel Bernoulli. Specimen Theoriae Novae de Mensura Sortis. Commentarii Academiae Scientiarum Imperialis Petropolitanae. 5(1738), 175-192.

[3] F. R. Ramsey. The Foundations of Mathematics and Other Logical Essays (London and New York, 1931), Ch. VII.

[4] Abraham Wald. Basic ideas of a general theory of statistical decision rules. Proceedings of the International Congress of Mathematicians 1952, I, 231-243.

[5] L. J. Savage. The Foundation of Statistics (New York and London, 1954).

[6] R. Duncan Luce. Individual Choice Behavior: A Theoretical Analysis (New York, 1959).

[7] Patrick Suppes. Behavioristic foundations of utility. Econometrica 29(1961), 186-201.

[8] Karl Borch. The utility concept applied to the theory of insurance. The Astin Bulletin I(1961), 245-255.

Three forms of preference hypothesis have been envisaged, which may be distinguished as the object-preference, distribution-preference, and utility-expectation hypothesis. They form a sequence in which each is more specific than its successor:

object-preference $\subset$ distribution-preference $\subset$ utility-expectation.

It will be seen how each of these can be examined for admissibility on given data, and given a specific construction.

When, by an act, one object is attained, when others might have been, so the one object appears as selected and the others as rejected, there is taken to be revealed a relation of preference, of the selected object to every rejected one. When a variety of acts is observed, and the sum is taken of all the preferences thus associated with them, it is asked if the relation obtained could be contained in an order. This is necessary and sufficient for the existence of a preference order which would have resulted in those acts and is therefore an admissible preference hypothesis. The condition is that the transitive closure of the observed preferences is irreflexive, and therefore itself an order containing those preferences, and contained in every order which contains them. The totality of admissible hypotheses on the given data appears as the class of orders which are refinements of this order.

The principle just stated can be applied to choice with certainty, between objects, and to choice with uncertainty, between probability distributions over objects. But in the latter case, with the expected utility hypothesis in view, concern is not with arbitrary order relations between distributions, but only with those of the special form required by this hypothesis. Thus, given any relation P whatever between probability distributions, it is asked if it is contained in a relation P_U

for some utility vector $U = \{U_r\}$ such that $\Sigma U_r = 0$. This is the condition for the admissibility of the expected utility hypothesis on the preference data P. Then further, if the hypothesis is admissible, it is asked what is the class U_P of utility vectors such that

$$P \subset P_U \quad (U \in U_P) .$$

The class U_P defines the range of admissibility of the hypothesis. The hypothesis is admissible in a unique fashion if $P \subset P_U$ for a unique relation P_U, or equivalently, if the vectors of U_P can all be expressed as positive multipliers $U\lambda$ $(\lambda > 0)$ of a single utility vector $U = \{U_r\}$ $(\Sigma U_r = 0)$.

It is imagined that the relation P here considered is such as might be obtained by direct observation of an agent, in any sphere of behaviour to which the expected utility hypothesis has relevance. The hypothesis is a possible model for determining behaviour; and then, proceeding backwards, from what is observed, there can be considered the existence and the totality of hypotheses on this model which are compatible with observation.

This procedure is on the same lines as the revealed preference method of Samuelson[9] and Houthakker[10] in consumer theory. But if preference is to be revealed in choices, the hypothesis that a preference system determines choices has first to be admitted. Then, since the objects of choice are infinite and observation can only be finite, the data of observation can only be fragmentary and there will always be an indeterminacy in the preference hypothesis to be accepted on the data. An investigation of the empirical admissibility and indeterminacy of consumer

[9] P. A. Samuelson. Consumption theory in terms of revealed preference. Economica 28(1948), 243-253.

[10] H. S. Houthakker. Revealed preference and the utility function. Economica 17(1950), 159-174.

preference systems has been made by Afriat[11], especially in connection with cost-of-living measurements. Here an analogous investigation is in view, in regard to the expected utility hypothesis for behaviour in which the consequence of an action is not certain, but only probable.

An order relation on the probability distribution space Δ which is of the form P_U for some utility vector U may be called an expected-utility relation. Such relations have been given an axiomatic characterization by von Neumann and Morgenstern, and subsequently by other writers. Aumann[12] has considered a relation P which satisfies all these axioms of von Neumann and Morgenstern but the completeness axiom, and he has shown that for such a relation there exist utilities U such that

$$\alpha P \beta \implies U(\alpha) > U(\beta) \ ,$$

that is,

$$P \subset P_U \ .$$

He has thus shown that such a relation P, which may be called a partial expected-utility relation, is contained in some expected-utility relation. Thus every partial expected-utility relation can be refined to what may be called a complete expected-utility relation; and this is parallel to the proposition that any partial order can be refined to a complete order (assuming the axiom of choice).

Here not just the completeness axiom will be dropped; but axioms will be dropped completely. That is to say, there will be considered a perfectly general relation P, and it is asked if it can be part of an

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[11] S. N. Afriat. Expenditure Configurations. Research Memorandum No. 21 (February 1961). Econometric Research Program, Princeton University. (Also Research Memoranda Nos. 24, 27 and 29.)

