STABLE PAYOFF CONFIGURATIONS FOR QUOTA GAMES

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1. Introduction.[1]

Cooperative n-person quota games are formed by assigning each player i a real number ω_i — called his quota — having the property that

(1.1) $$\omega_i + \omega_j \geq 0 , \quad i \neq j , \quad i, j = 1, 2, \ldots, n ,$$

and defining the characteristic function for 1- and 2-person coalitions to be

(1.2) $$v(i) = 0 , \quad v(ij) = \omega_i + \omega_j , \quad i \neq j , \quad i, j = 1, 2, \ldots, n .$$

Other coalitions are not permissible.

If all other subsets of the players were permissible coalitions as well, if the characteristic function were superadditive, and if

(1.3) $$v(12 \ldots n) = \omega_1 + \omega_2 + \ldots + \omega_n ,$$

one would obtain Shapley's quota games (see [3]).

In this paper we study the properties of some "stable payoff configurations" (= stable outcomes) for quota games as well as for Shapley's quota games. The definition of "stable payoff configurations" which we use here is one of several definitions which are proposed by R. J. Aumann and M. Maschler [1]. For the sake of completeness we state the necessary definitions in Section 2. We are mainly interested in the cases where not more than a 1-person coalition occurs. These are the most "practical" cases, because otherwise, two players can join and, in general, better themselves.

In Section 3 we characterize the stable payoff configurations of n-person quota games, $n \geq 3$, in the case where the players are partitioned into a maximal number of 2-person coalitions. We prove that these stable

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[1] Part of the content of this paper was presented in the Conference on Recent Advances in Game Theory, Princeton, 1961.

configurations are those and only those in which each player gets essentially his quota. The word "essentially" is used since a player in a 1-person coalition must get 0 , and so must a player whose quota is negative, while his associate (if any) then gets the value of the coalition. (The latter case is true only for $n \geq 4$ and $n \neq 5$.)

The various definitions for "stable payoff configurations," proposed in [1], correspond to various "levels" of stability that might be desired by the players. The definition which is chosen in this paper is the one which leads to the smallest class of stable payoff configurations. Intuitively, it applies to the case where the players wish "maximum" stability. This strong desire for stability may sometimes cause the players not to agree to participate in a coalition in a certain proposed coalition-structure. Indeed, we give such examples of Shapley's quota games (Section 4). One example shows that an imputation may be stable in one coalition-structure but not stable in another. Other examples show that a quota split is not always stable and sometimes the stable outcomes differ from any imputation which appears in the main solution given by Shapley in [3].

We choose to start with quota games, in order to show the basic properties of the stable payoff configurations. The results for Shapley's quota games are similar, under some additional restrictions (Section 4), but some "artificial" complications may arise, due to the fact that some k-person coalitions, $2 < k < n$, may possess enough power to "block" a quota split. I am grateful to Dr. H. Kesten for discussions that stimulated this paper.

2. Preliminary notations and definitions.

Let Γ be an n-person cooperative game, described as follows:

A set $N = (1, 2, \ldots, n)$ of n players is given, together with a set $\{B\}$ of subsets of N, called permissible coalitions. For each B, $B \in \{B\}$, a real number $v(B)$ is attached, which is called the value of the coalition B. $v(B)$ is the characteristic function of the game.

For simplicity, we assume that the 1-person coalitions are always permissible, normalized to have a zero value:

(2.1) $$i \in \{B\}, \quad v(i) = 0, \quad i = 1, 2, \ldots, n,$$

and that

(2.2) $$v(B) \geq 0, \quad \text{all } B, \; B \in \{B\}.$$

An outcome of the game, in which the players are partitioned to permissible coalitions $B_1, B_2, \ldots, B_m$ and each coalition shares its value among its members, can be described by a payoff configuration (p.c.)

(2.3) $$(x; \mathcal{B}) \equiv (x_1, x_2, \ldots, x_n; B_1, B_2, \ldots, B_m),$$

where

(2.4) $$B_j \cap B_k = \phi \text{ for } j \neq k, \quad \bigcup_{j=1}^{m} B_j = N,$$

and the x_i, $i = 1, 2, \ldots, n$, is a real number representing the amount which player i gets, $i = 1, 2, \ldots, n$. Therefore,

(2.5) $$\sum_{i \in B_j} x_i = v(B_j), \quad j = 1, 2, \ldots, m.$$

A coalitionally rational payoff configuration (c.r.p.c.) $(x; \mathcal{B})$ is a p.c. which satisfies the condition:

(2.6) $$\sum_{i \in B} x_i \geq v(B), \quad \text{whenever } B \subset B_j, \; B \in \{B\}.$$

Thus, if B is a subset of B_j, in a c.r.p.c. $(x; \mathcal{B})$, its members alone cannot themselves make together more than what they have in $(x; \mathcal{B})$.

Let K be any non-empty subset of N. The union of the sets

B_j , which intersect K , will be called the set of <u>partners</u> of K in $(x;\mathcal{B})$, and will be denoted by $P[K ; (x;\mathcal{B})]$:

$$(2.7) \qquad P[K ; (x;\mathcal{B})] \equiv \{ i \mid i \in B_j , \; B_j \cap K \neq \phi \} .$$

Note that in this terminology, each player of K is also a partner of K . If S is a subset of a coalition B , we shall refer to the players in the complement of S with respect to B as the <u>associates</u> of S in B .

