STATISTICAL ESTIMATION AND TESTING OF A REAL BUSINESS CYCLE MODEL

Gregory C. Chow
Princeton University

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Econometric Research Program
Princeton University
203 Fisher Hall
Princeton, NJ 08544-1021, U.S.A.

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ABSTRACT

This paper demonstrates that real business cycle models can and should be estimated and tested by classical econometric methods rather than by calibration. It presents a method of dynamic optimization and of maximum likelihood estimation that works for such models of optimizing agents. It provides empirical evidence unfavorable to a baseline real business cycle model which is uncovered by classical statistical methods but remains unnoticed through calibration.

1. INTRODUCTION

The purpose of this paper is threefold. First, contrary to the opinion of some researchers on real business cycle (RBC) models cited below who use calibration, this paper demonstrates that standard statistical methods can and should be used to estimate and test such models. Second, it presents a method of statistical estimation which is applicable to other economic models of a representative agent who is assumed to solve a dynamic optimization problem. Such an estimation problem is considered difficult in the literature. Third, it provides statistical evidence concerning an otherwise well-analyzed baseline RBC model which shows obvious flaws for the model not revealed by calibration.

In an RBC model, the behavioral (control) equations for the representative economic agent are derived by maximizing the expected value of a multiperiod objective function with respect to the control variables subject to the constraint of a stochastic model describing the dynamic process for the state variables. Given the parameters of the objective function and the dynamic model, control theory is applied to solve the optimization problem, yielding the required optimal control equations. One would expect the researcher to apply standard statistical techniques for the estimation and testing of such a model which includes the dynamic model for the state variables and the optimal control equations describing economic behavior, as described in Chow (1983, Chapter 12) for example. A major argument against using the standard approach is that the optimal control equations derived from dynamic optimization are nonstochastic. These equations cannot fit the data without error terms added to them. Some proponents of RBC models are unwilling to make additional statistical assumptions concerning these error terms (see, for example, Kydland and Prescott, 1982, King and Watson, 1991, and Watson, 1990). They argue, somewhat convincingly, that the model is intended to capture certain important, but no all, aspects of reality and that the researcher should not be forced to make arbitrary assumptions concerning the unexplained parts (the errors) of the model generating the economic data, such as statistical independence between the explained and the unexplained parts of the data. A main point of this paper is that such independence assumptions (as made in Altug, 1989, and Chow, 1983, for example) are not required for the estimation and testing of RBC models by standard statistical methods. This point is demonstrated in section 2 using the baseline RBC model discussed by King, Plosser and Rebelo (1988) and analyzed by Watson (1990). It is shown that standard statistical estimation and testing methods are applicable to this model without making the assumption of independence for the errors in the optimal linear control equations.

As a second objective, this paper advocates and applies a more effective method than dynamic programming for solving optimal control equations in dynamic optimization. Section 3 explains how this method can be applied to obtain optimal linear control equations for the model of section 2. This method of optimization is imbedded in a maximum likelihood algorithm for estimating the parameters of the model. A third objective is to perform an empirical analysis for a baseline RBC model using quarterly data for the United States and to evaluate its performance using standard statistical methods. Results of estimation without imposing the optimization constraints will be presented in section 4. Section 5 imposes the constraints in estimating the baseline RBC model. Section 6 concludes this paper.

2. STANDARD STATISTICAL ANALYSIS APPLICABLE TO A REAL BUSINESS CYCLE MODEL

The problem facing the representative economic agent of an RBC model is

(1)
$$\max_{\{u_t\}_{t=0}^{\infty}} E_0 \left[\sum_{t=0}^{\infty} \beta^t r(s_t, u_t) \right]$$

subject to

(2)
$$s_{t+1} = f(s_t, u_t) + \varepsilon_{t+1}$$

where s_t is a vector of p state variables, u_t is a vector of q control variables, β is the discount factor, ε_{t+1} is a vector of random shocks to the economy, E_t is expectation conditioned on information at time t and for convenience r and f are assumed to be differentiable. The solution takes the form of optimal control equations $u_t = g_t(s_t)$. For the purpose of statistical analysis, by standard or other methods, a common practice is to approximate $g_t(s_t)$ by a time-invariant linear control equation

