AN ANALYSIS OF THE REAL INTEREST RATE
UNDER REGIME SHIFTS

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ABSTRACT

This study considers the time series behavior of the U.S. real interest rate from 1961 to 1986. We provide an explicit statistical model of the series using the methodology of Hamilton (1989a), by allowing three possible regimes affecting both the mean and variance of the series. The results suggest that the ex–post real rate is essentially random around a mean that is different for the periods 1961–1973, 1973–1980 and 1980–1986. The variance of the process is also different in these episodes being higher in both the 1973–1980 and 1980–1986 sub–periods. The inflation rate series is also analyzed using a three regime framework and again our results show interesting patterns with shifts in both the mean and variance. Various model selection tests are run and both an ex–ante real rate and an expected inflation series are constructed based on our univariate models. Finally, we make clear how our results can explain some recent findings in the literature.

Key Words : Nonstationary series, inflation rate, unit root, stochastic process.

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1. INTRODUCTION

While a lot of effort has been devoted to studying the univariate time series properties of GNP [1], it is only recently that issues about the non–stationarity of the real interest rate series were investigated systematically. Using the Fisher relation and a rational expectations assumption for the expected inflation rate, Rose (1988) analyzed the non–stationarity of the ex–ante real interest rate by testing for the presence of a unit root in both the nominal interest rate and the inflation rate applying the Dickey–Fuller methodology. Based on various monthly, quarterly and annual series for many periods and different countries, he concludes that the real ex–ante interest rates must have a unit root, since nominal rates have a unit root while the inflation rates do not [2]. By looking at the ex–post real interest rate for various sample periods, Walsh (1987) also failed to reject the presence of an integrated component in the real rate. More recently, Corbae and Ouliaris (1989) found evidence for a unit root in the nominal interest rate and for stationarity of the inflation rate with annual data from 1920 to 1987 for both the US and the UK. However, conflicting evidence in favor of the non–stationarity of the inflation rate has recently been provided by Bollerslev (1988) and Gokey (1990).

This conclusion about real interest rates is indeed important. For instance, Fama's (1975) influential study analyzing the efficiency of the bond market rests on the postulate that the ex–ante real interest rate is constant, implying not only the absence of an integrated component but also the absence of additional serial correlation. On the other hand, a strand of literature (Fama and Gibbons (1982), Garbade and Wachtel (1978), Nelson and Schwert (1977)) criticizing Fama's paper and finding evidence against a constant real interest rate over the period 1953–1971, is based on the assumption that the real short rate follows an integrated process. In Nelson and Schwert (1977), the inflation rate is also assumed to have a unit root.

Other tests of the constancy of the ex–ante real interest rate can be found in the literature. For example, Mishkin (1981) rejected strongly the hypothesis that the real rate is constant for both the periods 1953–1979 and 1931– 1952, attributing Fama's results to the insufficient variation in the real rate over the period 1953–1971, as noted previously by Shiller (1980). Litterman and Weiss (1985), who also addressed the issue, deserve a closer look. They tested both the hypotheses that the ex– ante real rate is constant and that it follows a random walk against an alternative with four lags each for the nominal interest rate, output, money and inflation. They used three samples for their tests: 1949:2–1983:2

(full sample), full sample with 1950:2–1951:2 removed, and full sample with 1950:2–1951:2 and 1979:4–1982:1 removed, the rationale for taking out these sub–periods being the changes which occurred in the Federal Reserve's operating procedures. Although the hypothesis that the real rate is constant is strongly rejected in all samples, the hypothesis that the real rate is a random walk could only be rejected in the last two samples, i.e. when the periods of the Federal Reserve's new operating procedures are dropped. These results are interesting in light of Perron's (1990) recent work showing that unit root tests are biased towards non–rejection of the unit root hypothesis when the series contains a sudden change in the mean. In that paper, the U.S. ex–post real interest rate from 1961:1 to 1986:3 is analyzed choosing 1980:3 as the time of break.

The presence of regime shifts in the real interest rate process is also of interest to assess the impact of changes in monetary policy. Huizinga and Mishkin (1986) argued that monetary policy has important effects on the ex–ante real interest rate, showing that the shifts in the real rate process coincide with the changes in the Federal Reserve's operating procedures of October 1979 and October 1982. However, their methodology to test for shifts in the real rate series has been recently criticized by Walsh (1988) who argues that the real rate shift identified by Huizinga and Mishkin in October 1982 is in fact due to a shift in the inflation rate process.

In view of the importance of regime shifts in evaluating the random walk character of the real rate process and in assessing the impact of monetary policy changes, one would like a systematic way to model stochastic regime shifts in the time series of interest. In a number of recent papers, Hamilton (1988, 1989a,b,c) proposed to model the regime shifts as the outcome of an unobserved discrete–time, discrete–state Markov process. His algorithm has the distinct advantage of allowing for an endogenous assessment of the likely breaks in a series.

Our analysis will apply this methodology to both the ex–post real interest rate series and the inflation rate series in trying to shed some new light on the debates just mentioned. More specifically, we will investigate whether there was one or more regime shifts in both these series for the US during the period 1961–1986, and if so, when they occurred. To find the number of regimes in a series, we will follow a gradual model building procedure, starting with an autoregressive specification (one regime) and testing for a higher number of regimes until the optimal number of regimes is found. The algorithm used gives as a by–product the dates at which the regime shifts most probably occurred. This

dating will be most useful to determine the likely source of the changes observed in the ex–post real rate. For instance, a change in the real rate occurring near the last quarter of 1979 would point to monetary factors (i.e. the change in the Federal Reserve operating procedures) as the likely cause, while a change occurring near the end of 1980 or the beginning of 1981 would suggest that fiscal policies (i.e. the increase in current and expected future deficits) are responsible for the increase.

Our results support Fama's original characterization of the real rate as a stochastic process that is essentially random with, however, the important difference that the mean of the series is subject to occasional shifts. Indeed, our results strongly argue against the notion that the real interest rate is an integrated process. The endogenously determined shifts in the level of the series occur in 1973 and the end of 1980. The latter shift (by far the largest in magnitude) therefore points to fiscal factors as the likely cause of the increase.

On the inflation front, we also find evidence of important regime shifts in the series. In particular, some evidence is found for a shift in the last quarter of 1982, as Walsh claims. Our characterization of the inflation rate as a stochastic process with different means and variances is also useful to establish if higher rates of inflation are associated with higher variability of inflation, as conjectured by Friedman (1977) and rejected by Engle (1983), using an ARCH model. Our model will furthermore allow us to embed an ARCH specification in a Markov–switching model and test for the presence of remaining ARCH effects once different variances are allowed in different regimes.

Finally, our best specifications for the ex–post real interest rate and the inflation rate are used to construct, based on a rational expectations assumption, both an ex–ante real rate series and an expected inflation series. These in turn will help us assess the within–sample forecasting ability of the Markov–switching models compared with random walk models.

In this study we use two different data sets. The first consists of quarterly series at annual rates drawn for the Citibase data bank. It uses the U.S. 90–day Treasury bill rate for the nominal interest rate and a quarterly inflation rate series constructed from the U.S. CPI non–seasonally adjusted [3]. The sample period is from 1961:1 to 1986:3 which is the same as used by Perron (1990) to assess the impact of a change in mean in a series on unit root tests. This period is also long enough to allow for a comparison of our results to the

existing literature. We will also use for comparative purposes a data set used in Mishkin (1990). Here the nominal interest rate is the three–month Treasury bill rate obtained from the Center for Research in Security Prices (CRSP) at the University of Chicago. The inflation rate series is calculated from a CPI series with proper adjustments for treating housing costs on a rental–equivalence basis throughout the sample. The sample used is monthly from 1961:1 to 1986:12. However, we put more emphasis on the results obtained using the quarterly version of this data set. Since the nominal interest rate series is basically the same in both data sets, this difference in the inflation rate series will be directly reflected in the ex–post real interest rate series.

The plan of the paper is as follows. Section 2 discusses some preliminary issues. Section 3 presents the general model used to characterize the ex–post real interest rate and the inflation rate, the estimation method and the various tests performed. In section 4, the model building exercise is carried out for the ex–post real interest rate and the inflation rate. Section 5 derives the ex–ante real rate and the expected inflation series and within sample forecasts are compared to the random walk model forecasts. Section 6 provides some concluding comments. Two appendices contains some technical details.

2. PRELIMINARY ISSUES

Since Fisher (1930), the nominal interest rate is traditionally decomposed into two parts, the expected inflation rate (π_t^e) and the ex–ante real interest rate (r_t^e) : $i_t = r_t^e + \pi_t^e$. The ex–post real interest rate (r_t) which is the actual real return from holding a one–period bond from t to t + 1, is given by the difference between the nominal interest rate (i_t) and the inflation rate (π_t) from t to t + 1 , $r_t = i_t - \pi_t$. Hence, the ex–ante and ex–post real rates are related by $r_t = r_t^e - e_t$ where $e_t = \pi_t - \pi_t^e$ is the inflation forecast error. If agents use the available information efficiently, the inflation forecast error e_t is uncorrelated with variables known when forming expectations. Hence , $E[\pi_t - \pi_t^e \mid \Phi_{t-1}] = 0$ where Φ_{t-1} is the information set available at t − 1.

To test for the presence of a unit root in the ex–ante real interest rate, Rose (1988) analyzes separately the nominal interest rate and inflation rate series, and infers the properties of the real rate. While his methodology is correct if the series are not integrated of the same order (the nominal interest rate has a unit root but the inflation rate does not, for example, as supported by his results), a problem could arise if the series are both integrated of order one since nothing can be said about the unit root properties of the real rate without a test for cointegration of the nominal rate and the inflation rate. Given the conflicting evidence about the existence of a unit root in the nominal rate and the inflation rate series, it seems more appropriate to directly look at the ex–post real interest rate, as did Walsh (1987), since the ex–post real rate differs from the ex–ante real rate by a stationary component (given the assumption of rational expectations).

While Rose (1988) and Walsh (1987) seem to provide strong evidence for the existence of a unit root in the real rate, Perron (1990) shows that the presence of a regime shift in a series might make it very difficult to reject the hypothesis of a unit root even if the series is characterized by i.i.d. innovations around a shifting mean. In that paper, statistical procedures similar to those of Dickey and Fuller (1979) and Phillips and Perron (1988) are developed to test formally the null hypothesis of a unit root with the possible presence of a one–time change in the mean at a given point in time.

Furthermore, that paper shows that the OLS estimator in a first–order autoregression

is an inconsistent estimate of the true first–order autocorrelation coefficient of the noise function. The magnitude of the asymptotic bias depends upon the magnitude of the change in the mean (as well as the time of the change within the sample). As the change in the mean increases, the limit approaches one irrespective of the true first–order autocorrelation coefficient. This implies that the usual test statistics for the null hypothesis of a unit root are still consistent as the sample size increases to infinity. However, in finite samples, a bias will occur towards non–rejection of the null hypothesis of a unit root if the magnitude of the change in the mean of the series is large enough. If the change is small, the inference is unlikely to be much affected.

These results have the following implication for our analysis of the ex–post real interest rate. Figure 1 suggests the presence of two changes in the mean of the series : one around 1973 and one around 1980. However, the change associated with 1980 is much larger than the one associated with 1973. Hence, a test for a unit root allowing a single break around 1980 may be adequate in the sense that not taking into account the change in 1973 would not alter our inference. Also, if a second change is small, it may be preferable to use the procedure described above rather than allow for the presence of two breaks in the formal unit root testing procedure. Indeed, the latter is likely to be less powerful due to an increase in the absolute value of the critical values.

The results obtained in Perron (1990) using the methodology described above point to the fact that the ex–post real interest rate indeed appears to be characterized by stationary fluctuations around a shifting mean. While allowance was made for only one change around 1980, the test was still powerful enough given that the 1973 change was relatively small. It must be emphasized, however, that this earlier analysis was solely oriented towards assessing whether or not a unit root was present in the noise function of the series. It did not provide a statistical model of the series on which one could, for example, construct forecasts. The aim of the present paper is oriented towards this issue. Given the results discussed above, we shall specify a stochastic process with various regimes but with the noise (or cyclical) part specified as a stationary process.

3. MODEL, ESTIMATION METHOD AND TEST PROCEDURES

The previous discussion warrants the use of a model allowing possible changes in the mean and variance of a series in different sub–periods of the sample. We will use the following general model, an AR(r) allowing for shifts in the mean and variance, to describe a given time series of data y_t (t = 0, 1, ..., T) :

$$y_t - \mu(S_t) = \phi_1[y_{t-1} - \mu(S_{t-1})] + ... + \phi_r[y_{t-r} - \mu(S_{t-r})] + \sigma(S_t)\epsilon_t \qquad (3.1)$$

where the mean μ and the standard deviation σ of the process depend on the regime at time t, indexed by S_t, a discrete valued variable, and $\{\epsilon_t\}$ is a sequence of i.i.d. N(0,1) random variables. Given that Perron (1990) rejects the unit root hypothesis for the ex–post real interest rate allowing for a change in regime we specify that the roots of $(1 - \phi_1 z - ... - \phi_r z^r)$ = 0 are outside the unit circle. Theoretically, one might include any number of regimes in such a setting. We will follow a gradual model building procedure, starting with an AR(r) formulation with no change in mean and variance, to determine the number of regimes present in the series under study. Various tests described below will be used to assess the adequacy of the specified model and to test for the presence of another state. Since the two–state model has been most commonly used until now [4] , we will specify below a three–state model with two autoregressive parameters, which will be ultimately used for both the real rate and the inflation rate.