[12] Robert J. Aumann. Utility Theory Without the Completeness Axiom. Research Memorandum No. 26. Econometric Research Program, Princeton University (April 1961).

expected-utility relation. All the expected utility relations of which it can be part, if any, will be characterized, and construction will be made of relation $\check{P}$ which is the intersection of all these relations. The definition of $\check{P}$ from P is by a constructively defined closure operation which can always be performed. If $\check{P}$ is irreflexive, it will be an order which, like the relations considered by Aumann, satisfies all the von Neumann-Morgenstern axioms with the exception of the completeness axiom. It will follow by Aumann's results, and immediately by the arguments here, that $\check{P}$, and therefore also P, is contained in an expected-utility relation. The irreflexivity of $\check{P}$ is necessary and sufficient for the admissibility of the expected-utility hypothesis on data P. The relation $\check{P}$ is always transitive, by its construction, and thus contains the transitive closure $\vec{P}$ of P. The irreflexivity of $\vec{P}$, implied by but not implying the irreflexivity of $\check{P}$, is the condition for P to lie in an order, and thus to admit a preference hypothesis of the general form, though not necessarily of the special form provided by the expected-utility hypothesis.

In this discussion, the arguments are directly in terms of preference, rather than preference-or-indifference relations, contrary to what is often customary. The motive in doing this is in the supposition that observed choice is to be the data of analysis, that the method of analysis is to consider choice as determined by preference, and that indifference, which is the absence of preference, merely represents an inability for choice and is therefore without direct bearing on choice data. This is a particular point of view; and it is nevertheless easy, and has interest, to reformulate the arguments for the other point of view in terms of preference-or-indifference. Then a criterion is obtained for excluding the possibility of indifference in some instances, and allowing it in others. The criterion is based on the distinction between frontier and interior directions in a cone of preference-oriented directions. However, if a choice is achieved between objects viewed as indifferent, it has to be viewed as random, or inexplicable. But commitment is made to preference as the inevitable explanation of choice. If a choice is actually made by an agent, there cannot be denied the capacity on the part of that agent for making that choice, and this capacity is equated with possession of some preference system determining that choice. These considerations point to the alienation of the concept of indifference from practical analysis of choice, except perhaps as an essentially dispensable, auxiliary concept.

2. Let Δ be the space of probability distributions on n objects; so $\alpha \in \Delta$ means $\alpha = \{\alpha_r\}$, where $\alpha_r \geq 0$ and $\Sigma \alpha_r = 1$. Accordingly, Δ is a simplex of dimension $n - 1$, with vertices $\delta_r = \{\delta_{rs}\}$, lying in the hyperplane $\Sigma \alpha_r = 1$ in the n-dimensional cartesian space of coordinates $\{\alpha_r\}$.

Figure 1

(following)

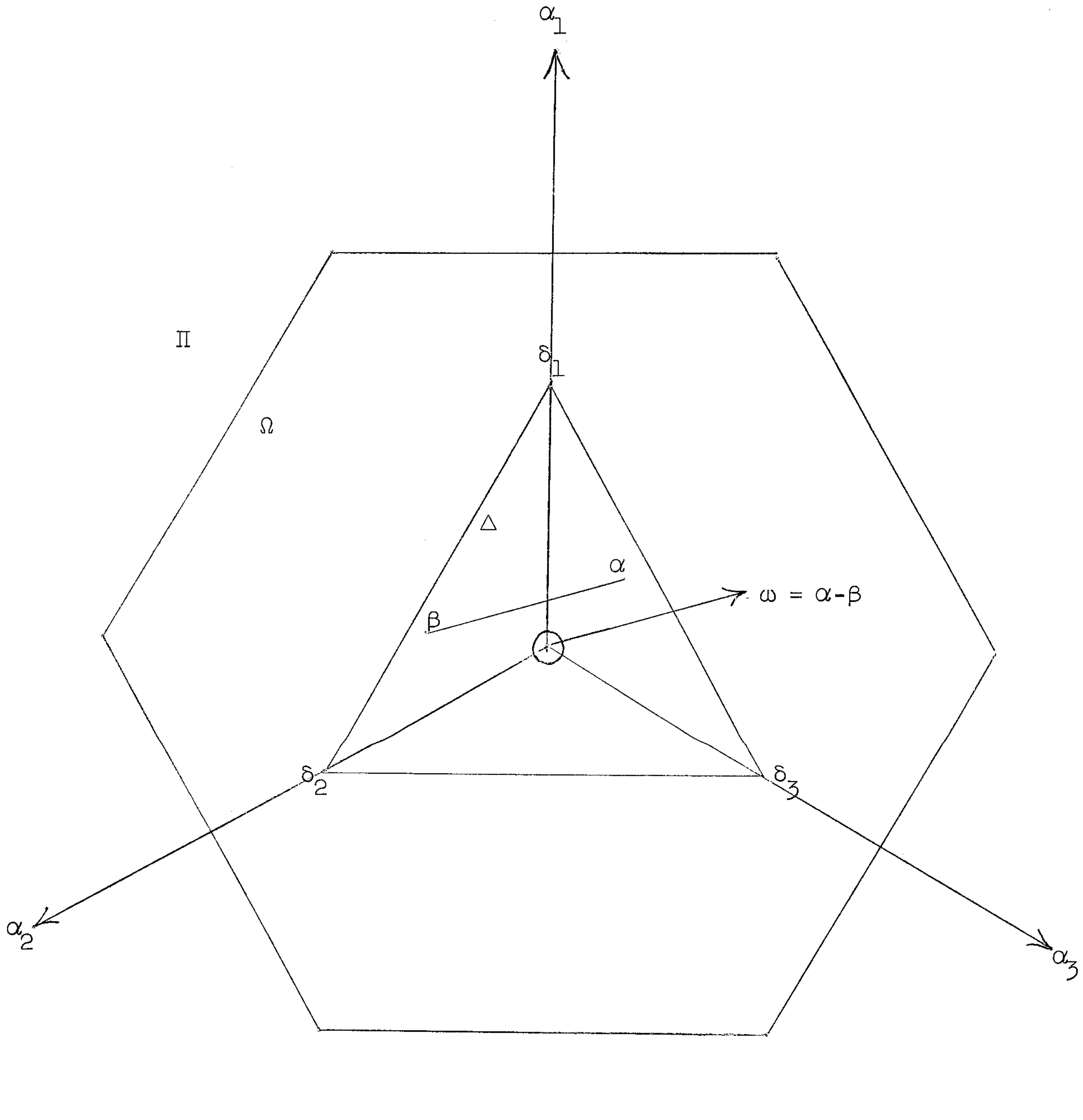
α_1
Π
δ_1
Ω
Δ
α
β
$\omega = \alpha - \beta$
δ_2
δ_3
α_2
α_3