(3)
$$u_t = Gs_t + g$$

For the derivation of such a linear control equation see Chow (1975, Chapter 12), Kydland and Prescott (1982), King, Plosser and Rebelo (1989), and Chow (1992) for example. (3) is said to explain only a part of the real world. To fit the observed data for u_{t+1}, a vector e_{t+1} of errors has to be added to (3), yielding for period t+1

(4)
$$u_{t+1} = Gs_{t+1} + g + e_{t+1}$$

One is not justified to assume the statistical independence of e_{t+1} and s_{t+1}, but there is no need to. A justifiable assumption is that ε_{t+1} is uncorrelated with s_t and u_t in equation (2). If e_{t+1} is correlated with ε_{t+1} it is also correlated with s_{t+1} through (2), violating the independence assumption. A main point of this paper is to show that there is no need to assume e_{t+1} to be independent of ε_{t+1} and s_{t+1} for the estimation and testing of the model consisting of (2) and (4) by standard statistical methods. This point is illustrated using the baseline RBC model presented by King, Plosser and Rebelo (1988).

The model consists of two control variables u_{1t} and u_{2t} representing consumption and labor input respectively, and two state variables s_{1t} and s_{2t} denoting respectively $\log A_t$ and capital stock at the beginning of period t, where A_t represents technology in the production function $q_t = s_{2t}^{1-\alpha}(A_t u_{2t})^{\alpha}$. The dynamic process is

$$s_{1t} = \gamma + s_{1,t-1} + \varepsilon_t \tag{5}$$

$$s_{2t} = (1-\delta)s_{2,t-1} + s_{2,t-1}^{1-\alpha}\exp(\alpha s_{1,t-1})u_{2,t-1}^{\alpha} - u_{1,t-1}$$

The first equation assumes $s_{1t} = \log A_t$ to be a random walk with a drift γ, ε_t being a random shock to technology. The second equation gives the evolution of capital stock s_{2t}, with δ denoting the rate of depreciation and investment being the difference between output q_{t-1} given by the production function and consumption $u_{1,t-1}$. The utility function r is assumed to be

$$r = \log u_{1t} + \theta \log (1-u_{2t}) \tag{6}$$

where $1-u_{2t}$ denotes leisure.

One can solve the optimal control problem for the representative economic agent to yield approximately optimal linear control equations of the form

$$u_{1t} = g_{11}s_{1t} + g_{12}s_{2t} + g_1 + e_{1t} \tag{7}$$

$$u_{2t} = g_{2t}s_{1t} + g_{22}s_{2t} + g_2 + e_{2t}$$

where e_{1t} and e_{2t} have been added to account for the deviations of the observed consumption u_{1t} and labor input u_{2t} from their partially explained values derived from an incomplete theory. Given time-series data on u_{1t}, u_{2t}, s_{1t} and s_{2t}, the econometrician wishes to estimate and test this RBC model.

To demonstrate our point we assume that the state of technology $s_{1t}=\log A_t$ can be observed. This assumption is relevant because RBC researchers have used the Solow residuals from a Cobb-Douglas production function as observations on s_{1t} (see Plosser, 1989). (There must exist models involving optimizing agents in which some parameters cannot be identified, but identification is a problem for standard simultaneous equations models also.) Given this assumption, we have a model consisting of four simultaneous equations (5) and (7) for four endogenous variables s_{1t}, s_{2t}, u_{1t} and u_{2t}. Note that the second equation for capital stock s_{2t} is nonstochastic and can be excluded from the system of simultaneous equations for the purpose of statistical analysis. It is a part of the deeper structure as the representative consumer-worker has utilized it, together with the utility function and the first equation of (5), in deriving the optimal control equations (7), thus imposing cross-equations restrictions on the parameters of the three remaining simultaneous stochastic equations. Since s_{2t} is capital stock at the beginning of the period and is a function of only lagged variables, it is a predetermined variable in the system.