To make the model (3.1) tractable, the econometrician must specify a stochastic process for the regime variable S_t. Hamilton (1988,1989a,b) proposes to model S_t as the outcome of an unobserved discrete–time, discrete–state Markov process, building on an original idea by Goldfeld and Quandt (1973) [5] . Assuming a three–state, first–order Markov process, where S_t can take the values 0, 1 or 2, we can write the transition probability matrix as follows :

$$P= \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} \qquad (3.2)$$

where $p_{ij} = \Pr[S_t = i \mid S_{t-1} = j]$ with $\Sigma_{j=1}^3 p_{ij} = 1$ for all i. The state–dependent means and variances are specified linearly as :

$$\mu(S_t) = \alpha_0 + \alpha_1 S1_t + \alpha_2 S2_t$$
$$\sigma(S_t) = \omega_0 + \omega_1 S1_t + \omega_2 S2_t \tag{3.3}$$

where Si_t takes the value 1 when S_t is equal to i and 0 otherwise. With the number of autoregressive lags restricted to 2, equation (3.1) can therefore be written as :

$$y_t = \alpha_0 + \alpha_1 S1_t + \alpha_2 S2_t + z_t \tag{3.4}$$

$$z_t = \phi_1 z_{t-1} + \phi_2 z_{t-2} + (\omega_0 + \omega_1 S1_t + \omega_2 S2_t)\, \epsilon_t \;. \tag{3.5}$$

The two–state model is defined similarly with $\alpha_2 = \omega_2 = 0$ and $\Pr[S_t = 0 \mid S_{t-1} = 0] = p$ and $\Pr[S_t = 1 \mid S_{t-1} = 1] = q$. If the states S_t, S_{t-1}, S_{t-2} were known, it would be possible to write the joint conditional likelihood function of the sequence $\{y_t\}$:

$$f(y_T,...,y_r \mid S_t, S_{t-1}, ...) = -(2\pi)^{-1/2} \Sigma_{t=r}^T \{\exp[-v_t^2/2\sigma(S_t)^2]/\sigma(S_t)\}$$

where $\sigma(S_t)$ is given by (3.3) and $v_t = (z_t - \phi_1 z_{t-1} - \phi_2 z_{t-2}) = \sigma(S_t)\epsilon_t$. Since we do not observe S_t, but only y_t $(t = 0, 1, ..., T)$, a way must be found to make an optimal inference about the current state based on the history of the observed values for y_t. This is the idea of the non–linear filter proposed by Hamilton. In a recursive fashion similar to the Kalman filter, this non–linear filter gives as a by–product the likelihood function of the y_t's :

$$f(y_T, \cdots, y_r) = \Sigma_{t=r}^T f(y_t \mid y_{t-1}, y_{t-2}, ..., y_0) \;. \tag{3.6}$$

Hamilton (1989a) proposes an algorithm to estimate the parameters α's, ω's, and ϕ's allowing for a prespecified number of states. In the three–state case, for the construction of the probability structure of the first r observations, we use the limiting unconditional probabilities of being in state i ($i=1, 2$ or 3) for the first observation to start the algorithm.

These are given by $\pi_i = A_{ii} / \Sigma_{j=1}^3 A_{jj}$ ($i = 1, 2, 3$), where A_{ii} is the ii^{th} cofactor of the matrix $A = I - P$, with I the 3 x 3 identity matrix and P as defined in (3.2) [6] . As a by−product of the algorithm, we also obtain the joint conditional probability $p(S_t, S_{t-1},...,S_{t-r+1} \mid y_t, y_{t-1},...,y_0)$. One can then construct a sequence of probabilities making it possible to draw inferences about the state S_t conditional upon information at date t. These are obtained using the relation :

$$p(S_t \mid y_t, y_{t-1},..., y_0) =$$

$$\sum_{S_{t-1}=0}^{1} \sum_{S_{t-2}=0}^{1} ... \sum_{S_{t-r+1}=0}^{1} p(S_t, S_{t-1},..., S_{t-r+1} \mid y_t,..., y_0) . \qquad (3.7)$$

The specification testing procedures in the context of Markov switching models also raises interesting issues. Let $\hat{\lambda} = (\hat{\mu}, \hat{\omega}, \hat{p})$ be the vector of MLE estimates for the parameters of the mean, variance and transition probabilities of the model. The score $h_t(\lambda)$ is defined as $h_t(\lambda) = \partial \log p(y_t \mid \Phi_t; \lambda)/\partial \lambda$ and is conditional upon the information until time t (Φ_t) and λ. In our model, Φ_t contains the present and past values of the state variable S_t [7]. The estimated scores are useful in the construction of various test statistics. Of particular interest to us are the LM test for omitted variables (either from the mean or from the variance) and the LM test for ARCH effects. The latter will be useful to test if there are some ARCH effects remaining in the series if one allows the variances to be different in different states. This test is interesting in the context of the inflation rate series as documented by Engle (1982).

The LM test for omitted variables is useful in the context of regime switching models since one can test for the presence of a change in mean or variance for a sub−period of the sample and determine if the series under study could be better characterized by a model with one more regime. Basically, the test amounts to putting a dummy variable in periods where a potential break might have occurred. There are two prerequisites though in such an approach : first, the states have to be persistent ; second, the econometrician has to form strong priors about the possible dating of the break in the series. Although computationally attractive, this approach seems to take away one of the advantages of the Markov−switching formalization, which is to identify endogenously the breaks in a series.

In this regard, if the states are highly persistent, one is better off splitting the sample into two overlapping sub–samples and apply the Markov–switching model say with k states to both sub–samples. If the algorithm identifies clearly k states in both sub–samples, it is likely that the k+1 state model applies to the full sample. This procedure is not however totally free of the econometrician's intervention since the choice of the sub–samples implicitly determines the end or the beginning of the period. One would like then to have a testing procedure which can directly compare competing models, i.e. a model selection procedure. In the first analysis, it seems that assuming the equality of the means and variances in say a two–state Markov–switching model will nest into it the AR(r) representation. However, a closer look at the problem reveals serious difficulties in applying the usual Lagrange Multiplier (LM), Likelihood Ratio (LR) or Wald procedures due to the non identification of parameters under the null hypothesis.

For the purpose of illustration, consider the case of testing the null hypothesis of one regime versus the alternative hypothesis of two regimes with transition probabilities p and q. First, the Lagrange multiplier test is undefined because p and q (in the two–state case) cannot be estimated when α_0 and ω_0 are equal to α_1 and ω_1, respectively. Second, even if theoretically one can compute the LR and the Wald tests, both the identification condition and the rank condition are violated under the null hypothesis of no change in regime. If the two means and the two variances are equal, then p and q are not identified since any two values will give the same value to the likelihood function. As for the rank condition, the score statistic (the derivative of the log likelihood with respect to the full vector of parameters under the Markov–switching alternative) will be identically zero since the full sample mean and the full sample variance will satisfy the normal equations of the regime switching model (see Hamilton (1989a) for more details). Similar issues arise when testing the null hypothesis of any number of regimes versus the alternative hypothesis of a higher number of regimes. Various avenues can be pursued to overcome these problems. The first two, which we will call the Davies test (1987) and the Gallant test (1977), start with the idea of giving a range of values to the parameters under the alternative, thus avoiding the problem of estimating them. The second step is to construct some statistics based on the value of the function obtained with these given parameter values. For the Davies test, one obtains an upper bound for the significance level of the likelihood ratio statistic under the null hypothesis consisting of the model with the lowest number of states [8]. Gallant's procedure, suggested by Hamilton (1989a) but to our knowledge never applied, consists in calculating the estimated values of the dependent variable for the various values chosen for

the unidentified parameters. One then adds these constructed variables [9] to the model with the lower number of states and compute an F–test for their significance. The formulas for both these tests are given in Appendix A.

Finally, contrary to the previous approach, one might still decide to estimate the model with the larger number of states and run tests for non–nested models (Davidson and MacKinnon (1981)). We will apply the so–called J–test which uses a t–test on $\hat{\alpha}$ in the model :

$$y_t = (1-\alpha) \, f_t(\beta) + \alpha \, \hat{g}_t + u_t$$

where $f_t(\beta)$ will represent in our case the model with the lower number of states and $\hat{g}$ the estimated value of y obtained by fitting the model with the larger number of states [10].

In the next Section, these estimation and testing procedures will be used to construct univariate Markov models for the ex–post real interest rate and the inflation rate series using the two different data sets described earlier.

4. RESULTS AND DISCUSSION

4.A : The Ex–post Real Interest Rate with Two Regimes.

We will use the model presented in Section 3 to test various specifications regarding the number of states and the number of autoregressive lags. We consider first the Citibase data series.We start with an AR(r) specification for the ex–post real interest rate. After the identification and estimation steps for AR(1) to AR(4) models, we opted for an AR(4). The estimation results of the AR(4) are presented in Table I. Over the sample, the series exhibits a mean of 1.927 and a variance of 2.395 with a relatively high persistence, the sum of the autoregressive coefficients being .871.

In the next step, we estimate and test a two–state Markov model with different means and variances. The estimation revealed the presence of a number of local optima, three of which are reported in Table II because they carry some information about the possible regime changes present in the series. The estimated model associated with the highest of these local minima (Fmin = 130.52) locates two regimes with very different means (–0.7% and 5.2%) and standard deviations (1.9% and close to 3%) in the low and high states. The transition probability parameters indicate very persistent states. The filter probabilities, a sub–product of the algorithm which give the probability that S_t equals one given the information at time t, are shown in Figure 2.a. We can see that the series was in state 0 up to the last quarter of 1979 (the well–documented change in the Federal Reserve procedures) and jumped to state one for the remainder of the period [11] . The other two local minima of interest do not, in a 2–state model, favor such a big change in mean. The filter probabilities for each model are presented in Figures 2.b and 2.c. These local minima (Fmin = 128.55 and 130.20) are interesting from an economic point of view, since they also show persistent states but locate the changes at different points in time : the first locates a single change in regime (from a high mean to a low mean state) in the 3rd quarter of 73. The second points to two abrupt changes, one from a low mean to a high mean state in the first quarter of 1980, and a return to a low mean state near the end of 1982. The first minimum can obviously be related to the rise in inflation in 1973, the second to the beginning and the end of the change in operating procedures at the Fed. These are the turning points respectively found in the inflation series and the nominal interest rate series if the 2–state model algorithm is applied to these series.

The global minimum in these estimated two state models (see Table I) does not, however, seem to have any ready economic interpretation. First, the transition probabilities are noticeably smaller and the filter probabilities (presented in Figure 2.d) identify all the extreme points as belonging to state 1 by attributing to that state a very large variance. As documented in Boldin (1989), this feature seems to be the result of the presence of four autocorrelation terms. Boldin shows that a series generated by a 2–state Markov model can be mistaken for an AR(1) process if only one state is allowed. Going one step further, it could also be the case that significant AR terms in 2 states are due to the fact that a three–state model is correct. In small samples, spurious AR terms can be found, since the algorithm will artificially increase the value of the function by changing states frequently to follow closely the ups and downs in the series. One diagnostic of this problem is to look at the value of the transition probabilities p and q which fall in our case to 0.246 and 0.446. This spurious effect seems to be supported by the fact that in a model with 2 states and 2 lags (reported in Table IV), the global minimum is now associated with high transition probabilities (persistent states) and the filter probabilities (not shown) point to a change in the 3rd quarter of 1973 (as in one of the local minima found before), the series remaining in this highly variable state until the end of the sample.

These results seem to indicate a misspecification of the Markov model, but more likely in the direction of a three–state model since the model selection tests all favor the two–state model over the AR(4), except for the AIC and SBC criteria which are very slightly in favor of the more parsimonious models. The quick rule for the Davies test (see Appendix A) leads to a probability close to zero ($\sim$ .3%) for the likelihood ratio test statistic to be greater than 17.53. The Gallant test was calculated by adding to the AR(4) formulation the filter probabilities of the three lowest minima in the 2–state Markov model [12]. The test strongly rejects the null hypothesis of a fourth order autoregression with a single regime. The J–test of Davidson and MacKinnon was calculated using the parameter estimates from the model corresponding to one of the local minima (Fmin = 127.19) to construct the variable $\hat{g}$. The estimate for the coefficient α is 0.998 with a standard error of 0.03, therefore concurring with the results of the previously reported tests.