Figure 1.

The distribution-displacement space Ω is now defined with elements of the form $\omega = \alpha - \beta$ where $\alpha, \beta \in \Delta$. This is a region in the hyperplane Π through the origin parallel to the hyperplane through Δ.

Let P be an arbitrary relation between the points of Δ, and define

$$\Omega_P = \{\alpha - \beta \ ; \ \alpha P \beta\}$$

so that Ω_P is a certain subset of Ω associated with P, and is the set of displacements directed between elements in the relation P. Let V_P be the convex cone spanned by Ω_P. Thus, $\omega \in V_P$ means

$$\omega = \omega_1\lambda_1 + \dots + \omega_k\lambda_k ,$$

where $\omega_1, \dots, \omega_k \in \Omega_P$ and $\lambda_1, \dots, \lambda_k > 0$. Let $\check{\Omega}_P$ be the intersection of V_P with Ω. Now let $\check{P}$ be the relation on Δ defined by

$$\alpha \check{P} \beta \equiv \alpha - \beta \in \check{\Omega}_P ,$$

which is to say by

$$\Omega_P = \check{\Omega}_P .$$

Any relation R which satisfies the axiom that, for all $\alpha, \beta, \gamma \in \Delta$ and $0 < \epsilon < 1$,

$$\alpha R \beta \iff \{\alpha\epsilon + \gamma(1 - \epsilon)\} \ R \ \{\beta\epsilon + \gamma(1 - \epsilon)\}$$

may be called translatable. The sense of translatability is explained by its equivalence to another form of condition, now to be shown.

Call ω an R-displacement at α if $(\alpha + \omega)R\alpha$, and say ω determines an R-direction at α if $(\alpha + \sigma)R\alpha$ for any $\sigma = \omega\lambda \in \Delta$ $(\lambda > 0)$. Now translatability is equivalent to the axiom: an R-displacement at any point determines an R-direction at every point. Two parallel displacements ω at α and ρ at γ $(\omega = \rho\lambda \ ; \ \lambda > 0)$ may be said to be in perspective at the intersection β of the joins of α, γ and $\alpha + \omega, \gamma + \rho$. The equivalence of the two considered conditions depends