The three remaining simultaneous equations can be written in matrix form as

$$(8)\quad \begin{bmatrix} 1 & 0 & 0 \\ -g_{11} & 1 & 0 \\ -g_{21} & 0 & 1 \end{bmatrix} \begin{bmatrix} s_{1t} \\ u_{1t} \\ u_{2t} \end{bmatrix} = \begin{bmatrix} \gamma \\ g_1 \\ g_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} s_{1,t-1} \\ u_{1,t-1} \\ u_{2,t-1} \end{bmatrix} + \begin{bmatrix} 0 \\ g_{12} \\ g_{22} \end{bmatrix} s_{2t} + \begin{bmatrix} \varepsilon_t \\ e_{1t} \\ e_{2t} \end{bmatrix}$$

Allowing the covariance matrix of the three residuals ε_t, e_{1t} and e_{2t} to be of full rank, all parameters of the system (8) can be identified. The parameters include 7 coefficients and 6 elements of the residual covariance matrix. The 7 coefficients of (8) are derived through dynamic optimization from 5 deep parameters of the optimization process, including the discount factor β, θ of the utility function (6), γ, δ and α of (5). The 6 elements of the residual covariance matrix are common between the derived model (8) and the optimization model appended by two error terms e_{1t} and e_{2t}. The 6 deep parameters including $var(\varepsilon_t)$ are related to the 8 parameters including 7 coefficients plus $var(\varepsilon_t)$ in (8). The latter eight parameters from (8) can be consistently estimated. They can be used to identify the six deep parameters. Given an optimal control algorithm to derive eight of the parameters of (8) from the six deep parameters, one can estimate the latter by maximizing a likelihood function using normal residuals (as in Chow, 1983, Chapter 12). The remaining 5 variances and covariances can also be estimated as we do not assume the independence of e_{1t} and e_{2t} from ε_t. Using this example, we have shown that there is no reason to abandon standard statistical inference in the estimation and testing of RBC models, and there is no need to assume the errors of the control equations to be uncorrelated with the state variables in these equations.

3. SOLUTION FOR THE CONTROL EQUATIONS BY LAGRANGE MULTIPLIERS

In this section, I derive a well-known approximate solution to the optimal control equations by the method of Chow (1992) using Lagrange multipliers. To solve the constrained maximization problem (1)-(2), the method introduces a p×1 vector λ_t of Lagrange multipliers and sets to zero the derivatives of the Lagrangean expression

$$\mathcal{L} = E_0\left[\sum_{t=0}^{T}\left\{\beta^t r(s_t,u_t) - \beta^{t+1}\lambda'_{t+1}\left[s_{t+1} - f(s_t,u_t) - \varepsilon_{t+1}\right]\right\}\right] \tag{9}$$

with respect to u_t, s_t and λ_t ($t = T,T-1,\ldots,0$). Denoting the q×1 vector $\partial r/\partial u$ by r_2, the q×p matrix $\partial f'/\partial u$ by f_2,etc., and setting to zero $\partial\mathcal{L}/\partial u_t$, $\partial\mathcal{L}/\partial s_t$ and $\partial\mathcal{L}/\partial\lambda_t$

respectively yield

(10) $$r_2(s_t,u_t) + \beta f_2(s_t,u_t)E_t\lambda_{t+1} = 0$$

(11) $$\lambda_t = r_1(s_t,u_t) + \beta f_1(s_t,u_t)E_t\lambda_{t+1}$$

and (2).