The results obtained with the Mishkin data set (also presented in Table I) are similar with respect to the estimated means in the different regimes but quite different with respect to the estimated variances. In the two–state quarterly model, the variance in both regimes is much smaller. Another important difference is found in the transition probability parameters where now one of the states is persistent. The filter probabilities

(not shown) do not lend themselves to any appealing economic interpretation and confirm the probable misspecification of the 2–state model. The two–state model was also estimated with the same data set at monthly intervals (Table I) [13]. Now the mean is almost the same in both states, but the variance is four times as large in the high state. Also the probability assignments reveal an interesting pattern. The series stays in the low variance state until the middle of 1973, jumps to the high variance state until the beginning of 1976 with a small lapse in between, goes down again to the low state until the end of 1979, jumps up until the end of 1982, down again until 1986 and up for the last year of the sample. These results tend to support the view of a constant real interest rate with higher variability in periods of high inflation and of changes in monetary policy. They also tend to concur with the findings of Huizinga and Mishkin (1986) about another shift in the real rate towards the end of 1982. With the Mishkin data set, the results for a test of the AR(4) versus a two–state model are not as clear as with the Citibase data set. Based on the reported results, not all test statistics reject the hypothesis that the series is characterized by an AR(4) model.

Given the results obtained with the 2–state model with four autoregressive parameters, one might suspect the presence of a third regime. As a step to assert the presence of such a third state, we applied the following analysis to the Citibase data series. We split the sample at 2 points in time (the choice of this particular split being suggested by the regimes identified by some of the estimated two–state models corresponding to local minima, see Table II and Figure 2) and run the 2–state algorithm for both these sub–samples. The results are shown in Table III. Both sub–samples 1961:1–1979:4 and 1973:1–1986:3 exhibit two persistent states. In the first, the mean turns negative (–2%) starting in 1973 until the end of the sub–sample with an associated higher variance. In the second sub–sample, the mean rises from –1.8% for the period 1973:1–1979:4 to 5% from 1980:1 until the end of the sample. Note, however, that the variance is not significantly different in both states and that the sum of the autoregressive coefficients in both sub–samples falls dramatically to 0.266 and 0.385 respectively. This evidence about the presence of two regimes in both sub–samples might explain the difficulty encountered by Walsh (1987) to reject the random walk hypothesis over the two subperiods 1961:1 –1979:3 and 1970:1–1985:3 [14] .

4.B : The Ex–post Real Interest Rate Under Three Regimes.

Given these preliminary results with two–regime models, it seems required to use the three–regime model to characterize the ex–post real interest rate over the period 1961:1–1986:3. Since the serial correlation in the autoregressive structure fell drastically when analyzing both sub–samples, we chose to limit the three–state specification to two autoregressive lags. The estimation results are presented in Table IV. Now α_0 denotes the mean for the high state, $\alpha_0 + \alpha_1$ the mean for the low state and $\alpha_0 + \alpha_2$ the mean for the middle state. Similarly, ω_0, $\omega_0 + \omega_1$, $\omega_0 + \omega_2$ denote the standard errors for the high, low and middle states respectively. The inferred probabilities for the current state are shown in Figure 3.a. The results are very similar to the split–sample results. The first break point (from the middle state to the low state) is the beginning of 1973 as identified before, but the second one (from the low state to the high state) is now located in the middle of 1981. This result is of importance in light of the alternative explanations offered for the high level of the real interest rate in the 80s. As mentioned by Walsh (1988), two explanations prevail : the first attributes the rise in the real rate to a restrictive monetary policy and identifies the last quarter of 1979 as its starting point, the second to current and expected federal budget deficits, especially since the 1981–1982 recession. Our three–regime model points in the direction of the second explanation as far as the dating is concerned.

Concerning the volatility of the real interest rate, the results tend to confirm our split–sample results since the standard error is about the same in the low and high states but is significantly smaller in the middle state. Hence, our results show an increased volatility after 1973, irrespective of the level of the real interest rate. This heteroskedastic behavior of the series contrasts with the results obtained by Bollerslev (1988) over the period 1951:1 to 1987:2. He shows, using an ARCH methodology, that the ex–post real rate exhibits no significant heteroskedasticity since the stochastic trend in variance present in the nominal interest rate series and the inflation rate is common to the two series. It is to be remembered however that his model, as all ARCH models, rests on a chosen specification for the conditional mean. His conclusions depend on the AR(6) specification in the first difference chosen for the nominal interest rate and the inflation rate. This choice is based on the non–rejection of a unit root in both series. The same argument as for the ex–post real rate can be made for this non–rejection if important changes in mean occur during this period for the nominal rate and the inflation rate. Also Lamoureux and Lastrapes (1990) showed recently, based on Monte–Carlo simulations, that GARCH

measures of persistence in variance can be affected by not taking into account structural shifts in the unconditional variance. This is indeed the case for both the mean and the variance of these two series [15].

Table IV also gives the results of the various model selection tests for a two–state, 2–lag Markov model against a three–state, 2–lag Markov model. The Gallant test is calculated by adding to the 2–state model the estimated $\hat{y}$ obtained from the maximum likelihood estimates of the 3–state model. All three tests reject the 2–state model against the 3–state alternative. The AIC also favors the 3–state model this time, but the SBC definitely favors parsimonious models [16].

The estimation results for the three–state, two–lag model with the Mishkin data set are also reported in Table IV and the probabilities for the states at date t in Figure 3.b. Except for the estimated variance, the results are very close to the ones obtained with the Citibase data. The dating of the changes in regimes is also the same. The estimation results with monthly frequency are very similar except, of course, for the autoregressive parameters. The filter probabilities locate the jump from a low state to a high state at the beginning of 1981, but there is more uncertainty about the starting point of the low state (more toward the end of 1973) and there are many switches between the low and the middle states until 1981. The test results strongly support the three state specification both for the quarterly and the monthly models. Since the AR(4) was not rejected against the two–state model for this data set, we also calculated statistics to test the AR(4) null hypothesis against a three–state two–lag Markov alternative. The results strongly reject the AR(4) specification (at less than the 1% level for the quarterly model).

The estimation results for the two data sets support a three–state Markov model for the ex–post real interest rate. For the quarterly model, neither the autocorrelation parameters nor ω_1 appear to be significant, leaving the series as better described as a random sequence with three different means and two different variances. In the logic of our gradual model building procedure, we should at this stage test our 3–state specification against a 4–state model. Instead, we analyzed, for the Citibase data set, the split samples containing two states each (1961:1–1979:4 and 1973:1–1986:3) and applied the three–state algorithm. These experiments provided no evidence for the presence of a possible fourth state [17].

4.C : A Three–State Model for the Inflation Rate.

As mentioned in the introduction, the evidence about the existence of a unit root in the inflation rate is mixed. Rose (1988) seems to find enough evidence over many periods and countries to reject it, while Bollerslev (1988) and Gokey (1990) are incapable of rejecting the hypothesis of a random walk. By assessing the presence of a large jump in mean, our model might explain why the random walk hypothesis is hard to reject in the case of the inflation rate. Our model will also prove useful in three respects. First, it will possibly indicate that there was a break in the last quarter of 1982 in the inflation rate. This hypothesis was advanced by Walsh (1988) to cast some doubt on Huizinga and Mishkin's (1986) argument about the influence of monetary policy over the real rate. Second, it will tell us if periods of high inflation are also periods of higher volatility in inflation. This is the point made by Friedman (1977). The empirical validity of this conjecture was challenged by Engle (1983) using an ARCH model [18]. Our model will give us direct estimates of the conditional variances (conditional on being in a high, middle or low state) and allow us to answer the question quite directly. Also, we can embed a form of conditional heteroskedasticity in our Markov specification and test if there remains any heteroskedasticity after allowing for discrete shifts in variance in different states. Finally, a model for inflation will permit, by endowing agents with rational expectations, to construct a series of predicted inflation.

The estimation results for the three–state inflation rate model are found in Table V and the corresponding inferred probabilities in Figure 4. Using the Citibase data set, our results document that the U.S. inflation rate went from a low state starting in 1961 (with a mean around 3% and a standard error of 1.3%) to a high state near the beginning of 1973. The mean in this high state is almost 9%, but more importantly, the standard error is 3.4%, almost three times its pre–1973 level. In the early eighties, according to the inferred probabilities, the series seems to oscillate between a high state and a middle state, except at the very end of the sample. Of particular interest is the probability of 0.62 associated with state 2 in 1982:4, which tends to confirm Walsh's (1988) argument. This uncertainty, in the end of the sample, regarding the prevailing state implied by the inferred probabilities parallels the debate among economists and policy makers during that period, some group claiming that inflation had gone down, others that the threat of high inflation was still present. In this regard, note that the middle state is characterized by a low mean, almost identical to the pre–1973 level, but by a much higher variance. It is this variability which induces the agents to differ in their expectations over future levels of inflation and which

impairs the coordination of economic activity by distorting the price signals.

We also report estimation results with the Mishkin series for both the quarterly and the monthly frequencies. For the quarterly frequency, the results are different in some respects since state 1 now has a mean of about 2% but a standard deviation of 1.5% and state 2 a mean of 4% with a standard deviation of 0.7%. In other words, state 1 is now a low mean higher variance state while state 2 has a higher mean and a lower variance than state 1. The classification of the points in the various states also differs: after being in state 1 until 1966, the series goes into state 2 until 1973, jumps into the high state until 1983 (with a similar 0.63 probability of being in state 2 in 1982:4) and alternates between states 1 and 2 until the end of 1986.

The monthly estimates with the Mishkin data set give a higher mean to the high state, but show the same pattern in terms of variance as the quarterly estimates. The probability assignments are quite different though: except for 3 months in the middle of 1973, state 3 (the high state) is not visited. The series is in state 1, with a mean of 5% and a standard error of 1.1%, for most of the sample. The main state 2 episode goes from 1967 to 1969.

The test results are also mixed. While the Davies quick rule fails to reject the 2–state model at any reasonable significance level, both the Gallant test and the J–test strongly favor the three–state model. Given our current ignorance about the power of these different tests, we cannot say much more than state the conflicting results, but it seems likely that a Gallant test, especially when performed by adding the estimated value of the alternative model at the MLE estimates, should have more power than a bound test such as the Davies test. Obviously, more work is needed to assess the size and power properties of these different tests in finite samples.

To test if there are any remaining ARCH effects in the inflation rate series after accounting for different variances in different states, we use the two–state 2–lag Markov model and construct a test of the null hypothesis of no first–order autoregressive conditional heteroskedasticity, as suggested by Hamilton (1989c). We obtained a value of 1.179 for this LM statistic which is asymptotically distributed as a χ^2_1. We therefore cannot reject the null hypothesis of no remaining ARCH effect of the type specified.

5. THE EX–ANTE REAL INTEREST RATE AND THE EXPECTED INFLATION.

The ex–ante real interest rate is of utmost importance, since it is the rate agents in the economy base their savings, investment, and portfolio decisions on. Some authors have in the past constructed ex–ante real series, e.g. Mishkin (1981) and Antoncic (1986). The recent history of the real rate is of particular interest because of the turbulence experienced in the 70s on the inflation front, which is alleged to have pushed the ex–ante real rate to negative levels, and because of the fiscal and monetary policy changes of the 80s which are cited as the sources for a high positive level of the rate [19]. Evidence of the first fact is found in Mishkin (1981), strong evidence of the second fact in Antoncic (1986). In her paper, the negativity of the real rate in the late 70s is just significant.

Using the ex–post real rate three–state model estimated in Section 4, we construct a series for the ex–ante real rate according to the following formula (derived in Appendix B), where $\{y_t\}$ denotes the past and present history of y_t :

$$E[y_{t+1} \mid \{y_t\}] = \alpha_0 + \alpha_1 \left\{ \Sigma^3_{j=1} \, p_{1j} \, \Pr[S_t = j \mid \{y_t\}] \right\} + \alpha_2 \left\{ \Sigma^3_{j=1} \, p_{2j} \, \Pr[S_t = j \mid \{y_t\}] \right\}$$

$$+ \, \phi_1 \Big\{ y_t - \alpha_0 - \alpha_1 \Pr[S_t = 1 \mid \{y_t\}] - \alpha_2 \Pr[S_t = 2 \mid \{y_t\}] \Big\}$$

$$+ \, \phi_2 \Big\{ y_{t-1} - \alpha_0 - \alpha_1 \Pr[S_{t-1} = 1 \mid \{y_t\}] - \alpha_2 \Pr[S_{t-1} = 2 \mid \{y_t\}] \Big\} . \tag{5.1}$$

This construction is of course open to the usual criticism about using the time series model estimated over the entire sample period, thereby endowing the agents with information not fully available to the market. It seems worthwhile, however, to be able to present estimates which give a precise picture of the evolution of the ex–ante real rate. As expected, the ex–post real rate is much more volatile than the ex–ante real rate, as illustrated in Figure 5 . The smooth behavior of the ex–ante real rate contrasts with the series previously constructed based on linear models. It is then seen as a constant subject to occasional jumps caused by important structural events. Both the negativity of the 70s and the high positive levels of the 80s are clearly present in these point forecasts. The results with the Citibase and Mishkin data sets are qualitatively similar, both showing that the ex–ante real interest rate is constant for sustained periods of time but subject to sudden changes in level. The main difference is that the mean level in the period 1973–1980 is close

to zero for the Mishkin data series while it is definitely negative for the Citibase data series.