<u>Definition 2.1</u> Let $(x;\mathcal{B})$ be a c.r.p.c., and let K and L be non-empty disjoint subsets of N , which intersect the <u>same</u> coalitions of $\mathcal{B}$; i.e.,

$$(2.8) \qquad K \cap L = \phi , \qquad K, L \neq \phi ,$$

$$(2.9) \qquad K \cap B_j \neq \phi \Longleftrightarrow L \cap B_j \neq \phi .$$

An <u>objection</u> of K against L in $(x;\mathcal{B})$, is a c.r.p.c. $(y;\mathcal{C})$, in which the members of K get more than in $(x;\mathcal{B})$, their partners get at least the amount they had in $(x;\mathcal{B})$, and moreover, the partners of K in $(y;\mathcal{C})$ do not intersect L ; i.e.,

$$(2.10) \qquad \begin{cases} y_i > x_i & i \in K \\ y_i \geq x_i & i \in P[K ; (y;\mathcal{C})] \\ P[K ; (y;\mathcal{C})] \cap L = \phi . \end{cases}$$

<u>Definition 2.2</u> A <u>counter objection</u> (to the above <u>objection</u>) is a c.r.p.c. $(z;\mathcal{D})$, in which not all of the members of K are among the partners of L , the partners of L (including L) get at least what they had in $(x;\mathcal{B})$, and those partners of L who happened to be partners of K in his objection get at least what they had in $(y;\mathcal{C})$; i.e.,

$$(2.11) \qquad \begin{cases} z_i \geq x_i & i \in P[L ; (z;\mathcal{D})] \\ z_i \geq y_i & i \in P[L ; (z;\mathcal{D})] \cap P[K ; (y;\mathcal{C})] \\ P[L ; (z;\mathcal{D})] \not\supset K . \end{cases}$$

Definition 2.3 A c.r.p.c. $(x;\mathcal{B})$ will be called stable, if for any objection of a K against an L in $(x;\mathcal{B})$, there is a counter objection of L against K.

The set of all stable p.c.'s will be called the bargaining set $\mathcal{M}_o$. Properties of this set, and other similar bargaining sets, are treated by R. J. Aumann and M. Maschler [1].

Note that only the players of $P[K ; (y;\mathcal{C})]$ and of $P[L ; (z;\mathcal{D})]$ appear in (2.10) and (2.11). Therefore, in order to show that there exists or that there does not exist an objection of a K against an L, it is enough to show that there exists or that there does not exist, respectively, a set $P[K ; (y;\mathcal{C})]$ whose members receive amounts which satisfy (2.10), subject to the conditions imposed on the c.r.p.c. $(y;\mathcal{C})$ restricted to these players only. The other players may split as they wish, e.g., form 1-person coalitions. Similarly, if the existence of the counter objection matters, it is enough to state the players of $P[L ; (z;\mathcal{D})]$ and tell what they get in $(z;\mathcal{D})$, and to check if (2.11) is satisfied and whether the conditions imposed on a c.r.p.c. $(z;\mathcal{D})$, restricted to these players only, are satisfied. We shall make use of this fact in the subsequent proofs.

3. Quota games.

Following L. S. Shapley [3], we shall make the following:

Definition 3.1 An n-person quota game is an n-person game, in which the set of permissible coalitions $\{B\}$ consists of the 1- and 2-person coalitions, and its characteristic function satisfies (2.1), (2.2) and

$$(3.1) \qquad v(ik) = \omega_i + \omega_k , \quad i, k = 1, 2, \ldots, n , \quad i \neq k .$$

Here ω_i, $i = 1, 2, \ldots, n$, are fixed real numbers. The number ω_i, $i = 1, 2, \ldots, n$, is called the quota of player i, and the vector

$\omega \equiv (\omega_1, \omega_2, \ldots, \omega_n)$ will be referred to as the quota of the game.

This definition is somewhat different from the one given by L. S. Shapley, and we shall therefore state some of his theorems modified to our case (Lemmas 3.1 - 3.4). Shapley's quota games will be treated in Section 4.

Lemma 3.1 An n-person game is always[1] a quota game for $n = 1, 2, 3$.

Lemma 3.2 The quota of a game, if such exists, is unique for $n > 2$.

Lemma 3.3 At most one player has a negative quota.

The proofs of these Lemmas are obvious.

A player with a negative quota will be called a weak player.

Lemma 3.4 A game, in which the set of permissible coalitions consists of the 1- and 2-person coalitions, is a quota game if and only if

(3.2) $$v(ij) + v(k\ell) = v(ik) + v(j\ell) ,$$

for all distinct i, j, k, ℓ.

Proof. The necessity is immediate. Suppose that (3.2) holds. There is nothing to prove if $n = 1, 2, 3$. If $n \geq 4$, then

(3.3) $$\omega_i \equiv \tfrac{1}{2}[v(ij) + v(ik) - v(jk)] \quad (i, j, k \text{ distinct})$$

is independent of j and k, and can serve as a quota for player i.

Remark 3.1 The conditions (3.2) are equivalent to the conditions:

(3.4) $$v(ij) + v(12) = v(1j) + v(2i) ,$$

for all distinct $1, 2, i, j$. These latter conditions are somewhat easier to check.

[1] Provided that the 3-person coalition is not permissible if $n = 3$.

Theorem 3.1 Let Γ be an n-person quota game which contains no weak player. Let n be even, $n \geq 4$, and let $\mathcal{B}^*$ be a partition of all the players into pairs. Then, a p.c. $(x;\mathcal{B}^*)$ is stable if and only if each player gets his quota.

Proof. Without loss of generality, we can assume that

$$\mathcal{B}^* = 12,\ 34,\ \ldots,\ (n-1)n\ . \tag{3.5}$$

Obviously,

$$(\omega_1,\ \omega_2,\ \ldots,\ \omega_n\ ;\ \mathcal{B}^*) \in \mathcal{M}_o\ , \tag{3.6}$$

since it is coalitionally rational and no player can object.