To justify the above method of solution, I make four observations. First, if the problem were nonstochastic, i.e., if E_0 were absent and ε_{t+1} were constants, the use of Lagrange multipliers is justified since variables in different time periods are simply treated as different variables and the constraint $s_{t+1}-f(s_t,u_t)-\varepsilon_{t+1}=0$ for each period requires a separate (vector) multiplier $\beta^{t+1}\lambda_{t+1}$, the scaling factor β^{t+1} being harmless but convenient. Second, if the problem were stochastic but unconstrained, the procedure is also justified because the expectation to be maximized is a function of the variables u_t, s_t and λ_t, and first-order conditions can be obtained by differentiation with respect to these variables, with the order of differentiation and taking expectation interchanged under suitable regularity conditions. Third, the method of Lagrange multipliers is to convert a constrained maximization problem to an unconstrained one by introducing the additional variables λ_t as is done above. Fourth, note that the problem is not to choose u_0, $u_1,\ldots,u_T$ all at once in an open-loop policy, but to choose u_t sequentially given the information s_t at time t in a closed-loop policy. Since s_t is in the information set when u_t is to be determined, the expectations in equations (10) and (11) for the determination of u_t and λ_t at period t are E_t and not E_0. (If s_t is not in the information set, the standard practice in the control theory literature is to estimate it by a Kalman filter; see Chow, 1975, Chapter 8, and Kydland and Prescott, 1982.)

We solve equations (10) and (11) using (2) to obtain the solution. The value function $V_t(s_t)$ and the Bellman equation in dynamic programming are not used. Equations (10) and (11) are similar to the result from applying Pontryagin's maximum principle except for the stochastic aspect. The above extension to the stochastic case appears obvious once pointed out. More credit should go to Lagrange than to Pontryagin. The method is better than dynamic programming both analytically and computationally because it avoids the need to solve (analytically or computationally) the value function V in the Bellman equation. The solution for V is unnecessary and is often too difficult, although V is useful if it can be obtained.

If we are interested in optimal linear control equations of the form (3), we will approximate the dynamic process (5) by a linear model of the form

(12) $$s_{t+1} = As_t + Cu_t + b + \varepsilon_{t+1}$$

We will approximate the vectors λ_{t+1}, $r_1(s_t,u_t)$ and $r_2(s_t,u_t)$ by linear functions

(13) $$\lambda_{t+1} = Hs_{t+1} + h$$

(14) $$r_1(s_t,u_t) = K_1s_t + K_{12}u_t + k_1$$

(15) $$r_2(s_t,u_t) = K_2u_t + K_{21}s_t + k_2$$

Equations (12) and (13) imply

(16) $$E_t\lambda_{t+1} = H\left[As_t + Cu_t + b \right] + h$$

(15) and (16) can be substituted into (10) to yield

$$r_2(s_t,u_t) + \beta C' E_t\lambda_{t+1} =$$

(17) $$K_2u_t + K_{21}s_t + k_2 + \beta C' H(As_t + Cu_t + b) + \beta C' h = 0$$

When (17) is solved for u_t, we obtain

(18) $$u_t = Gs_t + g$$

where

(19) $$G = -(K_2 + \beta C'HC)^{-1}(K_{21} + \beta C'HA)$$

(20) $$g = -(K_2 + \beta C'HC)^{-1}[k_2 + \beta C'(Hb + h)]$$

(18) and (14) imply

(21) $$r_1(s_t, u_t) = (K_1 + K_{12}G)s_t + K_{12}g + k_1$$

Substitution of (18), (21) and (16) in (11), with $f_1 = A'$ from (12), gives

$$\lambda_t = \left[K_1 + K_{12}G + \beta A'H(A + CG)\right]s_t + (K_{12} + \beta A'HC)g + k_1 + \beta A'(Hb + h)$$

(22) $$= Hs_t + h$$

implying

(23) $$H = K_1 + K_{12}G + \beta A'H(A + CG)$$

(24) $$h = (K_{12} + \beta A'HC)g + k_1 + \beta A'(Hb + h)$$

Equations (19), and (23) can be solved for G and H. Given G and H, (20) and (24) can be solved for g and h. Hence the parameters G and g in the optimal linear control function $u_t = Gs_t + g$ are derived from the deep parameters of the stochastic optimization problem.