We also used the estimated three–state model for the inflation rate series to construct inflation forecasts. These are presented in Figure 6. The expected inflation series follows closely the realized inflation series, but more interestingly, given the constancy of the ex–ante real rate in the 80s, it seems that the entire variation in the nominal bill rate during this period is due to the variation in the expected inflation rate.

To round up our univariate model building procedure for the real interest rate and the inflation rate series, we compare the within–sample forecasting ability of the Markov models to a simple random walk model in terms of the mean–squared error over the entire sample [20]. For the real interest rate series, the Markov model yields a mean–squared error of 5.58 and 3.36 for the Citibase and Mishkin quarterly series, respectively, while the random walk model gives values of 8.97 and 5.15. For the inflation rate, the corresponding figures for the Markov model are 7.20 and 3.36, compared to 8.73 and 4.12 for the random walk [21]. Given this criterion, the Markov model is therefore judged to be more appropriate.

6. CONCLUSIONS

The presence of a random walk component in the real interest rate is an important issue, both for public policy concerns and for its theoretical implications. If the real rate does not follow a random walk, then rate changes are temporary in nature and there is a tendency for the rate to revert to a constant value. What we have shown in this paper is that this constant value is subject to occasional jumps caused by important structural events. One such jump is concomitant with the oil price sudden rise at the beginning of 1973. The dating of the second jump in the beginning of the eighties is less clear for pointing to a possible explanation : a two–regime model will locate the jump at the end of 1979, suggesting that the change in monetary policy was at its origin, while a three–regime model will place it in the middle of 1981, a date more in line with the federal budget deficit explanation. Whatever their causes may be, these important jumps in the real interest rate series could well explain the systematic non–rejection of the random walk hypothesis in the recent tests performed by Walsh (1987), Bollerslev (1988), and Rose (1988).

The theoretical implications of the presence of either a unit root or a jump in the real rate series are important. Rose (1988) explored the implications of a unit root in the ex–ante real rate on the consumption CAPM model. The CCAPM implies that the time series properties of the growth rate of consumption and the real interest rate should be similar. Since for the U.S. data this is not verified, he questions the validity of the CCAPM. The presence of jumps in the real rate series is also very important for financial theoretical models, as recently demonstrated by Ahn and Thompson (1988). They find especially that jump diffusion processes in the underlying state variables tend to invalidate standard capital asset pricing models. Models of the type we presented might offer a direction in tests for the CAPM or the CCAPM which take into account these jumps in the series. As Fama noted, a test of market efficiency is not possible without a model for the ex–ante real interest rate. Our model offers an interesting avenue to perform new tests of efficiency of the market for U.S. Treasury bills.

FOOTNOTES

[1]   See, among others, Blanchard (1981), Nelson and Plosser (1982), Campbell and Mankiw (1987), Watson (1986), Hamilton (1989a) and Perron (1988,1989).

[2]   The results of Rose (1988) have recently been criticized by Gokey (1990). He argues that Rose used incorrect inferential procedures that when corrected show both the nominal interest rate and the inflation rate to be integrated of order one. Such a result does not per se imply anything about the time series behavior of the real interest rate since the latter would depend on whether or not the inflation rate and the nominal rate are cointegrated. As argued below, in a linear framework there appears to be no such cointegration (with a unit cointegrating vector) since one cannot reject that the ex–post real interest rate is integrated.

[3]   We also estimated the various models presented below with seasonally adjusted data. The results were qualitatively similar and the conclusions unchanged.

[4]   See Hamilton (1988, 1989a) for the two–state model.

[5]   This model has to be distinguished from the regime switching model used by Sanders and Unal (1988) in their modelization of the nominal interest rate. In their model, they use Goldfeld and Quandt's (1973) switching regression model with time as a classifying variable. They then obtain one, two or more switches depending on the number of regimes they choose for the series but what is crucial is that the resulting regimes are always persistent. In our model, we will also arrive in many instances at persistent states but it has to be remembered that frequent switches from one state to another can also result from the algorithm, and indeed do so in some of our specifications. Contrary to the previous methodology, the outcome of persistent states is endogenously determined.

[6]   To carry out the maximum likelihood estimation of the various models, we used the DFP and GRADX methods in the OPT and CONOPT subroutines of the GQOPT package, respectively for the AR and 2–state models and for the 3–state model (CONOPT stands for optimization under constraints). The covariance matrix of the estimates was obtained through the OPTMOV option, which gives the numerically computed negative inverse of the Hessian of the log–likelihood function, evaluated at the optimum. Also, contrary to the two–state case where the transition probabilities can be written as p and $1 - p$, and q and $1 - q$ respectively, for a three–state model one has to impose the constraint for the sum of the transition probabilities to be one

since even after writing p_{i3} as $1 - p_{i1} - p_{i2}$ one has still to limit the sum of p_{i1} and p_{i2} to be less than or equal to one.

7   As suggested by Hamilton (1989b), the scores can be used to construct asymptotic standard errors for the parameter estimates based on the outer product of the scores, $\text{Var}(\hat{\lambda}) = [\Sigma_1^T h_t(\hat{\lambda}) h_t(\hat{\lambda})']^{-1}$. For a number of estimated models reported in later sections, we have computed standard errors based on the product of the scores as well as the usual matrix based on numerically computed second derivatives evaluated at the optimum. The two procedures yielded essentially the same results and we therefore only report values using the latter method.

8   Boldin (1989) has recently applied the Davies test in testing for the number of states in Markov–switching models.

9   If there are too numerous, a few principal components of the matrix of these new variables can be used.

10   This approach is likely to run into computational difficulties only if the model with the lower number of states is the true model. In that case, estimation of a model with extraneous states involves the presence of unidentifiable parameters. Under the alternative hypothesis such problems do not occur. Hence, the method is best suited when the econometrician believes that the series is better described by a higher number of states. Once the model is judged adequate, the Davies or Gallant test can then be used to assess the possible existence of another state.

11   This dating of the change in regime is close to the point that Perron (1990) chose (1980:3) to illustrate his theoretical argument in the case of the ex–post real rate.

12   In that sense, we do not respect the original idea of the test, where estimation of the larger model is not carried out and one does not have this information, but having this information, the test should be more powerful than picking values of the parameters at random from a certain distribution or over a fine grid. Also, using the filter probabilities is, in our case, qualitatively equivalent to using the calculated $\hat{y}_t$.

13   The monthly estimates should be viewed as a check for the robustness of the quarterly results in terms of the number of states. A more appropriate monthly model would include a richer autocorrelation structure. In the present setting, this would increase unreasonably the computational difficulties in obtaining estimates associated with a global minimum, especially in the three–state model.

14   See Walsh (1987,Table 1, p.8). Note, however, that the t–statistics are generally smaller (weakening the evidence for a unit root) for the 1961:1–1979:3 period, where the change in mean is less dramatic than in the 1970–1985 period. This is in accordance with Perron's argument, whereby the bigger the jump in a series the more difficult it is to reject the unit root hypothesis.

15   We also estimated a three–regime model for the nominal interest rate series. The results (not reported) show an important jump in the mean and variance from 1979:4 until 1982:4. As argued in the next section, there is also an important jump in mean and variance in the inflation rate.

16   We also ran tests for the two–state, 4–lag Markov model against the three–state, 2–lag model. The results obtained were mixed. Both the Davies test and the J–test fail to reject the 2–state, 4–lag model but the Gallant test (performed again with the $\hat{y}$ calculated with the MLE estimates of the 3–state 2–lag model) rejects it overwhelmingly. Since nothing is known about the respective size and power of these tests in finite samples, we cannot say more than state the evidence.

17   With the Citibase series, convergence could not be achieved for the second sub–sample. After 200 iterations, the algorithm identified the two states found earlier, but the third state (a very high variance state with the same negative mean as the first state) is never reached (the filter probabilities for that state are very close to zero). For the first sub–sample, we found many local minima again with two persistent states and a third more volatile state which is reached only with low probability. We were never able to decrease the likelihood value under 75.23, the minimum value obtained with the 2–state 4–lag model. Given the nature of these results and the general similarity of the estimates for both data sets, we did not carry these split–sample estimations for the Mishkin data set.

18   Further evidence on the absence of heteroskedasticity in the variance of U.S. inflation based on the ARCH methodology was recently brought up by Cosimano and Jansen (1988).

19   See, for example, Blanchard and Summers (1984), for a discussion of the possible causes for the increase in the real interest rate in the 80s.

20   For the random walk model, the mean–squared error is defined as $MSE_{rw} = T^{-1}\Sigma_1^T(y_t - y_{t-1})^2$, and for the Markov model as $MSE_{mk} = T^{-1}\Sigma_1^T(y_t - E[y_t|\{y_{t-1}\}])^2$, where $E[y_t|\{y_{t-1}\}]$ is defined by (5.1).

21 For the monthly series, the random walk model beats slightly the Markov model. This is probably due to a too parsimonious characterization of the autoregressive structure for the Markov model.

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APPENDIX A : Description of the Test Statistics

1. LM test for ARCH effects.

The test for remaining ARCH effects after allowing for different variances in different regimes is based on the LM test proposed by Hamilton (1989c). In our 2–state 2–lag model, denote the vector of unknown parameters by $\theta = (\alpha_0, \alpha_1, \omega_0, \omega_1, \phi_1, \phi_2, p, q)$. Similarly, denote the maximum likelihood estimate of the vector of parameters by $\theta^* = (\alpha_0^*, \alpha_1^*, \omega_0^*, \omega_1^*, \phi_1^*, \phi_2^*, p^*, q^*)$. Under the hypothesis that the variance of the errors is characterized by the presence of a first–order ARCH effect, the variance of the errors, $\sigma^2(S_t)$ in (3.1), can be described by $h_t = (\omega_0 + \omega_1 S_t)^2[1 + \eta\, X^2_{t-1}/(\omega_0 + \omega_1 S_{t-1})^2]$ where $X_t = y_t - \alpha_0(1 - \phi_1 - \phi_2) - \alpha_1(S_t - \phi_1 S_{t-1} - \phi_2 S_{t-2}) - \phi_1 y_{t-1} - \phi_2 y_{t-2}$. Note that when $\eta = 0$, there is no ARCH effect. Using this notation, the conditional likelihood function of the data can be written as :

$$p(y_t \mid y_{t-1}, ..., y_{t-3}, S_t, S_{t-1}, ... S_{t-3}, \theta, \eta) = (2\pi h_t)^{-1/2}\exp(-X^2_t/(2h_t)).$$

Let $\sigma^*(S_t) = \omega_0 + \omega_1 S_t$, $\{S_t\} \equiv (S_t, S_{t-1}, S_{t-2}, S_{t-3})$, and define Ω_t to be the information set available at time t. The score with respect to η evaluated at $\eta = 0$ is given by

$$\partial \ln p(y_t \mid y_{t-1}, ..., y_{t-3}, \theta^*)/\partial\eta \mid_{\eta=0}$$

$$= \sum_{S_t=0}^{1} \sum_{S_{t-1}=0}^{1} \sum_{S_{t-2}=0}^{1} \sum_{S_{t-3}=0}^{1} [-1 + X^2_t/\sigma^*(S_t)^2][X^2_{t-1}/2\sigma^*(S_{t-1})^2]\, p(\{S_t\} \mid \Omega_t)$$

$$+ \sum_{\tau=4}^{t-1} \sum_{S_\tau=0}^{1} ... \sum_{S_{\tau-3}=0}^{1} [-1 + X^2_\tau/\sigma^*(S_\tau)^2][X^2_{\tau-1}/2\sigma^*(S_{\tau-1})^2][p(\{S_\tau\}\mid\Omega_\tau) - p(\{S_\tau\} \mid \Omega_{\tau-1})]$$

Denote by $\lambda_t(\theta^*, \eta) \equiv [\partial\log p(y_t\mid \cdot, \theta^*)\partial\theta, \partial\log p(y_t \mid \cdot)\partial\eta \mid_{\eta=0}]$ the stacked vector of scores evaluated at the restricted MLE estimates θ^*, the LM test is then defined as :

$$LM = (\Sigma^T_{t=3} \lambda_t(\theta^*,0))'[(\Sigma^T_{t=3} \lambda_t(\theta^*,0))(\Sigma^T_{t=3} \lambda_t(\theta^*,0))']^{-1}(\Sigma^T_{t=3} \lambda_t(\theta^*,0))$$

which is asymptotically distributed as χ^2_1 under the null hypothesis that $\eta = 0$.

