DYNAMIC OPTIMIZATION WITHOUT DYNAMIC PROGRAMMING

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Abstract

To solve a multiperiod optimization problem with a differentiable and concave objective function and a differentiable function for the dynamic process, this paper suggests an alternative to dynamic programming. It extends the method of Lagrange multipliers and Pontryagin's maximum principle to the stochastic case and proposes to solve for a Lagrangean function rather than the value function in dynamic programming. Since the value function is a solution to the partial differential equations given by the Lagrange functions, the method proposed is analytically simpler and computationally more efficient. Numerical methods of optimization using Lagrange multipliers and an illustrative example from the study of real business cycles are provided.

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To solve a multiperiod optimization problem with a differentiable and concave objective function and a differentiable function for the dynamic process, this paper suggests an alternative to dynamic programming. It is unnecessary, and often too demanding, to solve for the value function in a Bellman equation. It is analytically easier and computationally more economical to use Lagrange multipliers instead. In section 1, a standard dynamic optimization problem is solved by using Lagrange multipliers. In section 2, I compare the method of Lagrange multipliers with the method of dynamic programming and explain why the former is better. Section 3 provides numerical methods of optimization using Lagrange multipliers. Section 4 gives an illustrative example from the study of real business cycles.

1. DYNAMIC OPTIMIZATION BY LAGRANGE MULTIPLIERS

The problem is

$$\max_{\{u_t\}_{t=0}^{T}} E_0\left[\sum_{t=0}^{T} \beta^t r(s_t,u_t)\right] \tag{1}$$

subject to

$$s_{t+1} = f(s_t,u_t) + \varepsilon_{t+1} \tag{2}$$

where s_t is a p×1 vector of state variables at time t, u_t is a q×1 vector of control variables, β is the discount factor, E_t is the conditional expectation operator given information at time t which includes s_t, and ε_{t+1} is an i.i.d. random vector with mean zero and covariance matrix V. In (2), ε_{t+1} is assumed to be additive for convenience. If ε_{t+1} is inside f( ), we will put the expectation operator before f but can still approximate the right-hand side of (2) by a linear function of ε_{t+1}. Any higher-order process for s_t is converted to first-order as usual. Both r and f are assumed to be differentiable, and r is concave.

To solve this problem along the lines of Chow (1970, 1972 and 1975, pp. 157-159 and 280-383), we introduce the p×1 vector λ_t of Lagrange multipliers and set to zero the derivatives of the Lagrangean expression

$$\mathcal{L} = E_0\left[\sum_{t=0}^{T}\left\{\beta^t r(s_t,u_t) - \beta^{t+1}\lambda'_{t+1}\left[s_{t+1} - f(s_t,u_t) - \varepsilon_{t+1}\right]\right\}\right] \tag{3}$$

with respect to u_t, s_t and λ_t ($t = T,T-1,\dots,0$). Denoting the q×1 vector $\partial r/\partial u$ by r_2, the q×p matrix $\partial f/\partial u$ by f_2, etc., and setting to zero $\partial\mathcal{L}/\partial u_t$, $\partial\mathcal{L}/\partial s_t$ and $\partial\mathcal{L}/\partial\lambda_t$ respectively yield

$$r_2(s_t,u_t) + \beta f_2(s_t,u_t)E_t\lambda_{t+1} = 0 \tag{4}$$

$$\lambda_t = r_1(s_t,u_t) + \beta f_1(s_t,u_t)E_t\lambda_{t+1} \tag{5}$$

and (2). Note that the problem is not to choose u_0, $u_1,\dots,u_T$ all at once in an open-loop policy, but to choose u_t sequentially given the information s_t at time t in a closed-loop policy. Since s_t is in the information set when u_t is to be determined, the expectations in equations (4) and (5) for the determination of u_t and λ_t at period t are E_t and not E_0. We solve equations (4) and (5) using (2) backward in time to obtain the solution. The value function $V_t(s_t)$ and the Bellman equation in dynamic programming are not used. Equations (4) and (5) are similar to the result from applying Pontryagin's maximum principle except for the stochastic aspect. The above extension to the stochastic case appears obvious once pointed out. More credit should go to Lagrange than to Pontryagin.