The above method of finding a linear optimal control function $u_t = Gs_t + g$, to be called method I, employs a globally linear function $Hs+h$ to approximate the Lagrange function $\lambda(s)$ and may introduce sizeable errors. One can improve the accuracy of the solution by employing a locally linear function $H_t s + h_t$ to approximate $\lambda_t = \lambda(s_t)$ and $\lambda_{t+1} = \lambda(s_{t+1})$ in equation (10) and (11) for a given s_t. When we solve (10) and (11) for a given s_t, we linearize f, r_1 and r_2 as given by (12), (14) and (15) about s_t and some trial value u_t^* for u_t and solve (19), (23), (20) and (24) to obtain G_t, H_t, g_t and h_t associated with the given value of s_t. We replace u_t^* by $G_t s_t + g_t$ to perform the linearization of f, r_1 and r_2 again and iterate until convergence. For different values of s_t, we solve the optimal control problem as described above, obtaining a nonlinear optimal control function g(s) which is built up by the above locally linear optimal control functions. This method will be called method II. Both methods I and II will be applied to estimate the model in Section 5.

4. UNCONSTRAINED ESTIMATION OF THE BASELINE RBC MODEL

In this section I present estimates of the parameters of equation (8) without imposing the optimization constraints on the coefficients g_i and g_{ij} (i,j=1,2). Quarterly data for the United States in the Citybank data file citibase.rat from 1951.1 to 1988.4 are used, except for capital stock which I have constructed following the second equation in (5) for s_{2t}. By definition of this equation, the real output variable is per capital real gross national product excluding government expenditures and private residential investment. u_{1t} is per capita consumption expenditures (in 1982 dollars). u_{2t} is per capita total number of hours worked, adjusted by a constant to make it a fraction around .22 to .24. αs_{1t} is the difference between log per capita real gross national product and the sum of $(1-\alpha)$ times log per capita real non-residential fixed private capital s_{2t} and α times log per capita total number of hours worked, with $\alpha=.70$. Real private capital stock k is constructed by $k_t=.9775k_{t-1}$ + quarterly gross private non-residential investment$_{t-1}$ with starting value for the first quarter of each year given by the net capital stock figure reported in the *Survey of Current Business* (October 1986, p. 33). The equation for s_{1t} in (5) is estimated using the "Solow" residuals as constructed above for the sample period 1951.2 to 1988.4. Using $s_{1,t-1}$ as an instrument variable for s_{1t}, we estimated the equations for per capita real consumption u_{1t} and for per capita total number of hours worked u_{2t} given by (7), treating beginning-of-quarter capital stock s_{2t} as predetermined. The results from estimating these three equations are

$$s_{1t} = .004283 + s_{1,t-1} \qquad R^2 = .997$$
$$(.00092) \qquad DW = 2.178$$

$$u_{1t} = -\ 0.1681 + 0.4229\ s_{1t} + .1251\ s_{2t} \qquad R^2 = .990$$
$$(.0447) \quad (.0483) \quad (.00336) \qquad DW = .0857$$

$$u_{2t} = .2564 - .4229\ s_{1t} + .00984\ s_{2t} \qquad R^2 = .795$$
$$(.00689) \quad (.00745) \quad (.00052) \qquad DW = .123$$

Although the R^2's for the consumption and hours worked equations are high, the very low Durbin-Watson statistics indicate very strong positive serial correlations in the residuals. As alternative hypotheses to this RBC model one considers the introduction of lagged consumption in the equation for u_{1t} and lagged number of hours worked in the equation for u_{2t}. The results of estimation using $s_{1,t-1}$ as an instrumental variable are

$$u_{1t} = -\ .0424 + .0608\ s_{1t} + .0056\ s_{2t} + .9436\ u_{1,t-1} \qquad R^2 = .9986$$
$$(.0169) \quad (.02137) \quad (.0041) \quad (.0306) \qquad DW = 1.795$$

$$u_{2t} = .0027 + .0014\ s_{1t} + .0002\ s_{2t} + .9681\ u_{2,t-1} \qquad R^2 = .9754$$
$$(.0081) \quad (.0038) \quad (.0003) \quad (.02952) \qquad DW = 1.945$$