2. Davies' (1987) Bound Test

The procedure proposed by Davies applies when a q − vector of independent parameters, say ν, is present only under the alternative hypothesis. Define the likelihood ratio statistic as a function of ν :

$$LR(\nu) = 2(\ln L_1(\nu_g) - \ln L^*_0)$$

where $L_1(\nu_g)$ denotes the likelihood value of the objective function evaluated at ν_g (a given value for ν) under the alternative hypothesis, and L^*_0 denotes the maximized value obtained under the null hypothesis (where ν is not present). Let $\nu^* = \text{argmax } L_1(\nu)$ and $M = \text{max}$ $LR(\nu) \equiv 2(\ln L^*_1 - \ln L^*_0)$. Davies derives an upper bound for the significance level of M given by :

$$Pr[LR(\nu) > M] = Pr[\chi^2_q > M] + VM^{(q-1)/2}\exp(-M/2)(2)^{-q/2}/\Gamma(q/2)$$

where where $\Gamma(\cdot)$ denotes the gamma function and V is defined as :

$$V = \int^{\nu_u}_{\nu_a} |\partial LR(\nu)^{1/2}/\partial\nu| \, d\nu \qquad (A.1)$$

$$= |LR(\nu_1)^{1/2} - LR(\nu_a)^{1/2}| + |LR(\nu_2)^{1/2} - LR(\nu_1)^{1/2}| +...+ |LR(\nu_u)^{1/2} - LR(\nu_n)^{1/2}|$$

where ν_1, ν_2, ..., ν_n are the turning points of $LR(\nu)$. The quick rule is obtained upon making the assumption that the likelihood ratio has a single peak. In that case we have $V = 2M^{1/2}$. Our testing procedure uses this quick rule and estimates the model under the alternative hypothesis to obtain L^*_1 (and therefore M and V) to calculate the significance

level. Another procedure, which avoids estimating the model under the alternative hypothesis, would consider using a fine grid of values of the vector ν, find the maximized value $L_1(\nu^*)$ over this grid (with associated vector ν^*), and calculate V using these values substituted in (A.1).

3. Gallant's (1977) Procedure.

Consider the following models under the null and alternative hypotheses :

$$H_0 : y_t = g(x_t, \psi) + e_t$$

$$H_1 : y_t = g(x_t, \psi) + \tau\, d(x_t, \varphi) + e_t \,.$$

The basic idea of the test is straightforward. Let z_t be a given vector of variables which do not depend on unknown parameters. If τ_0, the true value of τ, is equal to 0, the least squares estimator of δ in the following regression :

$$y_t = g(x_t, \psi) + z_t'\delta + e_t \tag{A.2}$$

is estimating the zero vector. Let $\beta \equiv (\alpha_0, \alpha_1, \alpha_2, \omega_0, \omega_1, \omega_2, p_{ij} \ (i,j = 1,2,3))$ be the vector of parameters in the three–state model (in the two–state model the vector is defined similarly without α_2, ω_2 and $p_{i,j}$ $(i,j = 3)$). The Gallant procedure applied to determining the number of states in a Markov switching model follows four steps :

i) For a given set of values for β, calculate the fitted values $\hat{y}_t$ for the model with the larger number of states ($\hat{y}_t$ is a vector corresponding to different possible values for β).

ii) If the matrix $Y \equiv (y_1', ..., y_T')$ is too big, extract a few principal components, say d, (or the first few vectors of the orthogonal matrix in a singular value decomposition of Y).

iii) Add these principal components (call them z_t, a vector of dimension d) to the model with the lower number of states, i.e. estimate (A.2) with the function $g(x_t, \psi)$ being the model with the lower number of states. Denote the estimated parameters by $\tilde{\psi}$ and $\tilde{\delta}$.

iv) Compute the following residual sums of squares :

$$\tilde{\sigma}^2 = \Sigma^T_{t=1}(y_t - g(y_t, \tilde{\beta}) - \tilde{\delta} z_t)^2$$

and

$$\hat{\sigma}^2 = \Sigma^T_{t=1}(y_t - g(y_t, \hat{\beta}))^2$$

where $\hat{\beta}$ is the MLE of β under the null hypothesis ($\delta = 0$). The likelihood ratio test, with size α, rejects the null hypothesis if :

$$T \equiv \hat{\sigma}^2/\tilde{\sigma}^2 > 1 + d\, F_\alpha/(T - d - \mu)$$

where μ is the number of parameters estimated under the null hypothesis, d is the dimension of the vector z_t and F_α denotes the α percentage point of a $F(d, T - \mu - d)$ distributed random variable.

APPENDIX B : Derivation of the One Period Ahead Forecast

We write the model as :

$$y_t = \alpha_0 + \alpha_1 S1_t + \alpha_2 S2_t + z_t \tag{B.1}$$

$$z_t = \phi_1 z_{t-1} + \phi_2 z_{t-2} + (\omega_0 + \omega_1 S_{1t} + \omega_2 S_{2t})\epsilon_t \tag{B.2}$$

where $S1_t = 1$, if $S_t = 1$, and 0 otherwise ; $S2_t = 1$, if $S_t = 2$ and 0 otherwise. Denote by $\{y_t\}$ the information set available about the variable y at time t, i.e the set $(y_t, y_{t-1}, y_{t-2}, ...)$. Using (B.1) we have :

$$E[y_{t+1} \mid \{y_t\}] = \alpha_0 + \alpha_1 E[S1_{t+1} \mid \{y_t\}] \tag{B.3}$$
$$+ \alpha_2 E[S2_{t+1} \mid \{y_t\}] + E[z_{t+1} \mid \{y_t\}] .$$

We consider each term in (B.3) separately.

a) $E[S1_{t+1} \mid \{y_t\}] = Pr[S1_{t+1} = 1 \mid \{y_t\}] = Pr[S_{t+1} = 1 \mid \{y_t\}]$
$$= \Sigma_{j=1}^3 Pr[S_{t+1} = 1, S_t = j \mid \{y_t\}]$$
$$= \Sigma_{j=1}^3 \left\{ Pr[S_{t+1} = 1 \mid S_t = j] \, Pr[S_t = j \mid \{y_t\}] \right\} ;$$

using the Markov assumption about the transition probabilities. Hence :

$$E[S1_{t+1} \mid \{y_t\}] = \Sigma_{j=1}^3 P_{1j} \, Pr[S_t = j \mid \{y_t\}] . \tag{B.4}$$

Similarly :

$$E[S2_{t+1} \mid \{y_t\}] = \Sigma_{j=1}^3 P_{2j} \, Pr[S_t = j \mid \{y_t\}] \tag{B.5}$$

b) Consider now the last term in (B.3). Using (B.2) :

$$E[z_{t+1} \mid \{y_t\}] = \phi_1 E[z_t \mid \{y_t\}] + \phi_2 E[z_{t-1} \mid \{y_t\}] \tag{B.6}$$

$$+ E[(\omega_0 + \omega_1 S1_{t+1} + \omega_2 S2_{t+1})\, \epsilon_{t+1} \mid \{y_t\}] \,.$$

Consider the first expression in (B.6) :

$$E[z_t \mid \{y_t\}] = E[y_t - \alpha_0 - \alpha_1 S1_t - \alpha_2 S2_t \mid \{y_t\}]$$

$$= y_t - \alpha_0 - \alpha_1 E[S1_t \mid \{y_t\}] - \alpha_2 E[S2_t \mid \{y_t\}] \,.$$

Note that $E[S1_t \mid \{y_t\}] = Pr[S1_t = 1 \mid \{y_t\}] = Pr[S_t = 1 \mid \{y_t\}]$ and similarly $E[S2_t \mid \{y_t\}] = Pr[S_t = 2 \mid \{y_t\}]$. Hence :

$$E[z_t \mid \{y_t\}] = y_t - \alpha_0 - \alpha_1 Pr[S_t = 1 \mid \{y_t\}] - \alpha_2 Pr[S_t = 2 \mid \{y_t\}] \,. \tag{B.7}$$

Similarly :

$$E[z_{t-1} \mid \{y_t\}] = y_{t-1} - \alpha_0 - \alpha_1 Pr[S_{t-1} = 1 \mid \{y_t\}] - \alpha_2 Pr[S_{t-1} = 2 \mid \{y_t\}]. \tag{B.8}$$

Finally consider the last term in (B.6) :

$$E[\,(\omega_0 + \omega_1 S1_{t+1} + \omega_2 S2_{t+1})\, \epsilon_{t+1} \mid \{y_t\}]$$

$$= E\Big\{ E[\,(\omega_0 + \omega_1 S1_{t+1} + \omega_2 S2_{t+1})\, \epsilon_{t+1} \mid \{y_t\}, s_{t+1}] \mid \{y_t\}\Big\}$$

$$= E\Big\{ (\omega_0 + \omega_1 S1_{t+1} + \omega_2 S2_{t+1})\, E[\, \epsilon_{t+1} \mid \{y_t\}, s_{t+1}] \mid \{y_t\}\Big\}$$

$$= 0 \ ; \text{since } E[\, \epsilon_{t+1} \mid \{y_t\}, s_{t+1}\,] = 0 \,.$$

Combining (B.3) through (B.8) we obtain equation (5.1).

TABLE I : Estimation Results; Ex-Post Real Interest Rate (61:1–86:3)
1 versus 2–states models

	Citibase Data Set		Mishkin Data Set			
			Quarterly		Monthly	
	AR(4)	2–states	AR(4)	2–states	AR(4)	2–states
α_0	1.927 (1.851)	1.556 (0.944)	2.017 (1.177)	1.628 (1.444)	1.894 (0.840)	1.757 (0.646)
α_1	–	0.945 (0.643)		0.790 (0.341)	–	–0.202 (0.611)
ω_0	2.395 (0.170)	1.044 (0.263)	1.954 (0.145)	0.280 (0.086)	1.166 (0.047)	0.840 (0.055)
ω_1	–	2.490 (0.557)		1.800 (0.176)	–	0.898 (0.163)
ϕ_1	0.282 (0.098)	0.329 (0.075)	0.420 (0.100)	0.401 (0.066)	1.079 (0.051)	1.118 (0.056)
ϕ_2	0.117 (0.096)	–0.042 (0.093)	0.124 (0.107)	0.141 (0.053)	–0.166 (0.076)	–0.127 (0.084)
ϕ_3	0.289 (0.096)	0.201 (0.084)	0.157 (0.108)	0.151 (0.047)	–0.399 (0.076)	–0.491 (0.082)
ϕ_4	0.183 (0.098)	0.288 (0.087)	0.133 (0.106)	0.188 (0.041)	–0.407 (0.050)	0.412 (0.053)
p	–	0.246 (0.170)	–	0.946 (0.033)	–	0.948 (0.039)
q	–	0.446 (0.210)	–	0.494 (0.212)	–	0.978 (0.017)
Fmin	135.98	127.19	116.99	113.54	201.31	182.07
AIC	273.96	274.38	245.98	247.08	414.62	384.14
SBC	299.53	300.33	261.61	273.13	437.00	421.44

Tests :

Davies :	P[LR > 17.53] = .003	P[LR > 6.9] = .25	P[LR > 38.5] ≈ .00
J–test :	$\alpha = 0.998$ (.03)	$\alpha = .998$ (.011)	$\alpha = .053$ (.046)
Gallant :	$P[\sigma^{2*}/\hat\sigma^2 > 1.43] \approx .00$	$P[\sigma^{2*}/\hat\sigma^2 > 1.014] = .25$	$P[\sigma^{2*}/\hat\sigma^2 > 1.00] = .27$

Notes :   –Asymptotic standard errors are indicated in parentheses.

–Fmin refers to the minimized value of the objective function.

–AIC and SBC stand for the Akaike Information Criterion and the Schwartz Bayesian Criterion respectively.

For the Davies and Gallant tests, the given probabilities are p–values for the null hypothesis that the model is the one with the smallest number of regime.

For the J–test the asymptotic standard error is reported in parentheses.

TABLE II : Estimation Results ; Ex–Post Real Interest Rate (1961:1–1986:3)

2 states model ; Local Minima

	# 1	# 2	# 3
α_0	1.535 (1.278)	2.572 (3.068)	−0.706 (1.261)
α_1	−0.587 (1.839)	−0.401 (1.598)	5.873 (1.141)
ω_0	1.364 (0.170)	1.914 (0.156)	1.886 (0.169)
ω_1	1.608 (0.358)	2.272 (1.042)	1.074 (0.454)
ϕ_1	0.174 (0.105)	0.236 (0.094)	0.225 (0.095)
ϕ_2	0.175 (0.100)	0.110 (0.098)	0.077 (0.099)
ϕ_3	0.194 (0.098)	0.205 (0.095)	0.212 (0.104)
ϕ_4	0.266 (0.103)	0.373 (0.103)	0.296 (0.109)
p	0.987 (0.016)	0.887 (0.104)	0.986 (0.018)
q	0.986 (0.019)	0.985 (0.016)	0.992 (0.010)
Fmin	128.55	130.20	130.52

Notes : Standard errors are in parentheses and Fmin denotes the minimized value of the objective function.