In many applications one can assume the existence of and compute a steady-state solution $\bar{s}$, $\bar{u}$, $\bar{\lambda}$ for the deterministic control problem obtained by setting $\varepsilon_{t+1}=0$, i.e., by solving the three nonlinear equations (4), (5) and (2) for s, u and λ after dropping all time subscripts, omitting the operator E_t and setting ε_{t+1} in (2) equal to zero.

For any period t, the function $\lambda_{t+1}(s_{t+1})$ is assumed given. After s_{t+1} is replaced by $f(s_t,u_t)+\varepsilon_{t+1}$, (4) becomes an equation in s_t and u_t which can be solved to obtain a decision or control function $u_t=g_t(s_t)$. After substitution of $g_t(s_t)$ for u_t, the right-hand side of (5) is a function of s_t only. It provides $\lambda_t(s_t)$ for the next iteration at t-1. Therefore it is natural to start with the last period T. For period T, $\lambda_{T+1}=0$ and $u_T=g_T(s_T)$ is obtained simply by solving $r_2(s_T,u_T)=0$ using (4). If the iterations converge, the functions g(s) and λ(s) are the steady-state solutions. They are defined by two equations (4) and (5) with all time subscripts removed.

2. COMPARISON WITH DYNAMIC PROGRAMMING

By the method of dynamic programming one would solve for the value function V_t in the Bellman equation

$$V_t(s_t) = \max_{u_t} \left[r(s_t,u_t) + \beta E_t V_{t+1}(s_{t+1})\right] \tag{6}$$

For differentiable V, one can set to zero the derivative of the expression in brackets with respect to u_t, yielding

$$r_2(s_t,u_t) + \beta f_2(s_t,u_t) E_t dV_{t+1}/ds_{t+1} = 0 \tag{7}$$

which is identical with equation (4) with λ_{t+1} denoting dV_{t+1}/ds_{t+1}. Assuming the function V_{t+1} to be known, equation (7) can be solved for $u_t=g_t(s_t)$. Second, substituting the maximizing $g_t(s_t)$ for u_t in (6), we obtain

$$V_t(s_t) = r(s_t,g_t(s_t)) + \beta E_t V_{t+1}(f(s_t,g_t(s_t))) \tag{8}$$

which can be used in the next iteration for t-1 to solve for $u_{t-1}=g_{t-1}(s_{t-1})$ using (7).

The method using Lagrange multipliers is different because it replaces (8) by (5). With u_t understood to equal the maximizing value $g_t(s_t)$, one can obtain (5) by differentiating the Bellman equation (6) with respect to s_t to yield

$$dV_t/ds_t = r_1(s_t,u_t) + \beta f_1(s_t,u_t)E_t dV_{t+1}/ds_{t+1} \tag{9}$$

where dV_t/ds_t can be denoted by λ_t, yielding (5).

There are computational advantages in using the pair of equations (4) and (5), as compared with using (7) and (8). Using the Bellman approach, one needs to store the function V_t and evaluate its derivatives to solve equation (7) in each step. In the Lagrange approach one does not carry V_t and is not required to evaluate its derivatives; one deals with λ_t directly. Furthermore equation (5) may be easier to solve than equation (8) because the derivative of a function is simpler than the function itself. For example, dealing with a linear $\lambda=dV/ds$ in (5) is easier than dealing with a quadratic V in (8). One can deal with a quadratic λ in (5), as we illustrate in section 3, while dealing with a cubic V in (8) would be much more difficult. Also $r_1(s,u)$ in (5) is a simpler function than $r(s,u)$ in (8). If one is lucky $r_1(s,u)=0$ as in many applications in economics, including growth models with u denoting consumption in a utility function r and s denoting capital stock which does not enter. The first term on the right-hand side of (5) disappears, while one always carries the function r in (8). One can deal with a quadratic r_1 in (5), but perhaps not the corresponding r in (8). Similarly, evaluating the expectation of dV/ds is much easier than the expectation of V. Thus solving for V in the Bellman equation is unnecessary. V is a luxury which one often cannot afford but which is nice to have if one can afford it. One can always obtain V by substituting $g_t(s_t)$ for u_t in the objective function after equations (4) and (5) are solved. When $\lambda(s)$ has a solution in closed form, V(s) often does not for it is a solution to the partial differential equations $\partial V/\partial s_i=\lambda_i(s)$. Dealing with $\lambda(s)$ is much easier than with such a solution to the partial differential equations.