The own lagged variables help reduce the positive serial correlations in the residuals and dominate the remaining variables s_{1t} and s_{2t} provided by the RBC theory. The own lagged variables are important variables used in VAR's and in simultaneous-equation models of the Cowles Commission type. Although these models are sometimes considered *ad hoc* by RBC model builders, they fit the data so much better than the baseline RBC model, as the last two estimated equations demonstrate. Builders of RBC models need to come up with theories to explain the distributed lag relations found in time series data.

By using classical statistical methods we have shown that the two state variables selected by the baseline RBC model simply cannot explain the lag structures of consumption and labor supply. Such evidence is unnoticed by calibration. In the next section, we will present results from estimating the three equations by imposing the restrictions from dynamic optimization. However, even without imposing the constraints from the dynamic optimization theory, one finds that the data already strongly reject the model. Hence the main purpose of the next section is to present a workable method of maximum likelihood estimation for the deep parameters in a dynamic optimization model.

5. CONSTRAINED ESTIMATION OF THE BASELINE RBC MODEL

To estimate the baseline RBC model subject to the optimization constraints, we apply the method of maximum likelihood (see Chow, 1983, Chapter 12). Methods I and II of Section 3 require a linearized version of the model for the evolution of $s_t=(s_{1t},s_{2t})'$ as given in (12). Method I linearizes the equation for capital stock $s_{2,t+1}$ given by (5) about the means $\bar{s}_1$, $\bar{s}_2$, $\bar{u}_1$ and $\bar{u}_2$ of the sample period (1951.1-1988.4). Denoting $\exp(\alpha\bar{s}_1)\ \bar{s}_2^{-1-\alpha}\ \bar{u}_2^{-\alpha}$ by $\bar{q}$, we obtain the parameters of (12).

$$(25)\qquad A=\begin{bmatrix}1 & 0\\ \alpha\bar{q} & 1-\delta+(1-\alpha)\bar{q}/\bar{s}_2\end{bmatrix}\quad C=\begin{bmatrix}0 & 0\\ -1 & \alpha\bar{q}/\bar{u}_2\end{bmatrix}\quad b=\begin{bmatrix}\gamma\\ -\alpha\bar{q}\bar{s}_1\end{bmatrix}\quad \varepsilon_t=\begin{bmatrix}\varepsilon_t\\ 0\end{bmatrix}$$

The vector $r_1=\left[u_{1t}^{-1},\ -\theta(1-u_{2t})^{-1}\right]'$ can be linearized about $\bar{u}_1$ and $\bar{u}_2$ to yield the following parameters for (14) and (15)

$$K_1=0 \qquad\qquad K_{12}=0 \qquad k_1=0$$

$$(26)\qquad K_2=\begin{bmatrix}-\bar{u}_1^{-2} & 0\\ 0 & -\theta(1-\bar{u}_2)^{-2}\end{bmatrix}\quad K_{21}=0 \quad k_2=\begin{bmatrix}2\bar{u}_1^{-1}\\ -\theta[(1-\bar{u}_2)^{-1}-(1-\bar{u}_2)^{-2}\ \bar{u}_2]\end{bmatrix}$$

Method II performs similar linearizations about each s_t and a trial value u_t^*, the latter set equal to $G_ts_t+g_t$ at each iteration until convergence.

We first estimated model (8) by Method I with G and g derived from equations (19), (20), (23) and (24). Note that three of the deep parameters γ, α and δ are imbedded in the matrices A, C and b of the dynamic model. There is only one deep parameter θ in the utility function, affecting K_2 and k_2 in (26). Including the discount factor β, these five deep parameters appear in equations (19), (20), (23) and (24) for deriving G and g in the optimal linear control equation.