Suppose now that

$$(x;\mathcal{B}^*) \equiv (x_1,\ x_2,\ \ldots,\ x_n\ ;\ \mathcal{B}^*) \in \mathcal{M}_o\ . \tag{3.7}$$

Let K^* be the set of those players who get in $(x;\mathcal{B}^*)$ less than their quota. Certainly, the members of K^* belong to different coalitions, and their associates get more than their quotas. Let L^* be the set of these associates, then[1] $|K^*| = |L^*|$.

Case A. $|K^*| = 1$. We can assume that $1 \in K^*$. Since $n \geq 4$, player 1 can object against player 2, e.g., by joining player 3 in a quota split. Player 2 has no counter objection, either alone, because $x_2 > 0$, or by joining another player, because he received more than his quota in $(x;\mathcal{B}^*)$, and all other players except player 1 received their quotas in $(x;\mathcal{B}^*)$. This contradicts the assumption (3.7).

Case B. $|K^*| > 1$. We choose as the objecting K any fixed set of[2] $2\left[\frac{|K|}{2}\right]$ players from K^*, and let L be the set of their associates in $(x;\mathcal{B}^*)$, $|K| = |L| \geq 2$.

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[1] We denote by $|P|$ the number of elements in a set P.

[2] $[a]$ denotes the largest integer which not greater than a.

An objection $(y;\mathcal{C})$ of K against L in $(x;\mathcal{B}^*)$ is possible, in which the players K form pairs with a quota split. Here, $P[K ; (y;\mathcal{C})] = K$. In order to counter object, each player of L has to join an associate who will "agree" to accept less than his quota, because the members of L got more than their quotas in $(x;\mathcal{B}^*)$. As only 2-person coalitions are permissible, the number of these associates must be equal to $|L|$, i.e., greater than or equal to 2. But there is at most one player who may agree, namely the player in $K^* - K$ (if such exists), because any other player got his quota, either in $(x;\mathcal{B}^*)$ or as a member of $K = P[K ; (y;\mathcal{C})]$ in $(y;\mathcal{C})$. Therefore, no counter objection is possible, and again the assumption (3.7) is contradicted. Therefore, $|K^*| = 0$, and $(x;\mathcal{B}^*)$ is the p.c. (3.6).

<u>Remark 3.2</u> If $n = 2$, any p.c. of the form

$$(3.8) \qquad (x_1, x_2 ; 12) , \quad x_1 \geq 0 , \quad x_2 \geq 0 , \quad x_1 + x_2 = v(12) ,$$

belongs to $\mathcal{M}_0$. We can claim that even in this case each player received his quota, because we can define the quota to be $\omega_1 = x_1$, $\omega_2 = x_2$.

<u>Theorem 3.2</u> <u>Let</u> Γ <u>be an n-person quota game, which contains no weak player. Let</u> n <u>be odd,</u> $n \geq 1$, <u>and let</u> $\mathcal{B}^*$ <u>be a partition of all the players, but one, into pairs. Then, a p.c.</u> $(x;\mathcal{B}^*)$ <u>is stable if and only if each player gets his quota, except the one who is left alone and therefore gets</u> 0.

<u>Proof.</u> The theorem is trivial for $n = 1$. Let $n \geq 3$, and we can assume that

$$(3.9) \qquad \mathcal{B}^* = 12, 34, \ldots, (n-2)(n-1), n .$$

Certainly,

(3.10) $$(\omega_1, \omega_2, \ldots, \omega_{n-1}, 0; \mathcal{B}^*) \in \mathcal{M}_0 .$$

Indeed, only one kind of objection is possible, in which a player i, $i = 1, 2, \ldots, n-1$, objects against his associate, player j, by joining player n and offering him some amount which is <u>less</u> than ω_n. (Even this is impossible if $\omega_n = 0$.) But player j can counter object by offering the same player n a quota split.

Suppose that

(3.11) $$(x; \mathcal{B}^*) \equiv (x_1, x_2, \ldots, x_{n-1}, 0; \mathcal{B}^*) \in \mathcal{M}_0 .$$

Let K^* be the set of all the players, except player n, who get in $(x; \mathcal{B}^*)$ less than their quotas. Again, the members of K^* belong to different coalitions and their associates get more than their quota. If L^* denotes the set of these associates, then $|K^*| = |L^*|$.

Case A. $|K^*| = 1$. Player i, who belongs to K^*, can object against his associate, player j, by joining player n with a quota split. Player j has no counter objection, either as a 1-person coalition, because $x_j > \omega_j \geq 0$, or by joining any 2-person coalition, because each player received at least his quota either in $(x; \mathcal{B}^*)$ or as player i's partner in player i's objection. Thus, (3.11) is contradicted.

Case B. $|K^*| \geq 2$. Let the objecting K be any fixed $2\left|\frac{|K^*| + 1}{2}\right| - 1$ players from K^*, and let L be the set of their associates in $(x; \mathcal{B}^*)$, $|K| = |L| \geq 1$. K can object against L by allowing one member to join player n and the other members of K to form pairs with quota splits. At most one player in K^* is not in K, and therefore L cannot counter object, unless $|L| = 1$; in which case $|K^*| = 2$. In this case, however, K^* can object against L^* by forming a coalition with a quota split, and L^*, which has two members, cannot counter object, because only player n may perhaps "agree" to accept less than his quota. Thus, again (3.11) is contradicted. Therefore $|K^*| = 0$, and $(x;\mathcal{B}^*)$ is the p.c. (3.10).