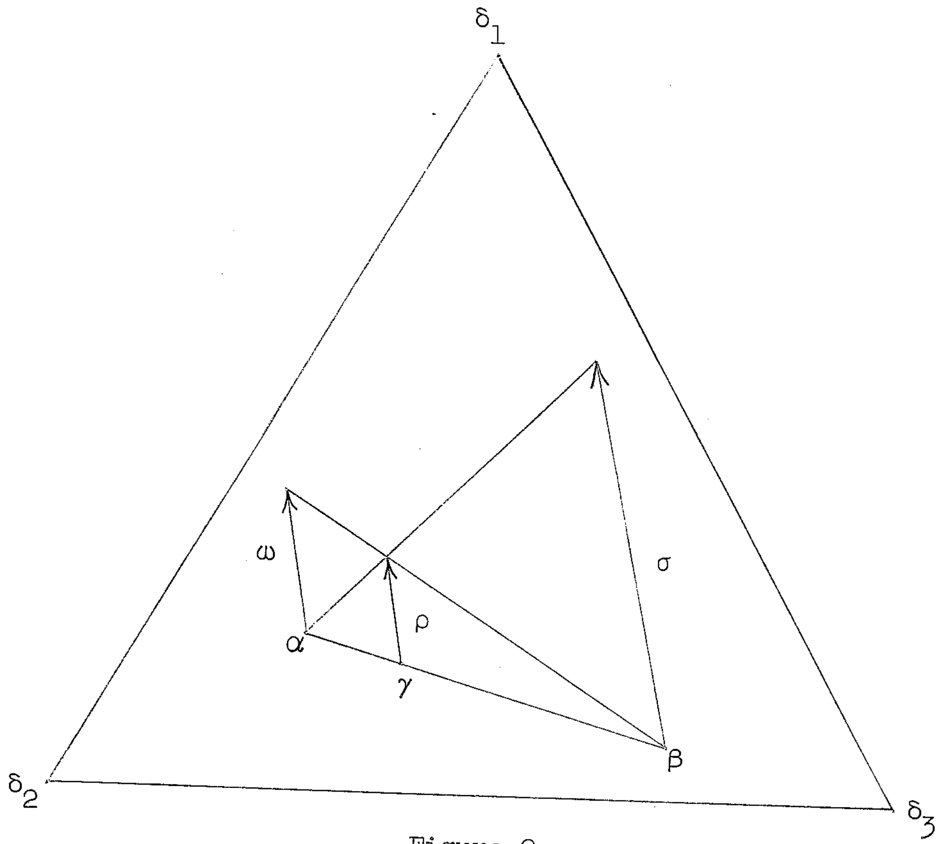

Figure 2.

just on any two parallel displacements, say ω at α and σ at β, being connected by perspective with a third, ρ at γ. For both these conditions are equivalent to the condition that for parallel displacements in perspective, either both or neither belong to the relation.

Now for a translatable relation R, transitivity is equivalent to all the directions of R forming a convex cone. For if R is translatable and $\alpha R \beta$ and $\beta R \gamma$, the $\alpha - \beta$ and $\beta - \gamma$ determines the R-direction of displacements $(\alpha - \beta)s$, $(\beta - \gamma)t$ $(s, t > 0)$. Convexity of R-direction requires $(\alpha - \beta) + (\beta - \gamma) = \alpha - \gamma$ to be an R-direction, and therefore that $\alpha R \gamma$, showing the transitivity of R. Conversely, transitivity with translatability requires the direction $(\alpha - \beta)s + (\beta - \gamma)t$ between $\alpha - \beta$, $\beta - \gamma$ to be an R-direction for all s, t and this is the considered convexity.

It is obvious that translatability and transitivity hold for an expected-utility relation. It is now seen that the same conditions hold for any relation of the form $\check{P}$. In fact, $\check{P}$ is the minimal transitive, translatable relation containing P, as is apparent from the manner of its construction, in view of the observations just made.

Now, with the cone V_P associated with any relation P in Δ, consider the dual cone[13] $U_P = V_P^*$ in the hyperplane Π through Ω. Thus $U \in U_P$ means $U \in \Pi$ and

$$U'\omega > 0 \quad \text{for all} \quad \omega \in V_P .$$

The conditions

$$0 \notin V_P$$

$$\check{P} \text{ is an order}$$

$$U_P \neq \varphi$$

are equivalent. For $\check{P}$ is transitive by construction and so it is an order if it is irreflexive, which is the condition $0 \notin V_P$. Also $0 \notin V_P$ is precisely the condition $V_P \cap (-V_P) = \varphi$, for V_P to be contained in the interior of a half space, and this is the condition for the dual cone U_P to exist. By the conditions given in these various equivalent forms, the relation P may be said to be consistent with the expected-utility model. For apparently

$$P \subset P_U \iff U \in U_P ,$$

so that $P \subset P_U$ for some U is equivalent to $U_P \neq 0$. Again, if $P \subset P_U$ for some U, then $\check{P} \subset P_U$, implying $\check{P}$ irreflexive and therefore an order, since anyway transitive. Conversely, if $\check{P}$ is an order $U_P \neq 0$, and there exists a U such that $P \subset P_U$.

If $U_P \neq 0$, obviously

$$\check{P} \subset \bigcap_{U \in U_P} P_U .$$

[13] A. J. Goldman and A. W. Tucker. Polyhedral convex cones. Linear Inequalities and Related Systems. Annals of Mathematics Studies No. 38 (Princeton, 1956).

Moreover, if $\alpha P_U \beta$ for every $U \in U_P$, then $U'(\alpha - \beta) > 0$ for all $U \in U_P$, and $\alpha - \beta$ belongs to the dual of U_P . But U_P is the dual of V_P ; and therefore V_P is the dual of U_P . Here $\alpha - \beta$ belongs to V_P , which is to say that $\alpha \check{P} \beta$. Accordingly

$$\bigcap_{U \in U_P} P_U \subset \check{P} ;$$

and hence

$$\check{P} = \bigcap_{U \in U_P} P_U .$$

Thus $\check{P}$, if it is irreflexive, appears as the intersection of all the $P_U \supset P$. The irreflexivity of $\check{P}$ is the condition for the admissibility of the hypothesis of an expected-utility relation containing P , and then it is the common part of all such relations. The latitude in choosing one of these relations as an hypothesis corresponds to the latitude in choosing a direction in the cone U_P . If there is only one direction, there is only one P_U with $U \in U_P$; in which case $\check{P} = P_U$, so $\check{P}$ is already the unique complete expected-utility relation containing P .