TABLE III : Estimation Results : Ex-Post Real Interest Rate , 2-states model.

Split-Sample Results

	Citibase Data Set		Mishkin Data set			
			Quatertly		Monthly	
	61:1–79:4	73:1–86:3	61:1–79:4	73:1–86:4	61:1–79:12	73:1–86:12
α_0	1.336	−1.842	1.775	−0.579	1.682	−0.080
	(0.262)	(0.774)	(0.202)	(0.254)	(0.207)	(0.741)
α_1	−3.460	6.974	−2.448	5.987	−2.371	4.212
	(0.599)	(1.111)	(0.358)	(0.450)	(0.387)	(0.944)
ω_0	1.261	2.237	1.239	1.606	0.786	1.135
	(0.142)	(0.354)	(0.165)	(0.246)	(0.054)	(0.134)
ω_1	1.031	0.553	0.461	0.351	0.425	0.307
	(0.342)	(0.539)	(0.311)	(0.396)	(0.125)	(0.233)
ϕ_1	−0.018	0.254	0.016	0.154	0.864	1.108
	(0.119)	(0.143)	(0.133)	(0.145)	(0.078)	(0.079)
ϕ_2	0.040	−0.174	−0.198	−0.321	0.068	−0.358
	(0.118)	(0.141)	(0.140)	(0.152)	(0.083)	(0.131)
ϕ_3	−0.012	0.247	−0.075	−0.097	−0.585	−0.207
	(0.116)	(0.141)	(0.135)	(0.143)	(0.082)	(0.128)
ϕ_4	0.256	0.058	0.076	−0.173	−0.273	0.271
	(0.117)	(0.139)	(0.133)	(0.174)	(0.069)	(0.086)
p	0.983	1.000	0.984	0.978	0.995	0.990
	(0.021)	(0.000)	(0.021)	(0.028)	(0.007)	(0.011)
q	0.986	0.965	0.985	0.981	0.996	0.982
	(0.016)	(0.036)	(0.019)	(0.023)	(0.006)	(0.015)
Fmin	75.23	76.57	65.60	61.15	101.70	132.55

Notes : Standard errors are in parentheses and Fmin denotes the minimized value of the objective function.

TABLE IV : Estimation Results ; Real Interest Rate ; 2 and 3 States Models

| | Citibase Data Set | | Mishkin Data set | | | |
| | | | Quatertly | | Monthly | |
	2 states	3 states	2 states	3 states	2 states	3 states
α_0	1.341	5.488	0.673	5.675	1.823	4.984
	(0.689)	(0.644)	(1.351)	(0.407)	(0.437)	(0.509)
α_1	0.0096	−7.269	1.313	−6.050	−1.503	−6.789
	(1.319)	(0.691)	(6.535)	(0.506)	(0.805)	(0.711)
α_2	−	−4.108	−	−3.807	−	−3.864
		(0.639)		(0.445)		(0.552)
ω_0	1.397	2.856	0.314	1.873	0.834	1.336
	(0.157)	(0.426)	(2.468)	(0.291)	(0.060)	(0.103)
ω_1	1.849	−0.335	1.800	−0.0001	0.991	−0.047
	(0.340)	(0.556)	(1.763)	(0.368)	(0.154)	(0.221)
ω_2	−	−1.586	−	−0.723	−	−0.508
		(0.449)		(0.319)		(0.114)
ϕ_1	0.305	−0.020	0.577	0.029	1.178	1.169
	(0.098)	(0.111)	(0.425)	(0.107)	(0.064)	(0.054)
ϕ_2	0.367	0.013	0.139	−0.043	−0.309	−0.414
	(0.091)	(0.116)	(0.742)	(0.103)	(0.060)	(0.052)
p_{11}	0.988	0.968	0.915	0.967	0.922	0.835
	(0.014)	(0.024)	(0.515)	(0.025)	(0.037)	(0.001)
p_{12}	−	0.013	−	0.000	−	0.165
		(0.011)		(0.000)		(0.001)
p_{21}	−	0.011	−	0.015	−	0.033
		(0.000)		(0.023)		(0.000)
p_{22}	0.988	0.988	0.329	0.985	0.962	0.962
	(0.016)	(0.000)	(0.207)	(0.002)	(0.018)	(0.001)
p_{31}	−	0.000	−	0.000	−	0.000
		(0.000)		(0.000)		(0.000)
p_{32}	−	0.017	−	0.048	−	0.011
		(0.000)		(0.001)		(0.005)
Fmin	135.95	125.43	120.74	105.43	209.17	200.75
AIC	287.90	278.86	257.48	238.86	434.34	429.50
SBC	308.82	315.47	278.48	275.61	464.23	481.81

Tests :

Davies P[LR > 21.04] = .002   P[LR > 30.62] ≈ .00   P[LR > 16.84] = .013

Gallant P[$\sigma^{2*}/\hat{\sigma}^2 > 1.71$] ≈ .00   P[$\sigma^{2*}/\hat{\sigma}^2 > 1.82$] ≈ .00   P[$\sigma^{2*}/\hat{\sigma}^2 > 2.12$] ≈ .00

J−test α = 1.350 (.175) α = 1.415 (.136) α = 1.476 (.074)

See notes to Table I.

	Citibase Data Set		Mishkin Data set			
			Quatertly		Monthly	
	2 states	3 states	2 states	3 states	2 states	3 states
α_0	3.374 (0.805)	8.776 (1.579)	3.033 (0.945)	7.520 (0.645)	4.401 (0.614)	10.189 (1.027)
α_1	3.411 (1.765)	−5.449 (1.657)	3.129 (1.280)	−5.217 (1.047)	0.794 (0.789)	−5.135 (0.556)
α_2	–	−6.028 (1.986)	–	−3.430 (0.734)	–	−6.593 (0.955)
ω_0	1.283 (0.147)	3.384 (0.444)	1.234 (0.148)	2.153 (0.250)	0.821 (0.063)	1.334 (0.552)
ω_1	2.126 (0.376)	−2.079 (0.484)	0.914 (0.251)	−0.640 (0.329)	0.890 (0.195)	−0.245 (0.556)
ω_2	–	−0.823 (0.949)	–	−1.414 (0.319)	–	−0.942 (0.562)
ϕ_1	0.386 (0.098)	0.351 (0.096)	0.456 (0.101)	0.283 (0.112)	1.213 (0.061)	1.262 (0.056)
ϕ_2	0.374 (0.099)	0.321 (0.108)	0.309 (0.100)	0.192 (0.123)	−0.301 (0.059)	−0.358 (0.054)
p_{11}	0.989 (0.014)	0.987 (0.015)	0.988 (0.015)	0.890 (0.073)	0.820 (0.076)	0.986 (0.018)
p_{12}	–	0.000 (0.000)	–	0.110 (0.072)	–	0.010 (0.016)
p_{21}	–	0.025 (0.028)	–	0.090 (0.002)	–	0.090 (0.004)
p_{22}	0.988 (0.015)	0.975 (0.006)	0.986 (0.017)	0.871 (0.026)	0.943 (0.029)	0.908 (0.010)
p_{31}	–	0.000 (0.000)	–	0.043 (0.139)	–	0.346 (0.069)
p_{32}	–	0.031 (0.001)	–	0.000 (0.000)	–	0.000 (0.000)
Fmin	135.97	134.61	108.81	106.08	176.52	171.37
AIC	287.94	297.22	233.62	240.16	369.04	370.74
SBC	308.86	333.83	254.62	276.91	398.93	423.05

Tests :

Davies	$P[LR > 2.72] \approx 1.00$		$P[LR > 5.46] = 0.80$		$P[LR > 10.3] = .17$	
Gallant	$P[\sigma^{2*}/\hat{\sigma}^2 > 1.51] \approx .00$		$P[\sigma^{2*}/\hat{\sigma}^2 > 1.68] \approx .00$		$P[\sigma^{2*}/\hat{\sigma}^2 > 1.19] \approx .00$	
J–test	$\alpha = .9999 \ (.0003)$		$\alpha = 1.79 \ (.053)$		$\alpha = 1.234 \ (.275)$	

See notes to Table I.
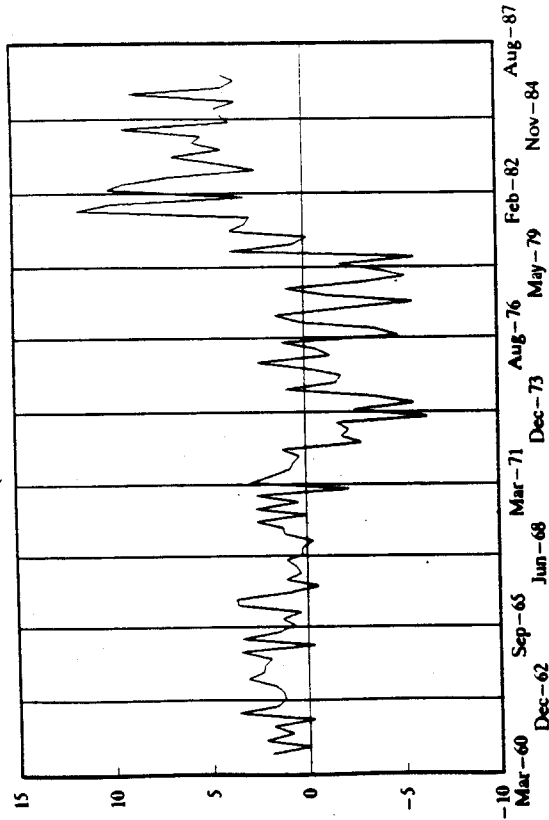
B) Real Ex–Post Interest Rate
(Mishkin Data Set)

A) Real Ex–Post Interest Rate
(Citibase Data Set)

D) Inflation Rate
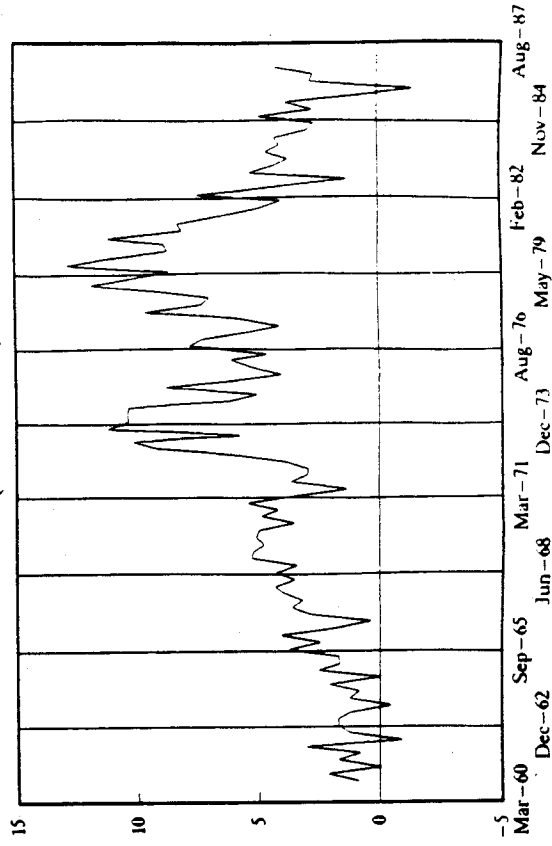(Mishkin Data Set)

C) Inflation Rate
(Citibase Data Set)

Figure 1. a) Real Ex-post Interest Rate (Citibase data set); b) Real Ex-post Interest Rate (Mishkin data set); c) Inflation rate (Citibase data set); d) Inflation rate (Mishkin data set). Data are quarterly from 61:1 to 86:3 and seasonally unadjusted.
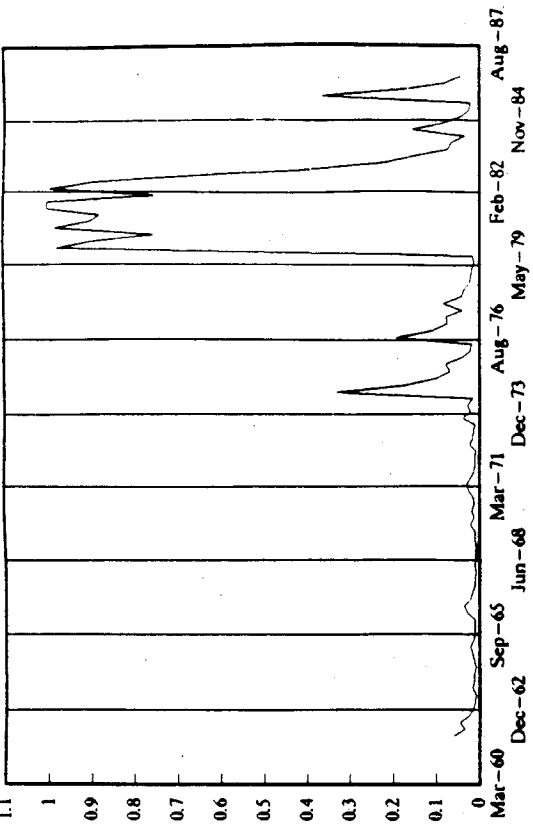
A) Filter Probabilities