Using a standard simultaneous-equations notation, we rewrite (8) as

(27) $$By_t + \Gamma x_t = e_t$$

where $y_t=(s_{1t}\ u_{1t}\ u_{2t})'$, x_t is a column vector consisting of $s_{1,t-1}$, $u_{1,t-1}$, $u_{2,t-1}$, s_{2t} and 1, and

(28) $$B = \begin{bmatrix} 1 & 0 & 0 \\ -g_{11} & 1 & 0 \\ -g_{21} & 0 & 1 \end{bmatrix} \qquad \Gamma = \begin{bmatrix} -1 & 0 & 0 & 0 & -\gamma \\ 0 & 0 & 0 & -g_{12} & -g_1 \\ 0 & 0 & 0 & -g_{22} & -g_2 \end{bmatrix} \qquad e_t = \begin{bmatrix} \varepsilon_t \\ e_{1t} \\ e_{2t} \end{bmatrix}$$

n observations of (27) can be written as

$$BY' + \Gamma X' = E'$$

where Y' is $3\times n$, X' is $5\times n$ and E' is $3\times n$.

Assuming normal and serially uncorrelated e_t with covariance matrix Σ, the concentrated log-likelihood function (see Chow, 1983, pp. 170-171) is well-known to be

(29) $$\log L = \text{const} + n \log |B| - \frac{n}{2} \log |n^{-1}(BY' + \Gamma X')(YB' + X\Gamma')|$$

with the maximum likelihood estimate of Σ given by

(30) $$\hat{\Sigma} = n^{-1}(BY' + \Gamma X')(YB' + X\Gamma')$$

To obtain maximum likelihood estimates of the five deep parameters by iterations, each iteration takes three steps: (a) given initial values of the deep parameters, compute G and g by solving (19)-(23) iteratively and (20)-(24) iteratively; (b) evaluate log L using (29) and the results of step (a); and (c) apply a maximization algorithm to revise the initial values of the deep parameters.

I report below the results obtained by using for step (c) a version of the Corana simulated annealing algorithm as implemented by Goffe, Ferrier and Rogers (1992) and translated into Gauss by Ron Miller and Chunsheng Zhou of Princeton University. The maximum likelihood estimates of α (labor coefficient in the Cobb-Douglas production function), β (discount factor), γ (drift in the random walk for the Solow residual), δ (depreciation rate for capital stock) and θ (relative coefficient of log leisure in the utility function) are 0.7309, 0.9732, 0.002636, 0.02207 and 2.7948, respectively. The algorithm has yielded a set of sensible estimates for all five parameters, results not often found in the estimation of dynamic optimization models. The computations took about two hours on a 486/25C personal computer.

The above deep parameters imply, through dynamic optimization, the following optimal linear control rules

$$u_{1t} = 1.1379 + .3809 s_{1t} + .007387\, s_{2t}$$

$$u_{2t} = 0.5065 - .1629\, s_{1t} - .003159\, s_{2t}$$

To compare the model with these restricted coefficients based on 5 deep parameters with the model with 6 unconstrained coefficients in the equations for u_{1t} and u_{2t} reported in section 4 plus a common γ, one may compute the log likelihood of the constrained model (27)-(28) minus the log likelihood of the model with the unconstrained coefficients given in section 4, with the first equation of (28) being that of the constrained model in both cases. Minus twice this log likelihood ratio turns out to be   . Since the unconstrained model has two extra parameters, with γ=.004283 common in both, one may test the null hypothesis of the constrained model being correct by a $\chi^2(2)$ test, rejecting it at a very low level of significance. However, this test is questionable as it is based on the incorrect assumption that both models have serially uncorrelated residuals.