As with the general preference model of behaviour, the reality of the expected-utility model has sometimes been made an object of doubt; but again, as before, there is no real question here. It is enough that the expected-utility model be adequate to represent a particular form of behaviour, and that it does it in the simplest possible fashion. It has a universal reference; and, with this, any more elaborate model would pretend to other _a priori_ necessities, beyond the bare form, which would be baseless. That one may not always find observations conforming to the model can be no objection. The model is just a way of viewing observations, even through their deviation from it. What has been shown here is the algebra for the examination of observations in the framework of this model.

The motive of such an examination is in the notion that, on the basis of observation of the agent's actions in different circumstances, and the pattern elicited from them by means of the proposed model, it is possible to have expectations of action by the agent when circumstances are not identically reproduced, provided there is reproduction of such pattern. As with all models applied to behaviour which could in principle violate any model, it is only a form with which to capture a supposed existing inertia which is inseparable from the continued identity which is preserved in the agent while acting in the different circumstances. It is only such inertia which could be a ground for the possibility of valid expectations. That such a method, by which observations, taken in certain circumstances, are related in a model which is applicable to any circumstances, should have any success in providing valid expectations cannot be deduced from any general considerations. It just has to be accepted as a condition of the world, which subsists in inertias, the more-or-less vague capturing of which is the object of statistical method. Any particular form this method takes arises itself out of consideration of form, which is always peculiar and for which there is no general account.

<u>ON BERNOULLIAN UTILITY FOR GOODS AND MONEY</u>

Bernoulli[1] introduced $\log e$ as a model for the utility of an amount e of money. If, as is fitting, money is measured as an increment above some level γ, the model then has the form $\log (e - \gamma)$ $(e > \gamma)$, where γ appears as a parameter.

Money is a uniform substance, which can be accumulated in interchangeable additive amounts, and which is a good, meaning that, within considered limits, more is always desirable. The same form of utility function is just as applicable to any such goods.

There will now be considered some n goods, obtainable on the market at prices $p_1, \ldots, p_n$ in any amounts $x_1, \ldots, x_n$ for an expenditure of money

$$e = p_1x_1 + \ldots + p_nx_n = p'x ,$$

where $p = \{p_r\}$ and $x = \{x_r\}$. With each there is associated a Bernoullian utility $\log (x_r - c_r)$, $(x_r > c_r)$. Now, as a model for the utility of any combination $x = \{x_r\}$, there is taken the function

$$\varphi(x) = \sum_r \omega_r \log (x_r - c_r)$$

which is a positively weighted sum of the utilities of the separate goods. With the weights $\omega_r > 0$, so that $\sum_r \omega_r > 0$, it can be assumed without loss in generality, or dividing first by $\sum_r \omega_r$, that $\Sigma \omega_r = 1$, since multiplication of a utility function by a constant provides an equivalent utility function.

It is going to appear that the Bernoullian model for the utility of money is implied by this model for the utility of n combinations of

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[1]Daniel Bernoulli. Specimen Theoriae Novae de Mensura Sortis. Comentarii Academiae Scientiarum Imperialis Petropolitanae 5(1738), 175-192. (Trans. <u>Econometrica</u> 22(1954), 23-36.

goods, on the principle that the utility of an amount of money at given prices is equal to the maximum utility of goods which can be got with that money at those prices. Also it will be seen how the parameter in the Bernoullian money-utility function depends on those prices.

The gradient g of the utility function φ has components which are by the marginal utilities

$$g_r = \frac{\omega_r}{x_r - c_r} .$$

Gossen's law (a special consequence of his Second Law), that marginal utilities are proportional to prices, in the equilibrium in which the maximum utility is attained under an expenditure-price constraint, is that $g = u\lambda$, where $u = \frac{p}{e}$, and $\lambda = x'g$. Accordingly, in such equilibrium,

$$\frac{\omega_r}{x_r - c_r} = u_r \lambda ,$$

where

$$\lambda = \sum_r \frac{\omega_r}{1 - \frac{c_r}{x_r}} .$$

Also

$$\omega_r = u_r(x_r - c_r)\lambda ,$$

so that

$$1 = (1 - u'c)\lambda ,$$

where $c = \{c_r\}$, in view of $\Sigma\, \omega_r = 1$ and $u'x = 1$.

Thus

$$(1 - u'c)\omega_r = U_r(x_r - c_r) ;$$

and hence

$$\varphi = \sum_r \omega_r \log \frac{(1 - u'c)\omega_r}{U_r}$$

$$= \log \Omega \frac{(1 - u'c)}{U} ,$$

where $\Omega = \prod_r \omega_r^{\omega_r}$, and $U = \prod_r u_r^{\omega_r}$. Equivalently

$$\varphi = \log (e - p'c) + \log \frac{\Omega}{P} ,$$

where

$$P = \prod_r p_r^{\omega_r} , \text{ since } \sum_r \omega_r = 1 .$$

Since Ω depends just on the weights, it is fixed; and if prices are fixed, then P and $\gamma = p'c$ are fixed. Therefore, but for an inessential additive constant, there is obtained the Bernoullian form $\log (e - \gamma)$ for the utility of money.