Real Interest Rate, two-state model, local minimum #1, fmin=128.55
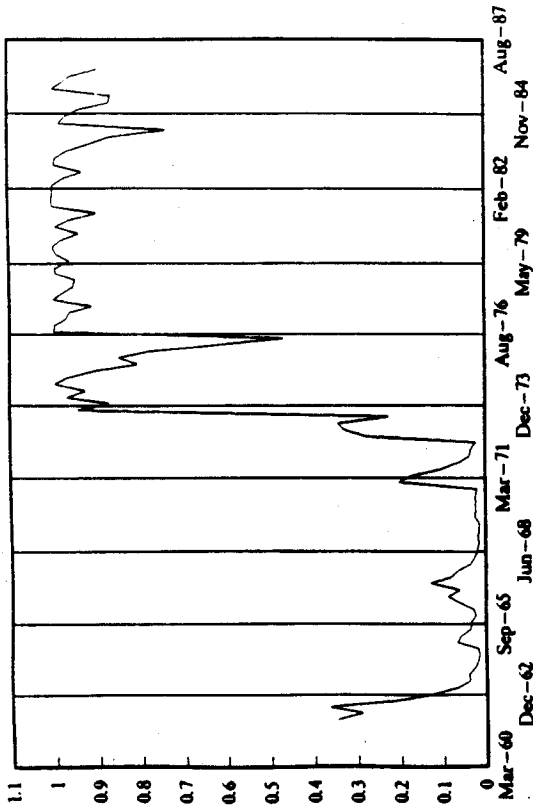
B) Filter Probabilities

Real Interest Rate, two-state model, local minimum #2, fmin=130.20

C) Filter Probabilities

Real Interest Rate, two-state model, local minimum #3, fmin=130.52
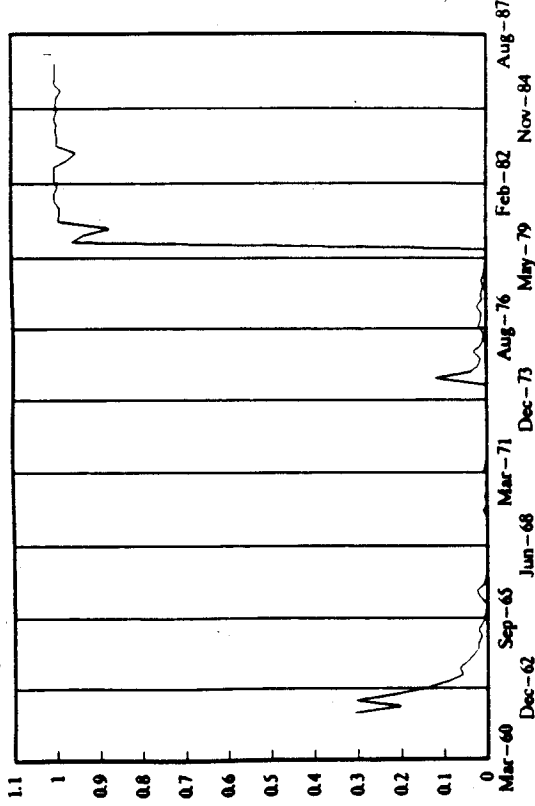
D) Filter Probabilities

Real Interest Rate, two-state model, minimum, fmin=127.19
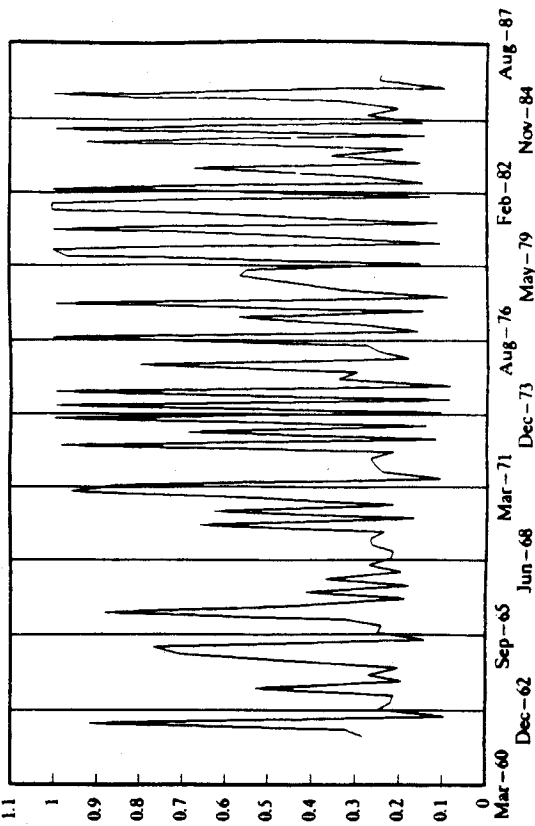
Figure 2. $P[S_t = 1 | r_t, r_{t-1}, \ldots]$; inferred probability that the real interest rate is in State 1 given the information available at date t obtained using a two-state model. Panels A, B and C correspond to estimates associated with local minima of interest; Panel D corresponds to estimates associated with the global minimum. In Panels A and B State 1 corresponds to the low mean-low variance state; in Panels C and D State 1 corresponds to the high mean-high variance state (see Tables I and II). The data used are from the Citibase data set.

A) Filter Probabilities
Real Interest Rate, Citibase Data Set, Three-state model
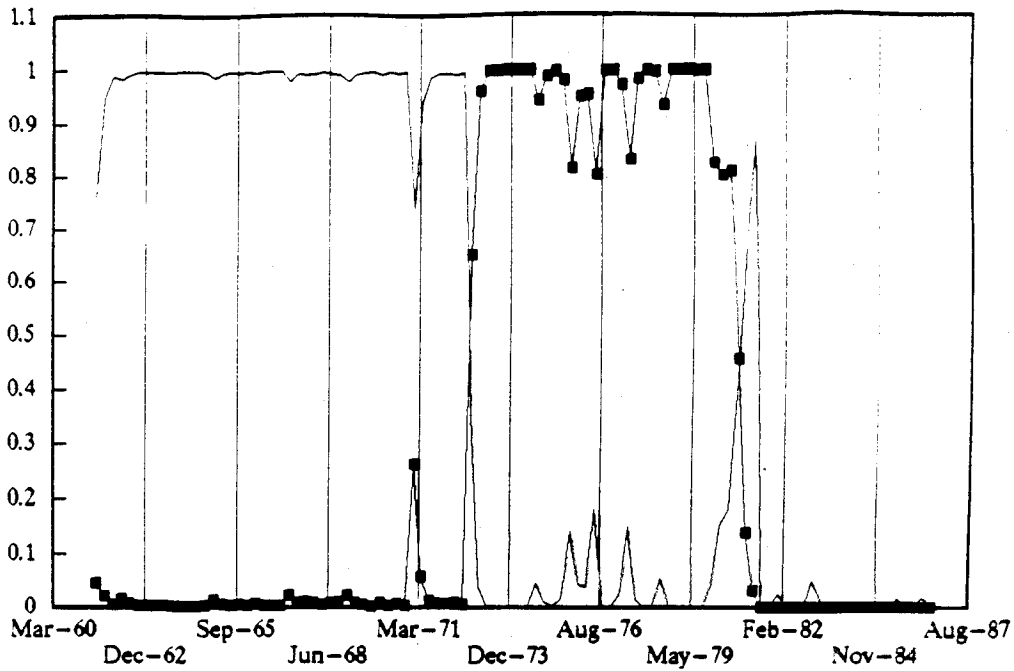
B) Filter Probabilities
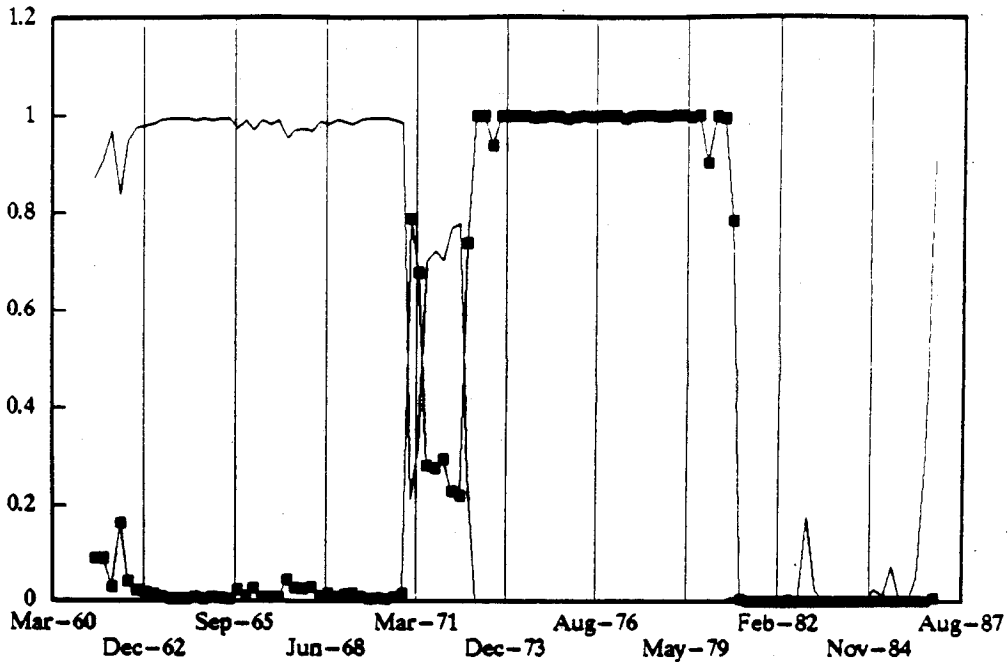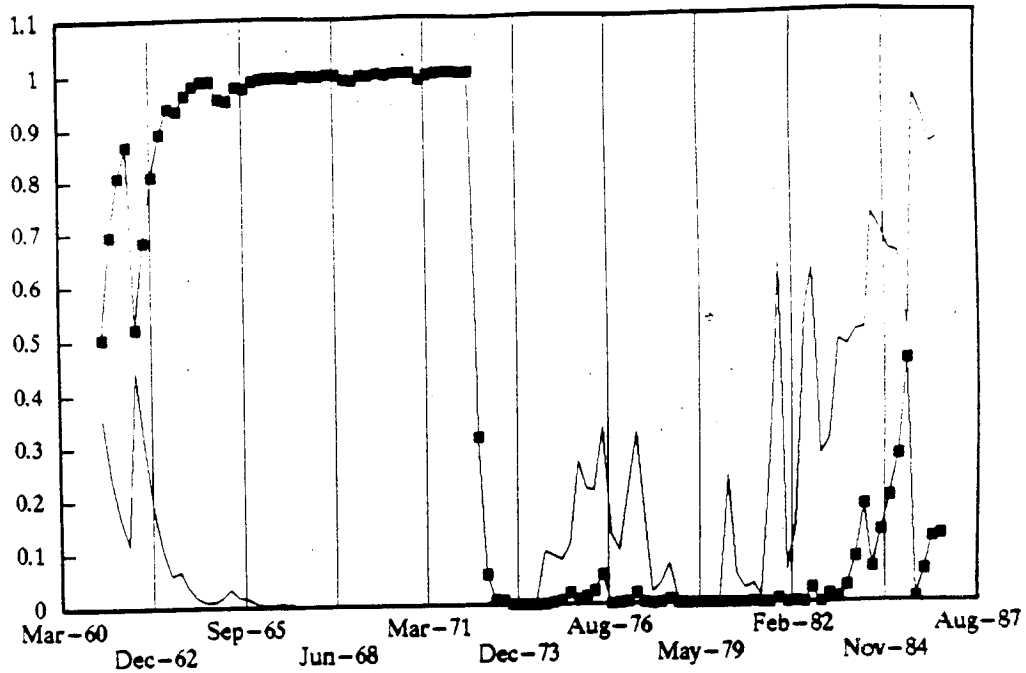### Real Interest Rate, Mishkin Data Set, Three-state model

Figure 3. Inferred probability that the real interest rate is in State 1 or in State 2 given the information available at date t (see eq. (3.7)), obtained using a three-state model. The thin curve corresponds to the probability of being in State 1 ($P[S_t = 1|\{r_t\}]$) associated with the intermediate mean level. The curve formed with dark boxes corresponds to the probability of being in State 2 ($P[S_t = 2|\{r_t\}]$) associated with the low mean state (see Table IV). The probability of being in State 0 associated with a high mean can be obtained as one minus the sum of these two probabilities. Panel A): Citibase data set; panel B): Mishkin data set.