3. NUMERICAL METHODS USING LAGRANGE MULTIPLIERS

Consider first the problem of finding numerically an optimal linear decision function

$$u_t = G_t s_t + g_t \tag{10}$$

This is an interesting problem in the literature of equilibrium business cycle models. See, for example, Kydland and Prescott (1982), Long and Plosser (1983), and the more recent work on computations summarized by Taylor and Uhlig (1990). In econometric testing of equilibrium business cycle models, e.g., Altug (1989), Plosser (1989), Baxter and King (1990), King (1990), and Watson (1990), one approach requires the computation of optimal linear decision functions for the representative agent in order to apply the statistical techniques for linear time series models.

In a popular method using equation (8) from dynamic programming, one approximates V_{t+1} and r_t by quadratic functions

$$V_{t+1} = \frac{1}{2} s'_{t+1} H_{t+1} s_{t+1} + s'_{t+1} h_{t+1} + c_{t+1} \tag{11}$$

and

$$r_t = \frac{1}{2} s'_t K_{1t} s_t + \frac{1}{2} u'_t K_{2t} u_t + s'_t K_{12,t} u_t + s'_t k_{1t} + u'_t k_{2t} + k_t \tag{12}$$

and f by a linear function, with the dynamic process written as

$$s_{t+1} = A_{t+1} s_t + C_{t+1} u_t + b_{t+1} + \varepsilon_{t+1} \tag{13}$$

all approximations being evaluated at the deterministic steady state $\bar{s}$ and $\bar{u}$. (10) and (13) give s_{t+1} as a linear function of s_t only. Substituting this function for s_{t+1} in (11) and taking expectation E_t, one finds $E_t V_{t+1}$ to be a quadratic function of s_t. This quadratic function plus (12) yields a quadratic function for V_t using (8). In the next step, equation (7) for t-1 is used to obtain a linear function $G_{t-1} s_{t-1} + g_{t-1}$ for u_{t-1}. This is easily achieved because in (7) both and

$$r_{2,t-1} = K_{2,t-1}u_{t-1} + K'_{12,t-1}s_{t-1} + k_{2,t-1} \tag{14}$$

$$f_2(s_{t-1},u_{t-1})E_{t-1}(dV_t/ds_t) = C'_t E_{t-1}(dV_t/ds_t) \tag{15}$$

are linear functions. The iteration continues by using $G_{t-1}s_{t-1}+g_{t-1}$ for u_{t-1} in (13) to give s_t as a linear function of s_{t-1} only, etc. Thus, as is well known, quadratic approximations for V and r and a linear approximation for f would yield linear decision functions (see Chow, 1975).

In the corresponding computations using Lagrange multipliers, one approximates λ_{t+1}, r_{1t} and r_{2t} by linear functions

$$\lambda_{t+1} = H_{t+1}s_{t+1} + h_{t+1} \tag{16}$$

$$r_{1t} = K_{1t}s_t + K_{12,t}u_t + k_{1t} \tag{17}$$

$$r_{2t} = K_{2t}u_t + K'_{12,t}s_t + k_{2t} \tag{18}$$

and f by a linear function given in (13). As before, (10) and (13) give s_{t+1} as a linear function of s_t. Substituting this function for s_{t+1} in (16) and taking expectations E_t, one finds

$$E_t\lambda_{t+1} = H_{t+1}\Big[(A_{t+1} + C_{t+1}G_t)s_t + C_{t+1}g_t + b_{t+1}\Big] + h_{t+1} \tag{19}$$