If prices are uncertain, Ω remains fixed, and so is inessential to the utility function. But there is a corresponding uncertainty in $\gamma = p'c$ and P , and then a corresponding uncertainty in the function $\log \frac{(e - \gamma)}{P}$, now representing the uncertain utility of money with uncertain prices.

Setting

$$\varphi = \log \Omega X ,$$

so X is a preference function, since the transformation between φ and X is monotone, but not a utility function, since that transformation is not linear, there is obtained the equation

$$PX = e - p'c .$$

Should it be the case that $c = 0$, this becomes

$$PX = e ,$$

with analogy to the market identity $p'x = e$. Here P and X appear as composite price and quantity indices, whose product is expenditure, which enter into a market identity as if there were just one good.

For two sets of prices and expenditures e_o, p_o and e_1, p_1 and the corresponding indices, there are the relations

$$P_o X_o = e_o - p_o'c , \quad P_1 X_1 = e_1 - p_1'c .$$

Setting $X_o = X_1$, it appears that the condition for money e_o at prices p_o to have the same purchasing power as e_1 at p_1 is

$$\text{(I)} \qquad \frac{e_o - p_o{}'c}{e_1 - p_1{}'c} = \frac{P_o}{P_1} .$$

The cost-of-living index which derives from this formula is associated with the one proposed by Klein and Rubin.[2] In fact, their formula is

$$\text{(II)} \qquad \frac{P_oC + p_o{}'c}{P_1C + p_1{}'c} = \frac{e_o}{e_1}$$

where $X_o = X_1 = C$ is their constant of integration. The value of this constant is crucial; and here its value is identified as the index X of the standard of living at which the index of the cost-of-living is defined. However, there is no need to have this enter explicitly, as it does in formula (II), since, with prices given, standard of living is defined by level of expenditure. Accordingly, there is a redundancy in formula (II), since, with p_o, p_1 given, e_o, e_1 are determined as functions of C. Instead, formula (I) directly gives not just the index at some level which has to be determined outside that formula (as with (II)), but the correspondence between all possible levels of expenditure at the two sets of prices, and hence a cost of living index corresponding to every standard of living.

The expenditure system associated with the utility function φ is

$$x_r = c_r + (1 - u'c) \frac{\omega_r}{u_r} ,$$

or equivalently

$$x_r = c_r + (e - p'c) \frac{\omega_r}{p_r} .$$

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[2] L. R. Klein and H. Rubin. A constant-utility index of the cost-of-living. Rev. Econ. Studies 15 (1947), 84-87.

It is the same as the one derived by Klein and Rubin from a more general one subjected to the Slutzky symmetry condition, which has been derived again by Frisch,[3] and which has been used by Stone[4] as a model for empirical analysis. In this model, expenditure first goes to purchase the fixed composition of goods $c = \{c_r\}$ and what is left is then distributed over the goods in the fixed proportions $\omega = \{\omega_r\}$.

The model can be written

$$x = c + (e - p'c)[\frac{\omega}{p}] ,$$

when $[\frac{\omega}{p}]$ denotes the vector with elements $\frac{\omega_r}{p_r}$ which are quotients of the elements of ω, p . With p fixed and e varying, x describes an expansion curve associated with p ; and this is a straight line passing through c in the direction of $[\frac{\omega}{p}]$. It follows that if just the direction of a linear expansion curve associated with any set of prices is given, then the weights ω in this model are determined. Further, if, with its direction, the curve is given, say by any one of its points, then it is known that c must be on this curve. So only one disposable parameter remains, giving the position of c on this expansion curve. A simple linear expansion curve thus fixes the model but for one parameter. That parameter is limited by the condition that $c_r < x_r$.

It is interesting to note the indeterminacy in the relation between e_o, e_1 defined by formula (I) corresponding to such an indeterminacy in c , which is left in the system when one expansion curve is specified at any prices. Thus, c will be of the form $\hat{c} - dt$ where $\hat{c}$ is in extreme position on the linear expansion curve, permitted by the condition $c_r < x_r$, depending on the considered region of validity of

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[3] Ragnar Frisch. Linear Expenditure Systems: An expository article. Econometrica 22(1954), 505-510.