Theorem 3.3 Let Γ be an n-person game which contains one weak player. Let n be even, $n \geq 4$, and let $\mathcal{B}^*$ be a partition of all the players into pairs. Then, a p.c. $(x;\mathcal{B}^*)$ is stable if and only if each player gets his quota, except the weak player who gets zero and his associate, who gets the value of their coalition.[1]

Proof. Without loss of generality, we can assume that player 1 is the weak player, and that

(3.12) $$\mathcal{B}^* = 12, 34, \ldots, (n-1)n .$$

We shall prove that

(3.13) $$(0, v(12), \omega_3, \ldots, \omega_n ; \mathcal{B}^*) \in \mathcal{M}_0 .$$

Indeed, in any possible objection, player 2 must join a player i, $i \geq 3$, because player 2 is the only player who gets in $(x;\mathcal{B}^*)$ less than his quota. Such an objection can be regarded either as an objection of player 2 against player 1, or as an objection of player i against his associate, player j, or as an objection of the players 2,i against the players 1,j. As player 2 cannot get more than his quota in the objection, any c.r.p.c., in which player 1 is a 1-person coalition, and players 2,j join in a quota split[2], is a counter objection for all the three cases.

Suppose that

(3.14) $$(x;\mathcal{B}^*) \equiv (x_1, x_2, \ldots, x_n ; \mathcal{B}^*) \in \mathcal{M}_0 .$$

Let K^* be the set of players who get in $(x;\mathcal{B}^*)$ less than their quota. Certainly, player 2 belongs to K^*, because $x_1 \geq 0$.

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[1]See Remark 3.2 for the case $n = 2$.

[2]The last coalition is not necessary if only player 2 objects.

If $|K^*| \geq 2$, let an objecting K be K^*, if $|K^*|$ is even, and $K^* - \{2\}$, if $|K^*|$ is odd. The players of K can object by forming pairs with a quota split. They so object against their associates in $\mathcal{B}^*$, who form a set L. Again, each player in L gets in $(x;\mathcal{B}^*)$ more than his quota, and $|L| \geq 2$. Only one player at most (player 1, if he is in L and if $x_1 = 0$,) can counter object by playing alone; therefore, L cannot counter object. This is a contradiction to the assumption (3.14). Thus, $|K^*| = 1$. Also $x_1 = 0$, because, otherwise, player 2 can object against player 1 by joining player 3 with a quota split, and player 1 has no counter objection. We have proved that $(x;\mathcal{B}^*)$ is the p.c. (3.13).

Theorem 3.4 *Let* Γ *be an n-person quota game which contains a weak player. Let* n *be odd,* $n \geq 7$. *Let* $\mathcal{B}^*$ *be a partition of all the players, but one, into pairs. Then, a p.c.* $(x;\mathcal{B}^*)$ *is stable if and only if the weak player gets* 0, *his associate, if such exists, gets the value of the coalition, and the player who is left alone gets* 0, *while all the other players get their quotas.*

Proof. Without loss of generality, we can assume that player 1 is the weak player. There are essentially two possible partitions:

(3.15) $$\mathcal{B}^* = \mathcal{B}^{(1)} \equiv 1,\ 23,\ 45,\ \ldots,\ (n-1)n\ ,$$

(3.16) $$\mathcal{B}^* = \mathcal{B}^{(2)} \equiv 12,\ 34,\ \ldots,\ (n-2)(n-1),\ n\ .$$

Certainly,

(3.17) $$(0,\ \omega_2,\ \omega_3,\ \ldots,\ \omega_n\ ;\mathcal{B}^{(1)}) \in \mathcal{M}_o\ ,$$

because no one can object. We shall prove that also

(3.18) $$(0,\ v(12),\ \omega_3,\ \ldots,\ \omega_{n-1},\ 0\ ;\mathcal{B}^{(2)}) \in \mathcal{M}_o\ .$$

Indeed, any objection must contain player 2, or player n, or both, as partners of the objecting K; because these are the only players who might get in (3.18) less than their quotas. Let player j and player m be the associates in $\mathcal{B}^{(2)}$ of players i and k , respectively. We have to consider only the following cases:

(i) Player 2 joins a player i , $i = 3, 4, \ldots, n$.

Here $K = \{2\}$ or $K = \{i\}$ (for $i \neq n$) , or $K = \{2,i\}$ (for $i \neq n$) .

(ii) Player n joins a player k , $k = 1, 3, 4, 5, \ldots, n-1$.

Here $K = \{k\}$.

(iii) Player 2 joins player i and player n joins player k , players 1, 2, i, j, k, m, n distinct.

Here $K = \{2,k\}$ or $K = \{i,k\}$ or $K = \{2,i,k\}$.

All these K's determine the L's , and all the objections can be countered as follows:

(i) Player 1 acts as a 1-person coalition, and if $i \neq n$, player j joins player n in a quota split.

(ii) Player m joins player n in a quota split.

(iii) Player m joins player n in a quota split, player j joins player 2 in a quota split,[1] and player 1 acts as a 1-person coalition.[2]

Suppose that

$$(x;\mathcal{B}^{(1)}) \equiv (0, x_2, x_3, \ldots, x_n ;\mathcal{B}^{(1)}) \in \mathcal{M}_o . \tag{3.19}$$

Let K^* be the set of those players who get less than their quotas in $(x;\mathcal{B}^{(1)})$. Certainly $1 \notin K^*$, and therefore $|K^*| = |L^*|$, where L^* is the set of the associates of the members of K^* .

<u>Case A.</u> If $|K^*| = 1$, there exists a player who has received his quota, because $n \geq 5$.[3] The player in K^* can therefore object by joining him in a quota split, and the player in L^* has no counter-objection.