which is linear in s_t. (10) and (17) imply

$$r_{1t} = (K_{1t} + K_{12,t}G_t)s_t + K_{12,t}g_t + k_{1t} \tag{20}$$

Substitution of (19) and (20) in (5), with $f_{1t}=A'_{t+1}$ from (13), gives

$$\begin{aligned}\lambda_t &= \Big[(K_{1t} + K_{12,t}G_t) + \beta A'_{t+1}H_{t+1}(A_{t+1} + C_{t+1}G_t)\Big]s_t \\ &\quad + (K_{12,t} + \beta A'_{t+1}H_{t+1}C_{t+1})g_t + k_{1t} + \beta A'_{t+1}(H_{t+1}b_{t+1} + h_{t+1}) \\ &= H_ts_t + h_t\end{aligned} \tag{21}$$

which is a linear function of s_t and defines H_t and h_t. In the next step equations (18) and (21) are substituted into (4) at t-1 with $f_{2,t-1}=C'_t$ to yield

$$r_2(s_{t-1},u_{t-1}) + \beta C'_t E_{t-1}\lambda_t =$$

$$K_{2,t-1}u_{t-1} + K'_{12,t-1}s_{t-1} + k_{2,t-1} + \beta C'_t H_t\left(A_t s_{t-1} + C_t u_{t-1} + b_t\right) + \beta C'_t h_t = 0 \tag{22}$$

(22) can be solved to give $u_{t-1}=G_{t-1}s_{t-1}+g_{t-1}$ and to define G_{t-1} and g_{t-1}. The above equations are displayed in order to compare the two methods, to show how the well-known results of linear-quadratic control theory can be derived by the method of Lagrange multipliers using equations (4) and (5), and to pave the way for a better approximation that follows.

We illustrate the computational advantage of the method of Lagrange multipliers, as compared with dynamic programming, by using quadratic approximations for λ, r_1 and r_2 in equations (4) and (5) in order to obtain linear or nonlinear control functions. To be specific, we change (16), (17) and (18) to quadratic functions

$$\lambda_{t+1} = h_{t+1} + H_{t+1}s_{t+1} + \frac{1}{2}\begin{bmatrix} s'_{t+1}\ Q_{1,t+1}s_{t+1} \\ \vdots \\ s'_{t+1}\ Q_{p,t+1}s_{t+1} \end{bmatrix} \tag{23}$$

$$r_{1t} = k_{1t} + K_{1t}s_t + K_{12,t}u_t + \frac{1}{2}\begin{bmatrix} (s'_t \quad u'_t)P_{1t}\begin{pmatrix} s_t \\ u_t \end{pmatrix} \\ \\ (s'_t \quad u'_t)P_{pt}\begin{pmatrix} s_t \\ u_t \end{pmatrix} \end{bmatrix} \tag{24}$$

and similarly for r_{2t}. (10) and (13) give

$$s_{t+1} = (A_{t+1} + C_{t+1}G_t)s_t + C_{t+1}g_t + b_{t+1} + \varepsilon_{t+1} \tag{25}$$

By (23) and (25), $E_t\lambda_{t+1}$ is quadratic in s_t, including linear terms and the following vector of quadratic terms

$$\begin{bmatrix} s_t' Q^*_{1t} s_t \\ \vdots \\ s_t' Q^*_{pt} s_t \end{bmatrix} ; \quad Q^*_{it} = (A_{t+1} + C_{t+1}G_t)' Q_{i,t+1}(A_{t+1} + C_{t+1}G_t) \tag{26}$$

Hence

$$f_1 E_t \lambda_{t+1} = A'_{t+1} E_t \lambda_{t+1} = \begin{bmatrix} s_t' \left(\sum_{j=1}^{p} a_{j1,t+1} Q^*_{jt} \right) s_t \\ \\ s_t' \left(\sum_{j=1}^{p} a_{jp,t+1} Q^*_{jt} \right) s_t \end{bmatrix} + \text{linear function of } s_t \tag{27}$$

where $a_{jk,t+1}$ denotes the j-k element of A_{t+1}.