[4] Richard Stone. Linear Expenditure Systems and Demand Analysis: An Application to the Pattern of British Demand. Econ. Journal 64(1954), 511-527.

the preference model, d is in the expansion direction, and $t > 0$. Hence e_o, e_1 could have any correspondence in the system

$$\frac{e_o - p_o'\hat{c} + p_o'dt}{e_1 - p_1'\hat{c} + p_1'dt} = \frac{P_o}{P_1} \qquad (t > 0) ,$$

and the cost of living index $\frac{e_o}{e_1}$, at the standard of living defined by e_o and p_o, has a corresponding indeterminacy. A similar kind of indeterminacy on given data for an index defined in regard to a quadratic preference model, instead of the present additive logarithmic model, has been investigated by Afriat.[5]

A linear expansion curve is determined when two points, say x and y, are given, corresponding to the same prices p. But suppose now that, more generally, they correspond to different prices, say p and q. Let $u = \frac{p}{e}$, $v = \frac{q}{f}$ where $e = p'x$, $f = q'y$. Then it is required of the c and ω in the model that

$$x_r - c_r = (1 - u'c)\frac{\omega_r}{u_r} , \quad y_r - c_r = (1 - v'c)\frac{\omega_r}{v_r} .$$

By subtraction,

$$\text{(I)} \qquad x_r - y_r = \left(\frac{\alpha}{u_r} - \frac{\beta}{v_r}\right)\omega_r ,$$

where

$$\text{(II)} \qquad \alpha = 1 - u'c , \quad \beta = 1 - v'c .$$

With (I) the condition $\Sigma\,\omega_r = 1$ is equivalent to a certain condition, which may be written

$$\text{(III)} \qquad \Omega(\alpha, \beta) = 0 ,$$

which determines each of α, β when their ratio is arbitrarily assigned. The condition $\omega_r > 0$ then requires this ratio to lie in a certain interval $I(\alpha : \beta)$.

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[5] S. N. Afriat. Preference Analysis: A General Method With Application to the Cost of Living Index. Research Memorandum No. 29 (August 1961), Econometric Research Program, Princeton University.

Now let α, β be chosen arbitrarily satisfying (III). Then (I) determines ω_r such that $\Sigma \omega_r = 1$; and then it follows from (I) that there is a unique c with the property

$$\text{(IV)} \qquad x_r - \frac{\alpha \omega_r}{u_r} = c = y_r - \frac{\beta \omega_r}{v_r} \quad ;$$

and from this (II) immediately follows. It appears thus that ω and c describe certain curves parameterized by the ratio $\alpha : \beta$, which is confined to an interval $I(\alpha : \beta)$.

When $p = q$ there is a return to the case just considered, in which a single linear expansion curve is given, in respect to which a cost of living index has a certain analysable indeterminacy. In the case $p \neq q$, such an indeterminacy can also be analysed, but the problem is more complicated.

In general, if, instead of just two conditions, there is taken a third condition, say

$$z_r - c_r = (1 - w'c) \frac{\omega_r}{w_r} \quad ,$$

then the model parameters are over-determined, and the model has to be rejected as inconsistent.

This model, with a utility function which is a weighted sum of Bernoullian utility functions, is a special case of the already very special case of "independent wants," which historically is the case originally considered by Gossen,[6] and which has recently been examined further by Houthakker,[7] in which the utility function of a combination of goods is

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[6] H. Gossen. Entwickelung der Gesetze des Menschlichen Verhehrs, und der darans fliessenden Regeln für menschliches Handeln. (Braunschweig, 1854.)

[7] H. S. Houthakker. Revealed preference and the utility function. Economica 17 (1950), 159-174.

a sum of utility functions for the separate goods. It is therefore an inadequate model insofar as the phenomenon of dependence between wants cannot be ignored. A model without this defect, which is appropriate for when there is concern simply with preference structure, and not with the further specification on that structure which defines a utility function, deals with preference structures which can be represented in some convex region by a quadratic function. Such a model, already familiar in the literature, has its peculiar conveniences; and also it has formal justifications which recommend it without any prejudice on grounds of convenience.[6] It is interesting that, with this model, there is again obtained an identity

$$PX = e - p'c ,$$

of the same form as the one which has been shown for the additive Bernoullian model, and with the analogous meaning. Although the quantities in it have analogous roles, they are nevertheless defined in a different framework.

The Bernoullian model $\log (e - \gamma)$ for the utility of money could be taken without commitment to an underlying model $\Sigma\, \omega_r \log (x_r - c_r)$ for the utility of goods, from which it was derived with the identification $\gamma = p'c$. Utility is an elaboration on preference structure, and a utility hypothesis for money is in no way constrained by what preference hypothesis for goods is assumed. But from preference structure can be deduced a relation $F(e_o, e_1) = 0$ for expenditures e_o at prices p_o to be equivalent in purchasing power to e_1 at p_1. Then if a utility function $u_o(e_o)$ is given for money e_o at prices p_o, the utility function $u_1(e_1)$ of money e_1 at prices p_1 can be deduced, by the condition

$$u_o(e_o) = u_1(e_1) \quad \text{when} \quad F(e_o, e_1) = 0$$

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[6] Afriat, _op. cit._

on the principle of identifying the utility of money with the utility of the preference level attained with it at given prices. The level depends on the prices; and therefore a change to new prices will induce a transformation of the utility function to a new function belonging to those new prices. It will now be seen how the Bernoullian utility function is transformed by a price change, when certain data, and preference structure on a certain model, is assumed.