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[1]This is not necessary if $K = (2,k)$.

[2]This is not necessary if $K = (i,k)$. Similar remarks apply to case (i).

[3]In this case only the restriction $n \geq 5$ is used.

This case contradicts the assumption (3.19) and is therefore ruled out.

Case B. If $|K^*| \geq 2$, the players of K^*, with the possible exception of at most one of them, can object by forming pairs with a quota split. Their associates cannot counter object because there are at least two of them, and there is at most one player available who may "agree" to accept less than his quota. This, again, contradicts assumption (3.19). Therefore, $|K^*| = 0$, and $(x; \mathcal{B}^{(1)})$ is the p.c. (3.17).

Suppose that

$$(3.20) \qquad (x; \mathcal{B}^{(2)}) \equiv (x_1, x_2, \ldots, x_{n-1}, 0; \mathcal{B}^{(2)}) \in \mathcal{M}_o .$$

Let K^* be the players, different from player n, who get in $(x; \mathcal{B}^{(2)})$ less than their quotas. Certainly, player 2 belongs to K^*, because $x_1 \geq 0$. Let L^* be the set of the associates of the members of K^*. Again, $|K^*| = |L^*|$.

If $|K^*| \geq 2$, then the players of K^*, with the exception of at most one of them, can form an objection as follows: One player joins player n and the others form pairs, and they all have a quota split. Denote the objecting players by K; then their associates, who form the set L, cannot counter object unless $|L| = 1$. In this case, however, $|K^*| = 2$, and K^* can object against L^* by having its members form a coalition with a quota split. If $x_1 = 0$, and $\omega_n > 0$, a player who received his quota[1] in (3.20) is added to the objection, by joining player n and offering him some positive amount. The objection cannot be countered. This contradicts the assumption (3.20), and therefore $|K^*| = 1$.

If now $x_1 > 0$, then player 2 can object against player 1 by joining player n with a quota split, and there exists no counter objection, because player 1 has to "defend" more than his quota. Thus we have proved that $(x; \mathcal{B}^{(2)})$ is the p.c. (3.18).

[1]This is the only case in which the restriction $n \geq 7$ is used.

Remark 3.3 In the case $n = 3$, with player 1 being weak, one gets a continuum for the partition $\mathcal{B}^{(1)}$. In fact, it is easy to see that

$$(3.21) \qquad (0, x_2, x_3 ; 1, 23) \in \mathcal{M}_0 ,$$

if and only if

$$(3.22) \qquad x_2 \geq \omega_1 + \omega_2 , \quad x_3 \geq \omega_1 + \omega_3 , \quad x_2 + x_3 = \omega_2 + \omega_3 .$$

In this case we are in the core of the game.[1]

Remark 3.4 To be sure, there are other partitions which are represented in $\mathcal{M}_0$. Take, for example, the case $n = 4$, where no player is weak; it is easy to see that if, e.g., $\omega_3 \geq \omega_1, \omega_2$ and $\omega_4 \geq \omega_1, \omega_2$, then

$$(3.23) \qquad (x_1, x_2, 0, 0 ; 12, 3, 4) \in \mathcal{M}_0 ,$$

whenever

$$x_1, x_2 \geq 0 , \quad x_1 + x_2 = \omega_1 + \omega_2 .$$

Remark 3.5 In R. J. Aumann-M. Maschler [1], another bargaining set was considered, which differs from $\mathcal{M}_0$ only by the additional requirement that K and L will always belong to the same coalition, among $B_1, B_2, \ldots, B_m$, of the c.r.p.c. $(x ; \mathcal{B})$. The resulting bargaining set $\mathcal{M}$ always contains $\mathcal{M}_0$. The following example will show that this inclusion may be strict for quota games.

Example 3.1 Let $n = 6$, and let the quotas be 5 for each player; then, e.g., the p.c. $(5 - x, 5 + x, 5 - x, 5 + x, 5 - x, 5 + x; 12, 34, 56) \in \mathcal{M}$, $0 \leq x \leq 5$. Indeed, if, for example, player 1 objects against player 2 by joining, say, player 3 in any possible split, then player 2 can counter-object by joining player 5 with the $5 + x, 5 - x$ split.

In practical situations, coalitions do not form simultaneously. In many cases, people negotiate and coalitions start to form one after the other. Once a coalition is formed, it is relatively hard to break it,

[1] A counter example for $n = 5$ is, e.g., $(0, 5, 8, 12, 0 ; 12, 34, 5)$ which is stable, if the quota is $(-5, 10, 10, 10, 10)$.

and one can consider the situation as a new game for the rest of the players, whose characteristic function is unchanged for subsets of the remaining players and has no value for other coalitions. Assume that the participants of a certain quota game desire the stability as demanded by the definition of $\mathcal{M}_o$. If a coalition is formed, there still remains a quota game for the rest of the players, and the quotas are unchanged. Therefore, as long as more than one or two players[1] remain in the game, each one of the members of a forming coalition will accept his quota. This situation will continue until one or two players[1] will be left. Having reached this stage, the remaining players will no longer be subjected to the outside "social pressure" to remain in a quota split, and they can end up anywhere in the core. This situation resembles (and also explains, perhaps) the feature of the benefactor and the beneficiaries which appears in the solutions to Shapley's quota games (given in [3]).

It is interesting to note that exactly this situation has been observed in some experiments on n-person quota games (without a weak player). See M. Maschler [2].