Equations (10) and (24) give r_{1t} as a quadratic function of s_t

$$r_{1t} = (K_{1t} + K_{12,t}G_t)s_t + K_{12,t}g_t + 2k_{1t} + \begin{bmatrix} s_t' P^*_{1t} s_t \\ \vdots \\ s_t' P^*_{pt} s_t \end{bmatrix} \tag{28}$$

where

$$P^*_{it} = P_{1t,11} + P_{1t,12}G_t + G_t' P_{it,21} + G_t' P_{it,22} G_t \quad (i=1,\ldots,p) \tag{29}$$

with $P_{it,jk}$ denoting the j-k (j,k=1,2) submatrix of P_{it} in (24). (28) and (27) can be combined on the right-hand side of (5) to express λ_t as a quadratic function of s_t. This function, together with (13) for s_t, makes $f_2(s_{t-1},u_{t-1})E_{t-1}\lambda_t$ or $C_t' E_{t-1}\lambda_t$ in (4) a quadratic function of s_{t-1} and u_{t-1}. If we approximate $r_2(s_{t-1},u_{t-1})$ in (4) by a quadratic function, we can linearize (4) about the deterministic steady state $\bar{s}$ and $\bar{u}$ to solve for a linear control function $u_{t-1}=G_{t-1}s_{t-1}+g_{t-1}$, or approximate the control function by a specified nonlinear function, in order to continue with the iteration. The corresponding computations using dynamic programming appear very difficult.

Using (4) and (5) an iterative solution for g(s) and λ(s) consists of (a) using (5) and $u_t=g_t(s_t)$ to evaluate $\lambda_t(s_t)$ given $\lambda_{t+1}(s_{t+1})$; (b) evaluating $E_{t-1}\lambda_t(s_t)$ as a function of s_{t-1} and u_{t-1}; and (c) solving (4) at t-1 for $u_{t-1}=g_{t-1}(s_{t-1})$. An approximate solution can be conveniently obtained if λ and r_1 belong to the same family of functions, if linear combinations of them belong to the same family, and if f is approximated by a linear function so that f_1 is a matrix and $f_1 E_t \lambda_{t+1}$ belongs to the same family. Obviously some variables may be logs or other monotone transformations of the original economic variables. Linear or quadratic functions of these transformations can be employed to approximate λ and r_1. Linearization of (4) can be performed to obtain a linear control function.

4. AN ILLUSTRATIVE EXAMPLE

To illustrate our method we use the baseline real business cycle (RBC) model discussed by King, Plosser and Rebelo (1988) and analyzed by Watson (1990). In this model there are two control variables $u_{1t}=C_t$ and $u_{2t}=N_t$ (consumption and labor input) and two state variables $s_{1t}=\log A_t$ and $s_{2t}=K_t$ (capital stock at beginning of period) with a production function $Q_t=K_t^{1-\alpha}(A_t N_t)^{\alpha}$. The function r in the objective function is

$$r = \log u_{1t} + \theta \log (1-u_{2t}) \tag{30}$$

The dynamic process is

$$\begin{aligned} s_{1t} &= \gamma + s_{1,t-1} + \varepsilon_t \\ s_{2t} &= (1-\delta)s_{2,t-1} + \exp(\alpha s_{1,t-1})\, s_{2,t-1}^{(1-\alpha)}\, u_{2,t-1}^{\alpha} - u_{1,t-1} \end{aligned} \tag{31}$$

where ε_t is i.i.d. with variance v, δ is the rate of depreciation for capital stock and the last two terms of (31) are output minus consumption or investment in the preceding period. Note that the equation for s_{2t} in (31) has no stochastic residual.

Equations (4) and (5) for this model are respectively

$$\begin{bmatrix} u_{1t}^{-1} \\ -\theta(1-u_{2t})^{-1} \end{bmatrix} + \beta \begin{bmatrix} 0 & -1 \\ 0 & \alpha\exp(\alpha s_{1t})\, s_{2t}^{1-\alpha}\, u_{2t}^{\alpha-1} \end{bmatrix} E_t \begin{bmatrix} \lambda_{1,t+1} \\ \lambda_{2,t+1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{32}$$

$$\begin{bmatrix} \lambda_{1t} \\ \lambda_{2t} \end{bmatrix} = \beta \begin{bmatrix} 1 & \alpha\exp(\alpha s_{1t})\, s_{2t}^{1-\alpha}\, u_{2t}^{\alpha} \\ 0 & 1-\delta + (1-\alpha)\exp(\alpha s_{1t})\, s_{2t}^{-\alpha}\, u_{2t}^{\alpha} \end{bmatrix} E_t \begin{bmatrix} \lambda_{1,t+1} \\ \lambda_{2,t+1} \end{bmatrix} \tag{33}$$