When Method II was applied, the log likelihood value increased to 9.38 from 8.6737 for Method I. The five parameters are estimated to be 0.6368, 0.8453, 0.003041, 1.770×10^{-9} and 3.5198, taking about eight hours of a 486/25C computer. Note that local linearization was performed for only 19 values of the vector of state variables in a sample of 152 quarterly observations, each being average values of eight consecutive quarters. Table 1 gives the parameters (g G) of the optimal control function obtained by Method II corresponding to the 19 values of the state variables and shows how poor a globally linear optimal control function is.

Much progress has been made in performing maximum likelihood estimation of the deep parameters of models of a representative agent who is assumed to solve a dynamic optimization problem since the early work of Blanchard (1983). From the methodological point of view, this paper has provided two useful algorithms to perform maximum likelihood estimation. It would be interesting to compare our Method II with the method of Christiano and Eichenbaum (1992). After using logs and detrending by the nonstationary s_{1t}, they produce a deterministic version of their model which has a steady-state solution about which they approximate the loss function by a quadratic function and apply linear-quadratic control theory to obtain optimal decision rules. Their method implies a globally linear approximation (though about the important steady-state values) to the Lagrangean function in our terminology, for all values of the state variables which may be different from the steady-state values. Furthermore, their statistical method of estimation differs from ours. From the substantive point of view, this paper has provided further evidence on a baseline RBC model. Even without examining the estimates of the coefficients of the optimal decision rules in equation (7), one can conclude from the estimates obtained from unconstrained maximization of the last section that the baseline real business cycles model is highly inconsistent with the data.

Table 1

Optimal Control Variables Corresponding to Selected State Variables

	g_1	g_2	G_{11}	G_{12}	G_{21}	G_{22}
1.	0.7406	0.5018	0.2843	0.0142	-0.1761	-0.0088
2.	0.8204	0.5111	0.2720	0.0129	-0.1724	-0.0082
3.	0.8374	0.5392	0.2870	0.0131	-0.1750	-0.0080
4.	0.9810	0.4965	0.2573	0.0112	-0.1645	-0.0072
5.	1.0031	0.5121	0.2643	0.0113	-0.1648	-0.0070
6.	1.0603	0.5161	0.2590	0.0108	-0.1631	-0.0068
7.	1.0554	0.5584	0.2818	0.0114	-0.1681	-0.0068
8.	1.0203	0.6191	0.3282	0.0125	-0.1777	-0.0068
9.	1.0731	0.5937	0.3501	0.0123	-0.1738	-0.0061
10.	1.1408	0.5767	0.3668	0.0119	-0.1708	-0.0056
11.	1.2059	0.5613	0.3784	0.0117	-0.1670	-0.0052
12.	1.1654	0.5663	0.4214	0.0122	-0.1704	-0.0049
13.	1.3174	0.5092	0.4059	0.0112	-0.1598	-0.0044
14.	1.0648	0.5736	0.5061	0.0135	-0.1719	-0.0046
15.	1.0943	0.5480	0.5259	0.0133	-0.1699	-0.0043
16.	1.3564	0.4849	0.4726	0.0114	-0.1590	-0.0038
17.	1.2316	0.5207	0.5327	0.0126	-0.1636	-0.0039
18.	0.9721	0.5692	0.6409	0.0145	-0.1720	-0.0039
19.	0.7933	0.5791	0.7198	0.0158	-0.1744	-0.0038

Perhaps Robert Solow (1988, page 311) was too generous to say that "E. Prescott's theory is hard to refute...." The limited statistical analysis of this paper seems to have refuted strongly a baseline RBC model.

6. CONCLUSION

Pursuant of its objectives, this paper reaches three conclusions. First, there is no excuse for not applying standard econometric methods to estimate and test real business cycle models. Second, two methods have been suggested for computing maximum likelihood estimates of the deep parameters of a model based on a representative agent acting as if she/he is solving a dynamic optimization problem. We have implemented both methods. Third, empirical evidence obtained by classical statistical analysis has revealed weaknesses in the dynamics of a baseline RBC model which have not been evident from calibration analysis.

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