4. Shapley's quota games.

In [3], Shapley defines quota games to be games in which all the subsets of the players are permissible coalitions, the characteristic function is superadditive, i.e.,

$$v(B \cup S) \geq v(B) + v(C) \quad \text{if} \quad B \cap C = \phi , \tag{4.1}$$

and satisfies

$$v(i) = 0 , \quad v(ik) = \omega_i + \omega_k , \quad i,k = 1,2, \ldots, n , \quad i \neq k , \tag{4.2}$$

$$v(N) = \omega_1 + \omega_2 + \cdots + \omega_n . \tag{4.3}$$

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[1] Or three players, if there is a weak player among them. One assumes that the weak player will be the last one to join a coalition, or that he will remain alone.

Here again, the quota ω_i of player i, $i = 1, 2, \ldots, n$, is a fixed real number. At most one player can be weak, i.e., can have a negative quota.

In this section we shall study some stable payoff configurations of these games. Again, we shall mainly be interested in the coalition structures in which at most one player acts as a 1-person coalition.

Definition 4.1 A coalition in a game Γ is called effective, if it is possible to share its value among its members in such a way that no sub-coalition is alone able to provide more for its members.[1]

By (2.1) and (2.2), all 1- and 2-person coalitions are effective. It follows from the coalitional rationality demand that only effective coalitions occur in a c.r.p.c. A non-effective coalition cannot enter any stable payoff configuration, or any objection or any counter objection, and therefore no change in the bargaining set will result if we declare such a coalition non-permissible.

Definition 4.2 Let Γ be a Shapley's quota game. A coalition B will be called superior, regular, or inferior, if $v(B)$ is greater than, equal to, or smaller than $\sum_{i \in B} \omega_i$, respectively.

Certainly, the 2- and n-person coalitions are regular. It will turn out later that the existence of superior coalitions may change considerably the bargaining set of the game. The following lemma will furnish some of the basic connections between the various kinds of coalitions.

Lemma 4.1 Let B be a coalition in a Shapley's n-person quota game.

a. If $|B|$ is even, then B is not an inferior coalition.

b. If B is a superior coalition, then any coalition obtained by adding two new members to B is also a superior coalition.

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[1] Such a share of the value will be called an effective share.

c. *If* B *is a superior coalition, then* $N - B$ *is an inferior coalition.*

d. *If* n *is even and if* $|B|$ *is even, then* B *is a regular coalition* (L. S. Shapley [3]); *therefore,* B *is superior or inferior only if* $|B|$ *is odd.*

e. *If* n *is even, then any two superior coalitions, if such exist, must intersect each other.*

f. *If* n *is odd, then* B *is a superior coalition only if* $|B|$ *is even.* (*Therefore, there can be no weak player in this case* (L. S. Shapley [3]).)

The proofs follow immediately from (4.1), (4.2) and (4.3).

Lemma 4.2 *Let* Γ *be a Shapley's quota game.* *An inferior coalition* B , *where* $|B| > 1$, *cannot be effective.*

Proof. Let B be an effective coalition in Γ . If $|B|$ is even, it is certainly not inferior (Lemma 4.1, *a.*). If $|B|$ is odd, let T_j be a subcoalition $T_j \equiv B - \{j\}$, $j \in B$. T_j contains an even number of players. Let $\{x_i\}$, $i \in B$, be an effective share of $v(B)$ among the members of B . Certainly,

$$\sum_{i \in T_j} x_i \geq \sum_{i \in T_j} \omega_i , \quad j \in B , \tag{4.4}$$

because the players in T_j can command at least $\sum_{i \in T_j} \omega_j$, by forming 2-person coalitions. Summing up these inequalities and dividing by $|B| - 1$, we find that B is a superior or a regular coalition.

Lemma 4.3 *Let* Γ *be a Shapley's quota game, and let* B *be a regular coalition, which contains at least three players.* *If* B *is effective, then the effective share must assign each player exactly his quota.*

Proof. If B is regular and effective, any two members of B must, in an effective share, get a total sum equal to the sum of their quotas. This is obvious if $|B|$ is even. If $|B|$ is odd, this follows from the fact that equalities must hold in (4.4). Therefore, since $|B| \geq 3$, any three members of B must get a total sum equal to the sum of their quotas, in any effective share. This implies that any three players in B must share their total sum in a quota split.

Clearly, if B is regular, it need not be effective. It is never effective if B is regular and contains a weak player, and $|B| \geq 3$. Examples 4.1, 4.2, and 4.3 will furnish other cases.

Corollary 4.1 If B and N - B are effective coalitions, $|B|$, $|N - B| \geq 2$, then they must be regular coalitions also.

Theorem 4.1 Let $(x ; \mathcal{B})$ be a c.r.p.c. in a Shapley's quota game, and suppose that each coalition in $\mathcal{B}$ contains at least two players; then the members of any k-person coalition in $\mathcal{B}$, where $k \geq 3$, get exactly their quotas.

Proof. The coalitions in $\mathcal{B}$ are effective; hence, by Lemma 4.2, they are not inferior. If one of them was superior, then, by (4.1), (4.3) would be violated. Therefore, the coalitions in $\mathcal{B}$ are all regular. Lemma 4.3 now completes the proof, because x induces an effective share in each coalition of $\mathcal{B}$.

Corollary 4.2 Note that a weak player cannot belong to a k-person coalition, $k \geq 3$, if $(x ; \mathcal{B})$ is subject to the conditions of Theorem 4.1.

Theorem 4.2 Let Γ be an n-person Shapley's quota game, $n \geq 3$, in which no coalition is superior.[1] Let $\mathcal{B}$ be a partition of the players into coalitions in which no two 1-person coalitions occur. In order that a p.c. $(x ; \mathcal{B})$ is stable, it is necessary and sufficient that:

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[1] Therefore, the game does not contain a weak player.