We first approximate $\lambda_t(s_t)$ by a linear function $h+Hs_t$. Using this linear function to evaluate $E_t(\lambda_{t+1})$, we can rewrite (32) and (33) as

$$u_{2t}^{-1} = \beta\Big[h_2 + H_{21}(\gamma+s_{1t}) + H_{22}\Big\{(1-\delta)s_{2t} + \exp(\alpha s_{1t})\, s_{2t}^{1-\alpha}\, u_{2t}^{\alpha} - u_{1t}\Big\}\Big] \tag{34}$$

$$\theta(1-u_{2t})^{-1} = \beta\alpha\, \exp(\alpha s_{1t})\, s_{2t}^{1-\alpha}\, u_{2t}^{\alpha-1} \times$$
$$\Big[h_2 + H_{21}(\gamma+s_{1t}) + H_{22}\Big\{(1-\delta)s_{2t} + \exp(\alpha s_{1t})\, s_{2t}^{1-\alpha}\, u_{2t}^{\alpha} - u_{1t}\Big\}\Big] \tag{35}$$

$$h_1 + H_{11}s_{1t} + H_{12}s_{2t} =$$
$$\beta\Big[h_1 + H_{11}(\gamma+s_{1t}) + H_{12}(1 - \delta + (1-\alpha)\exp(\alpha s_{1t})\, s_{2t}^{1-\alpha}\, u_{2t}^{\alpha})\Big] +$$
$$\beta\alpha\, \exp(\alpha s_{1t})\, s_{2t}^{1-\alpha}\, u_{2t}^{\alpha} \times$$
$$\Big[h_2 + H_{21}(\gamma+s_{1t}) + H_{22}\Big\{(1-\delta)s_{2t} + \exp(\alpha s_{1t})\, s_{2t}^{1-\alpha}\, u_{2t}^{\alpha} - u_{1t}\Big\}\Big] \tag{36}$$

$$h_2 + H_{21}s_{1t} + H_{22}s_{2t} = \beta\Big[1 -\delta + (1-\alpha)\, \exp(\alpha s_{1t})\, s_{2t}^{-\alpha}\, u_{2t}^{\alpha}\Big] \times$$
$$\Big[h_2 + H_{21}(\gamma+s_{1t}) + H_{22}\Big\{(1-\delta)s_{2t} + \exp(\alpha s_{1t})\, s_{2t}^{1-\alpha}\, u_{2t}^{\alpha} - u_{1t}\Big\}\Big] \tag{37}$$

Given the six parameters h_i and H_{ij} (i,j=1,2) of the linear functions for λ_1 and λ_2, we solve equations (34) and (35) for $u_{1t}=g_{1t}(s_{1t},s_{2t})$ and $u_{2t}=g_{2t}(s_{1t},s_{2t})$. If g_{1t} and g_{2t} are linear, they can be obtained by linearizing (34) and (35) and solving for u_{1t} and

u_{2t}. Replacing u_{1t} and u_{2t} on the right-hand sides of (36) and (37) by $g_{1t}(s_{1t},s_{2t})$ and $g_{2t}(s_{1t},s_{2t})$ respectively, we linearize the right-hand sides of (36) and (37) to yield the coefficients h_i and H_{ij} for the next iteration. The result of the iterations gives the desired linear control equations and the linear approximations to λ_1 and λ_2.

To illustrate the method using a quadratic approximation to λ, we write equation (4) or (32) as

$$\begin{bmatrix} u_{1t}^{-1} \\ -\theta(1-u_{2t})^{-1} \end{bmatrix} + \beta \begin{bmatrix} 0 & -1 \\ 0 & \alpha\ \exp(\alpha s_{1t}) s_{2t}^{1-\alpha} u_{2t}^{\alpha-1} \end{bmatrix} \times$$

$$\left\{ \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} + \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \gamma + s_{1t} \\ (1-\delta)s_{2t} + \exp(\alpha s_{1t}) s_{2t}^{1-\alpha} u_{2t}^{\alpha} - u_{1t} \end{bmatrix} + \right.$$