Assume a Bernoullian utility function $\log (e_o - \gamma_o)$ for money e_o at prices p_o. On the additive Bernoullian model for goods, it has been shown that

$$\frac{e_o - p_o{}'c}{e_1 - p_1{}'c} = \frac{P_o}{P_1}$$

where $P = \prod_r p_r^{\omega_r}$. Therefore

$$e_o = \frac{P_o}{P_1} e_1 + (p_o{}'c - \frac{P_o}{P_1} p_1{}'c)$$

so that

$$\log (e_o - \gamma_o) = \log \left[\frac{P_o}{P_1} e_1 + (p_o{}'c - \frac{P_o}{P_1} p_1{}'c) - \gamma_o \right]$$

$$= \log \frac{P_o}{P_1} + \log (e_1 - \gamma_1)$$

where

$$\gamma_1 = \frac{P_1}{P_o} (\gamma_o - p_o{}'c) + p_1{}'c .$$

It appears thus that, with this further Bernoullian function for an underlying preference structure for goods, a Bernoullian function $\log (e_o - \gamma_o)$ at prices p_o implies a Bernoullian function $\log (e_1 - \gamma_1)$ at prices p_1, where the parameters γ_o, γ_1 in these functions determine each other by the relation

$$\frac{\gamma_o - p_o'c}{\gamma_1 - p_1'c} = \frac{P_o}{P_1} .$$

To consider another approach, using Bernoullian utility but quadratic preference, suppose now that a pair of expansion curves is given for prices p_o, p_1. A necessary and sufficient condition for the admissibility on such data of an hypothesis of preferences on the quadratic model is that these expansion curves intersect, and moreover that the point c of intersection satisfy the condition

$$p_o'c > p_o'x_o , \quad p_1'c > p_1'x_1$$

for any points x_o, x_1 in the region of validity of the hypothesis. Such data determines a positive number I_{o1}, given by

$$I_{o1}^2 = \frac{p_o'(x_o-c)\, p_o'(x_1-c)}{p_1'(x_1-c)\, p_1'(x_o-c)}$$

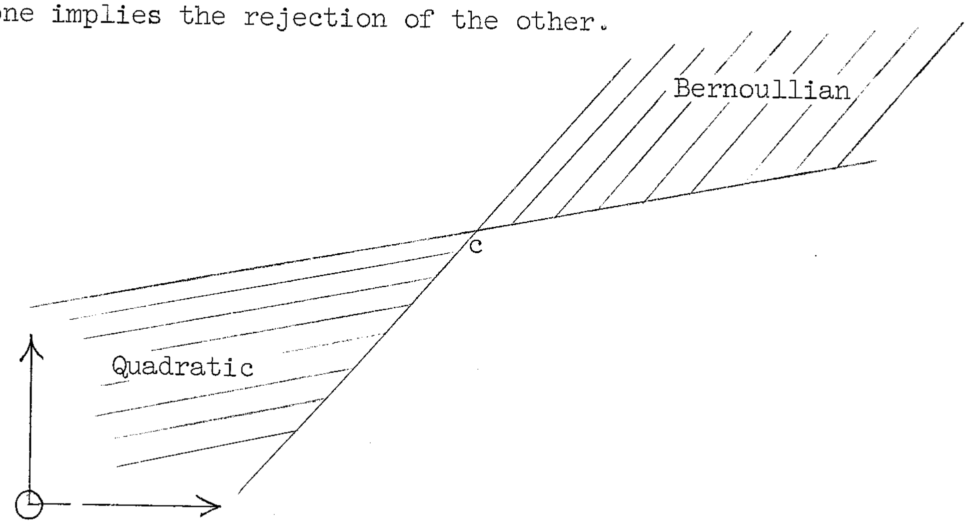
and then, invariantly in respect to every admissible quadratic hypothesis,

$$\frac{e_o - p_o'c}{e_1 - p_1'c} = I_{o1}$$

is the condition for any money e_o, e_1 at prices p_o, p_1 to be equivalent in purchasing power, and therefore to correspond to the same level of utility. Again, a Bernoullian utility function $\log (e_o - \gamma_o)$ transforms to another $\log (e_1 - \gamma_1)$, when prices change from p_o to p_1; and the parameters γ_o , γ_1 are now connected in a relation

$$\frac{\gamma_o - p_o'c}{\gamma_1 - p_1'c} = I_{o1} .$$

In both these cases the point c appears as the intersection of expansion curves. But it is a curious thing that, in the first case, the region of validity is over the point, away from the origin, whereas in the second case, it is under this point, towards the origin. So it appears that the two preference models considered for goods, additive Bernoullian and quadratic, are incompatible with each other, in that, with given expansion curves, and a certain region of validity, the acceptance of either one implies the rejection of the other.



Illustrating the disjoint validity regions for the Bernoullian and quadratic hypotheses with a given pair of intersecting linear expansion curves.

Assuming a utility function $u_o(e_o)$ of an arbitrary form, the same transformation, when prices change to p_1, can be determined. Thus

$$u_1(e_1) = u_o(K_{o1}(e_1 - p_1{}'c + K_{1o}p_o{}'c))$$

where, in the one case or the other, $K_{o1} = \frac{P_o}{P_1}$ or I_{o1}, so in either case $K_{o1}K_{1o} = 1$.

Still more generally, e_o could be quite a general monotonic function $E_{o1}(e_1)$ of the equivalent e_1. Then

$$u_1(e_1) = u_o(E_{o1}(e_1)) .$$

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