(i) Each coalition in $\mathcal{B}$, which contains more than 1-person, is regular.

(ii) Each player gets his quota, except the "exceptional player," who belongs to a 1-person coalition (if any). If there is an "exceptional player," he gets 0 .

Proof. Let $(x;\mathcal{B})$ satisfy (i) and (ii). It is certainly a p.c. It is also coalitionally rational, because the game has no superior coalitions. The only possible objection can arise if there exists a 1-person coalition, which consists of a player with a positive quota. In this case the objecting K is one player, who joins the "exceptional player," offering him some amount less than this player's quota.[1] If $|L|$ is even, the players in L can counter object by forming pairs with quota splits. Otherwise, one player of L joins the "exceptional player," and the others form pairs, again with quota splits.

Suppose now that $(x;\mathcal{B})$ is stable. Certainly (i) is satisfied. By Lemma 4.3, the players who perhaps do not satisfy (ii) must belong to 2-person coalitions. To prove that such players do not exist, we apply similar arguments to those used in Theorem 3.1,[2] if no 1-person coalition occurs in $\mathcal{B}$, and in Theorem 3.2,[3] otherwise. This is evident, because all the objections were made there by forming two-person coalitions, and so they can similarly be made here; and no additional counter objections are now possible, because the players cannot use k-person coalitions, $k \geq 3$, in their counter objections since they got more than their quotas in $(x;\mathcal{B})$.[4]

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[1] In any effective coalition which contains more than two players, each player must get his quota in an effective share, because there are no superior coalitions in the game (see Lemma 4.3).

[2] Starting after (3.7).

[3] Starting after (3.11).

[4] See the first footnote in this proof.

If superior coalitions exist in a Shapley's quota game, an analogue of the previous theorem no longer holds. It is instructive to study the following examples for which each player has a quota equal to 5 .

Example 4.1 A 6-person Shapley's quota game:

$v(B) = 0, 10, 12, 20, 25, 30$ if $|B| = 1, 2, 3, 4, 5, 6$, respectively, except that $v(135) = 18$, $v(135\ ij) = 28$ for $1, 3, 5, i, j$ distinct. Certainly,

$$(5,\ 5,\ 5,\ 5,\ 5,\ 5\ ;\ 123456) \tag{4.5}$$

is not stable because it is not coalitionally rational. The same holds true for the c.r.p.c.

$$(5,\ 5,\ 5,\ 5,\ 5,\ 5\ ;\ 12,\ 34,\ 56)\ , \tag{4.6}$$

because the objection

$$(6,\ 0,\ 6,\ 0,\ 6,\ 0\ ;\ 135,\ 2,\ 4,\ 6) \tag{4.7}$$

of $K = \{1,3,5\}$ against $L = \{2,4,6\}$ cannot be countered. On the other hand,

$$(5,\ 5,\ 5,\ 5,\ 5,\ 5\ ;\ 13,\ 24,\ 56) \tag{4.8}$$

is stable, because the only[1] possible objections are for $K = 5$ against $L = 6$, in which the set of partners of K is $\{1,3,5\}$ or $\{1,3,5,2,4\}$. In the latter case, players 2 and 4 must get exactly 5 each, because of the coalitional rationality of the objection. Therefore, player 6 can always counter object by

$$(0,\ 0,\ 0,\ 5,\ 0,\ 5\ ;\ 46,\ 1,\ 2,\ 3,\ 5)\ . \tag{4.9}$$

This example shows that the same imputation may be stable for one coalition-structure, and unstable for another.

[1]By (2.8) and (2.9), K cannot contain player 1 and|or player 3.

Example 4.2 A 4-person Shapley's quota game:[1]

$v(B) = 0, 10, 10, 20$ for $|B| = 1, 2, 3, 4$, respectively,

except that $v(123) = 18$.

The bargaining set[2] $\mathcal{M}_o$, which in this game is identical with the bargaining set $\mathcal{M}$ (see Remark 3.5), is given, up to obvious permutations of the players 1, 2, 3, by:

$$(4.10) \quad \begin{cases} (0, 0, 0, 0 ; 1, 2, 3, 4) ; & \\ (x_1, 10-x_1, 0, 0 ; 12, 3, 4) , & 0 \leq x_1 \leq 10 ; \\ (x_1, 0, 0, 10-x_1 ; 14, 2, 3) , & 6 \leq x_1 \leq 10 ; \\ (5, 5, x_3, 10-x_3 ; 12, 34) , & 6 \leq x_3 \leq 6\frac{1}{2} . \end{cases}$$

Note that no 3- or 4-person coalition is stable, and no quota imputation appears in $\mathcal{M}_o$.

Example 4.3 An 8-person Shapley's quota game:

$v(B) = 0, 10, 10, 20, 20, 30, 30, 40$ for $|B| = 1,2,3,4,5,6,7,8$, respectively, except that $v(123) = 18$; $v(123\ ij) = 28$, for $1,2,3,i,j$ distinct, and $v(N-k) = 38$, for $k \neq 1,2,3$.

It is easy to verify that

$$(4.11) \qquad (6, 6, 6, 5, 4, 4, 4, 5 ; 15, 26, 37, 48)$$

belongs to $\mathcal{M}_o$. The imputation in this p.c. differs from any imputation in Shapley's "quota-solution" [3].

We shall now turn to the study of Shapley's quota games which contain a weak player. This can happen only if n is even (L. S. Shapley [3]; see also Lemma 4.1, f.).

Lemma 4.4 Let Γ be a Shapley's quota game, which contains a weak player; then

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[1] A similar example was suggested in a different context by Dr. H. Kesten (see [2]).