$$\left. \frac{1}{2} \begin{bmatrix} \begin{pmatrix} \gamma + s_{1t} \\ (1-\delta)s_{2t} + \ldots \end{pmatrix}' \begin{pmatrix} Q_{1,11} & Q_{1,12} \\ Q_{1,21} & Q_{1,22} \end{pmatrix} \begin{pmatrix} \gamma + s_{1t} \\ (1-\delta)s_{2t} + \ldots \end{pmatrix} \\ \begin{pmatrix} \gamma + s_{1t} \\ (1-\delta)s_{2t} + \ldots \end{pmatrix}' \begin{pmatrix} Q_{2,11} & Q_{2,12} \\ Q_{2,21} & Q_{2,22} \end{pmatrix} \begin{pmatrix} \gamma + s_{1t} \\ (1-\delta)s_{2t} + \ldots \end{pmatrix} \end{bmatrix} + \begin{bmatrix} Q_{1,11}v \\ Q_{2,11}v \end{bmatrix} \right\} = 0 \qquad (38)$$

In this equation, the parameters of λ_t, i.e., h_i, H_{ij}, $Q_{1,ij}$ nd $Q_{2,ij}$ are given. We solve u_{it} (i=1,2) as a function $g_i(s_{1t},s_{2t})$. The right-hand side of equation (5) or (33) is

$$\beta \begin{bmatrix} 1 & \alpha\exp(\alpha s_{1t}) s_{2t}^{(1-\alpha)} u_{2t}^{\alpha} \\ 0 & 1-\delta+(1-\alpha)\exp(\alpha s_{1t}) s_{2t}^{(1-\alpha)} u_{2t}^{\alpha} \end{bmatrix} \left\{ \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} + \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \gamma + s_{1t} \\ (1-\delta)s_{2t} + \ldots \end{bmatrix} + \ldots \right\} \qquad (39)$$

where the expression inside curly brackets for approximating λ_{t+1} is the same as given in (38). Using the solution $g_i(s_{1t},s_{2t})$ provided by (38), we approximate (39) by a quadratic function of s_{1t} and s_{2t} about the deterministic steady state and use the resulting parameters for h_i, H_{ij}, $Q_{1,ij}$ and $Q_{2,ij}$ in the next iteration to solve (38) for g_1 and g_2.

The steady state of the deterministic version of this model is obtained by solving for the six unknowns in the three pairs of equations (31), (32) and (33) with t subscripts, ε_{t+1} and E_t omitted and $\gamma=0$ in (31). If we are interested in linear control functions, we may use only a linear approximation to λ by setting the matrices Q_1 and Q_2 to zero and retaining only six parameters h_i, H_{i1} and H_{i2} (i=1,2). We then linearize (38) about the steady-state values for u and s and solve for linear control functions u=g(s). (39) is linearized to obtain a linear $\lambda(s)$, and so forth. If we use a quadratic approximation of λ by including six more parameters in the symmetric matrices Q_1 and Q_2, we may approximate the control function g(s) by another specified function in the solution of (38).

The main message of this paper is that the pair of equations (4) and (5) can be conveniently used to solve optimal control problems involving differentiable objective and state-transition functions and that the Lagrangean function $\lambda(s)$ should replace the value function V(s) for solving such dynamic optimization problems. The method proposed is simply an application of the method of Lagrange multipliers to multiperiod problems with an extension to the stochastic case by taking conditional expectations appropriately. The extension to continuous-time models should be straightforward as it is the same extension of Pontryagin's maximum principle using co-state variables $\lambda(s,t)$ by taking conditional expectations appropriately. The results are analogous to (4) and (5), with $E_t\lambda_{t+1}$ replaced by $\lambda(s,t)+E_t d\lambda$ where the stochastic differential $d\lambda$ is evaluated by Ito's calculus and λ satisfies a stochastic differential equation as the value function V does (see, e.g., Chow, 1979, sections 4 and 5). Since λ represents the shadow price vector of s, economists would hardly require convincing to use it as the centerpiece of analysis and computation rather than the total value of s under the condition of constrained maximization.

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