A) Filter Probabilities
Inflation Rate, Citibase Data Set, Three-state model

B) Filter Probabilities
Inflation Rate, Mishkin Data Set, Three-state model
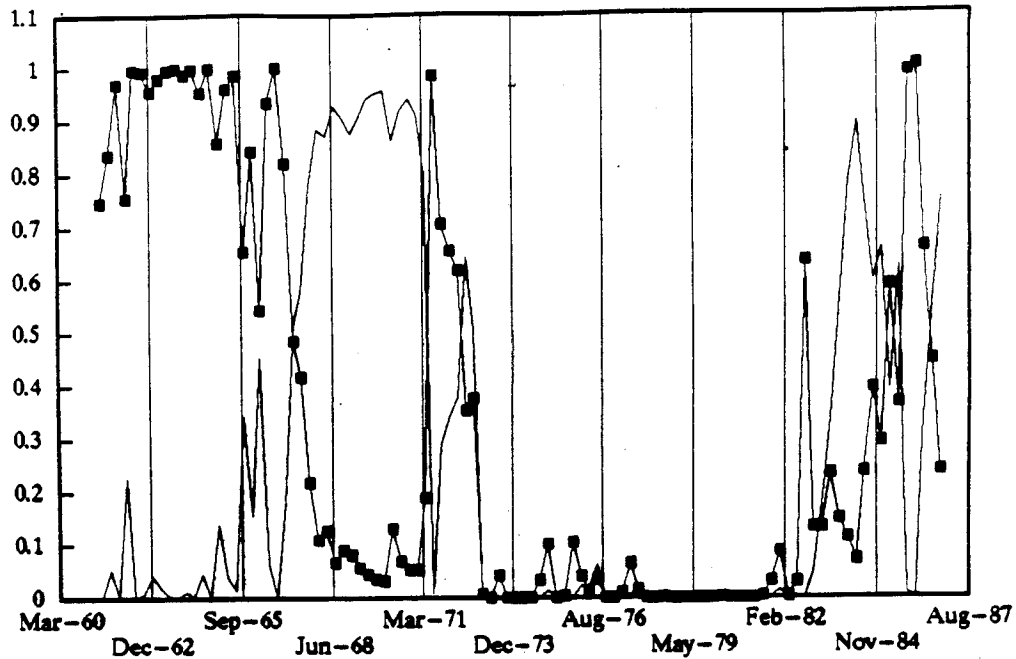

Figure 4.   Inferred probability that the inflation rate is in State 1 or State 2 given the information available at time t (see eq. (3.7)), obtained using a three-state model. The thin curve corresponds to the probability of being in State 1 (P[S_t = 1|(π_t)]) associated with the intermediate mean level. The curve formed with dark boxes corresponds to the probability of being in State 2 (P[S_t = 2|(π_t)]) associated with the low mean state (see Table V). The probability of being in State 0, associated with a high mean can be obtained as one minus the sum of these two probabilities. Panel A): Citibase data set; panel B): Mishkin data set.

A) Real Interest Rate
ex ante and ex post (Citibase Data Set)
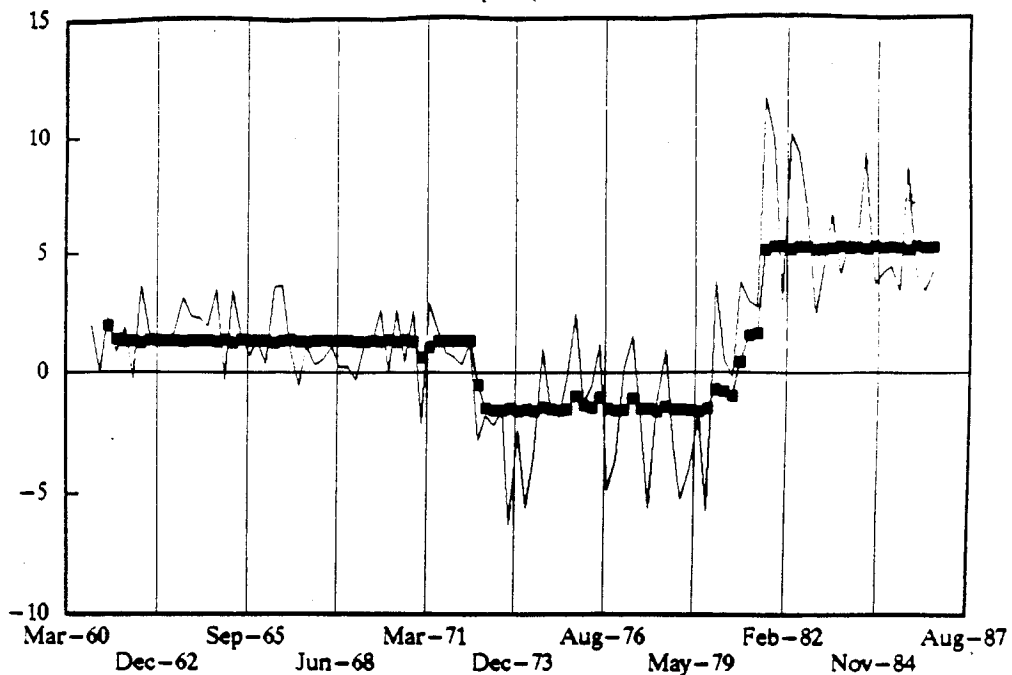
B) Real Interest Rate
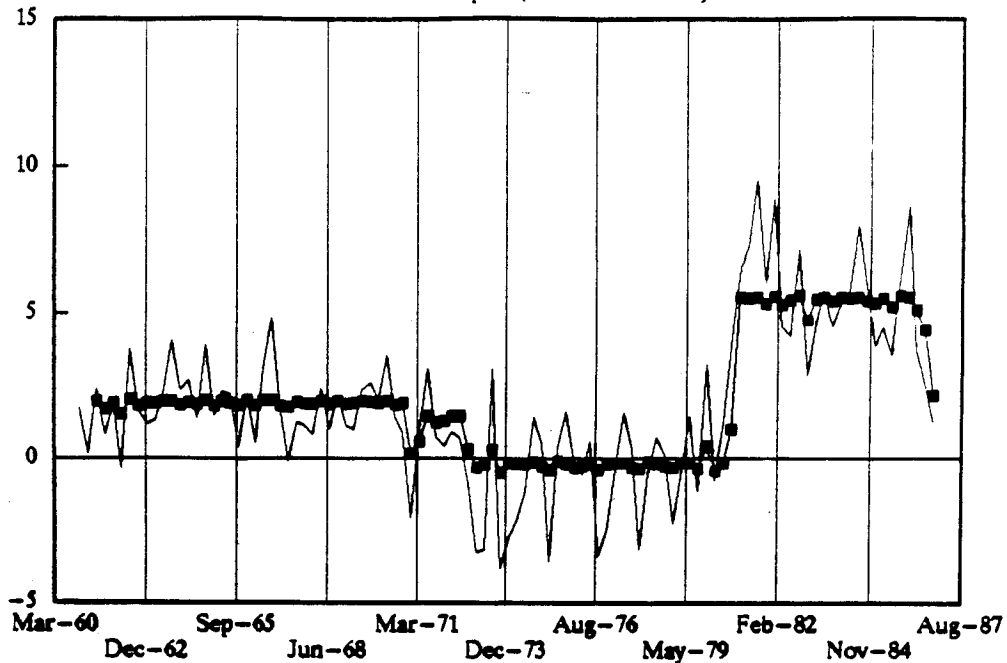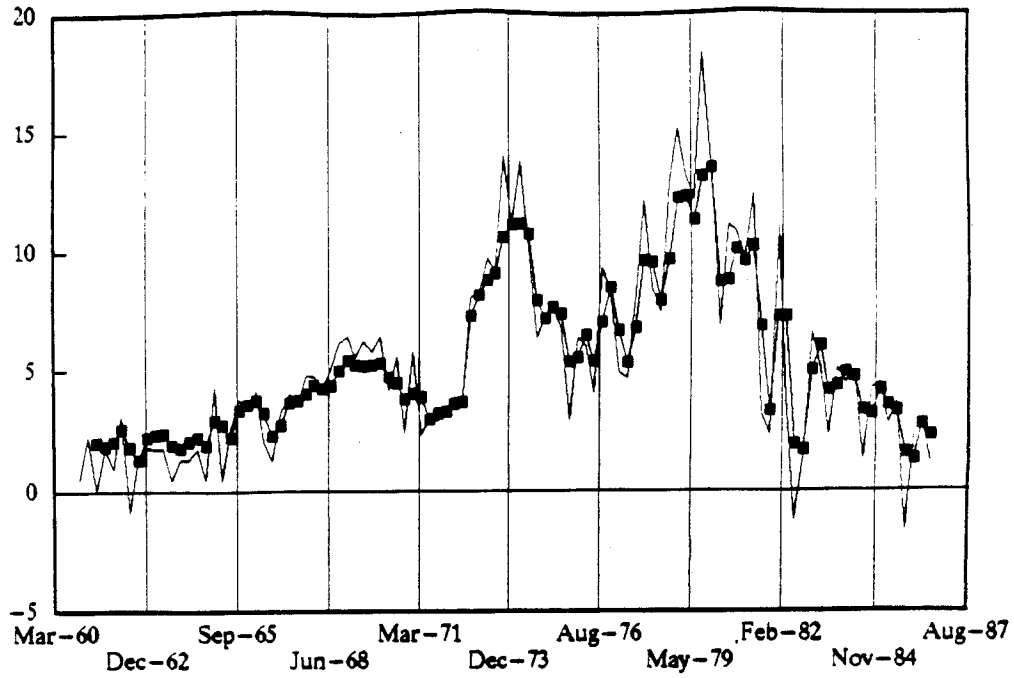### ex ante and ex post (Mishkin Data Set)

Figure 5. Ex-ante real interest rate, $E[r_{t+1}|(r_t)]$ (represented as dark boxes), constructed using eq. (5.1) with the estimated three-state model (Table IV) and the filter probabilities for each state (Figure 3). Panel A): Citibase data set; panel B): Mishkin data set.

A) Inflation Rate
expected and realized (Citibase Data Set)

B) Inflation Rate
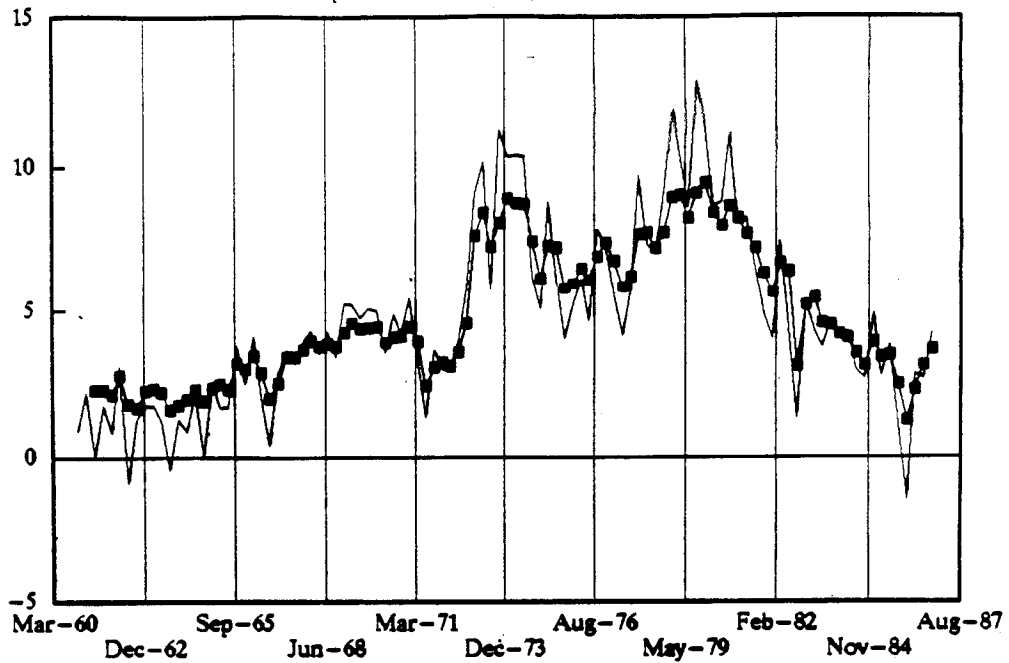### expected and realized (Mishkin Data Set)

Figure 6: Ex-ante inflation rate, $E[\pi_{t+1}|(\pi_t)]$ (represented as dark boxes, constructed using eq. (5.1) with the estimated three-state model (Table V) and the filter probabilities for each state (Figure 4). Panel A): Citibase data set; panel B): Mishkin data set.