[2] A method for computing the bargaining set is described in R. J. Aumann-M. Maschler [1].

(i) a coalition B is superior if and only if $|B|$ is odd and contains the weak player;

(ii) a coalition B is inferior if and only if $|B|$ is odd and does not contain the weak player.

The proof follows from Lemma 4.1 b. c. d. e., due to the fact that the weak player forms a 1-person superior coalition.

As in the case of games with no weak players, one can construct examples in which some non-weak players in 2-person coalitions do not get their quotas in stable p.c.'s. This is not the case if the superior coalitions are not "too superior." This will be clarified in Theorem 4.3.

Definition 4.3 Let B be a coalition in a Shapley's quota game. The quantity

(4.12) $$c(B) \equiv v(B) - \sum_{i \in B} \omega_i$$

will be called the excess of B.

Lemma 4.5 Let B be a superior coalition in a Shapley's quota game; then, the excess of B is not greater than the smallest among the quotas of the players in $N - B$.

Proof. By Lemma 4.1, the coalition $N - B$ is inferior, hence it contains an odd number of players. We delete from $N - B$ that player whose quota is minimum and let the others form 2-person coalitions. The result now follows from (4.1), (4.2) and (4.3).

Lemma 4.6 Let B_1 and B_2 be two superior coalitions in a Shapley's quota game. If $B_1 \cap B_2 = \phi$, then $c(B_1) + c(B_2)$ is not greater than the smallest among the quotas of the players in $N - (B_1 \cup B_2)$. If $B_1 \subset B_2$, then $c(B_1) \leq c(B_2)$.

Proof. If $B_1 \subset B_2$, then $|B_2 - B_1|$ is even, because, by Lemma 4.1,

$|B_1|$ and $|B_2|$ are both even or both odd. $c(B_1) \le c(B_2)$ now follows from the superadditivity of the characteristic function. Similarly, if $B_1 \cap B_2 = \phi$, then $B_1 \cup B_2$ is also a superior coalition with an excess not smaller than $c(B_1) + c(B_2)$, and the result follows from Lemma 4.5.

Lemma 4.7 Let B be a superior coalition in a Shapley's quota game, which contains a weak player; then the excess of B is not smaller than minus the quota of the weak player.

The proof is an immediate consequence of Lemma 4.4, (4.1), (4.2), and (4.3).

Definition 4.4 A superior coalition in a Shapley's quota game which contains a weak player will be called a superior-regular coalition, if its excess is equal to minus the quota of the weak player.

Theorem 4.3 Let Γ be an n-person Shapley's quota game, $n \ge 4$, which contains a weak player. Suppose, also, that all the superior coalitions in Γ are superior-regular. Let $\mathcal{B}$ be any partition of the players into coalitions, in which no two 1-person coalitions occur. In order that a p.c. $(x;\mathcal{B})$ is stable, it is necessary and sufficient that:

(i) If $B \in \mathcal{B}$, and B does not contain the weak player, then $|B|$ is even or $|B| = 1$.

(ii) If $B \in \mathcal{B}$, and B contains the weak player, then either $|B|$ is odd and ≥ 3, or $|B| = 2$.

(iii) The weak player gets 0. If he has only one associate, then the associate gets the value of their coalition.

(iv) A player in a 1-person coalition (if such exists) gets 0.

(v) All other players get their quotas.

Proof. We shall make use of the fact that if a superior-regular coalition occurs in Γ, then any effective share of its value among its members must assign the value 0 to the weak player, and if, in addition, this coalition contains more than 3 players, each non-weak player must get his quota. Similarly, an effective share of the value of a regular coalition, which contains more than 2 players, must assign each player of this coalition his quota (Lemma 4.3).

Let $(x;\mathcal{B})$ satisfy (i) - (v). It is easy to check that it is a c.r.p.c. The proof that $(x;\mathcal{B})$ is stable is similar to the proofs given in Theorem 4.2 and Theorem 3.3,[1] taking into account that if the weak player is in L, he counter objects by playing alone.

Suppose now that $(x;\mathcal{B})$ is stable. Certainly (iv) is satisfied. $\mathcal{B}$ must contain only regular, superior, and 1-person coalitions (Lemma 4.2), and therefore (i) is satisfied (Lemma 4.4). Let the weak player belong to a coalition B^o, $B^o \in \mathcal{B}$. If $|B^o|$ is even and greater than 2, then B^o is regular (Lemma 4.4), and by the remark at the beginning of the proof, each player of B^o, including the weak player, must get his quota. This is impossible because the weak player can alone make more. Therefore, $|B^o|$ must be equal to 2. If $|B^o| = 1$, there will exist, as a result of (i), another 1-person coalition in $\mathcal{B}$, contrary to our assumption. We have thus proved that (ii) is satisfied. Also, if $|B^o|$ is odd, the weak player must get 0. By the remark at the beginning of this proof, the players other than the player in a 1-person coalition (if any), who get in $(x;\mathcal{B})$ less than their quotas, must belong to different coalitions in $\mathcal{B}$, all of which are 2-person coalitions, except, perhaps, a 3-person coalition, which contains the weak player. The set of these players will be denoted by K^*. We shall denote by L^* the set of their associates, except that the weak player will not be included in L^*, if he belongs to a 3-person coalition. Evidently, $|K^*| = |L^*|$. The rest of the proof is similar to the proof given in

[1]If a 1-person coalition exists in $\mathcal{B}$, or not, respectively.

Theorem 3.3,[1] if the weak player belongs to a 2-person coalition, and to Theorem 3.2[2] otherwise.

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[1] Starting after (3.14).

[2] Starting after (3.11).

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