MULTIPERIOD COMPETITION WITH SWITCHING COSTS: SOLUTION BY LAGRANGE MULTIPLIERS

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Econometric Research Program
Research Memorandum No. 368

August 1993

Abstract

This paper applies the method of Lagrange multipliers to solve a model of dynamic games of Beggs and Klemperer (1992) on price determination of duopolists facing a market with no consumer switching products. The consumers are first assumed to be myopic, basing their choice of products on only current prices, and then allowed to take future prices into account. The solutions in the two cases illustrate that the method is simpler than dynamic programming because there are fewer parameters to solve and one saves the trouble of differentiating the value function in solving the first order condition for the optimum control function.

Acknowledgement

The author would like to thank a referee for pointing out an error in an earlier version of this paper.

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This note suggests that the method of Lagrange multipliers is more convenient to use than dynamic programming in solving problems of dynamic games. It applies the method to solve the model of Beggs and Klemperer (1992) on price determination of duopolists facing a market with consumer switching costs.

Let v be the number of new consumers entering the market in each period. The new consumers' tastes are distributed uniformly along a line segment $[0,1]$, with duopolists A and B located at 0 and 1 respectively. If a new consumer with taste y chooses A's product and is assumed not to change product in the future, she will have discounted life-time utility

$$r - \tau y + \sum_{t=1}^{\infty} \delta_c^t R - \sum_{t=0}^{\infty} \delta_c^t \, p_{At}$$

as she obtains utility $r-\tau y$ (τ being "transport cost" per unit distance) in the first period and utility R in each subsequent period with discount factor δ_c and with firm A charging price p_{At} in period t. Choosing B's product she will have discounted life-time utility

$$r - \tau(1-y) + \sum_{t=1}^{\infty} \delta_c^t \, R - \sum_{t=0}^{\infty} \delta_c^t \, p_{Bt}$$

I will first assume the consumer to be myopic, i.e., $\delta_c = 0$, and later drop the assumption to allow for the effects of future prices on her choice of products. A myopic consumer will be indifferent between choosing the two products if her taste is

$$z = (2\tau)^{-1} \left[(-p_A + p_B) + \tau \right] \equiv \beta(-p_A + p_B) + \alpha$$

Since y is distributed uniformly along $[0,1]$, the above expression is the fraction of new customers buying A's product. The remaining fraction $\beta(-p_B+p_A)+\alpha$ will buy B's product.

Profit of firm i ($i = A,B$) at time t is

(1)
$$\pi_{it} = (p_{it}-c_i)\left[\, x_{it} + \nu\beta(-p_{it}+p_{jt}) + \nu\alpha \,\right]$$

where c_i is unit cost for firm i, p_{At} is understood to be a function of $x_{At}=x_t$, the number of A's old customers, and p_{Bt} is a function of $x_{Bt}=S-x_t$, S being the constant stock of old customers in the market. Since only a fraction ρ of all customers is assumed to remain and become old customers after one period, x_t and x_{Bt} evolve according to

(2)
$$x_{t+1} = \rho x_t + \rho\nu\left[\, \beta(-p_{At}+p_{Bt}) + \alpha \,\right]$$
$$x_{B,t+1} = \rho x_{B,t} + \rho\nu\left[\, \beta(-p_{Bt} + p_{At}) + \alpha \,\right]$$

Both firms are assumed to maximize expected total discounted profits in infinitely many periods with discount factor δ by choosing price $p_i(x_{it})$ and taking the other firm's price function $p_j(x_{jt})$ as given. A problem is to find the equilibrium price functions.

Given $p_B(x_B)$, firm A's optimization problem can be solved by the method of Lagrange multipliers as suggested by Chow (1992, 1993). To apply the method we differentiate the following Lagrangean expression (with λ_t as Lagrange multiplier)

$$\mathcal{L}_A = \sum_{t=0}^{\infty} E_t \left\{ \delta^t \pi_{At} - \delta^{t+1}\lambda_{t+1}\left[\, x_{t+1} - \rho x_t - \rho\nu\left(\beta\left(-p_{At}+p_B(S-x_t)\right) + \alpha \,\right)\right]\right\}$$

with respect to p_{At} and x_t (t = 0,1,2,...), yielding the first-order conditions

(3)
$$\delta^{-t}\frac{\partial\mathcal{L}_A}{\partial p_{At}} = x_t + \nu\beta\left(-p_{At} + p_B\,(S-x_t)\right) + \nu\alpha$$
$$-\nu\beta(p_{At}-c_A) - \rho\nu\beta\delta E_t\lambda_{t+1} = 0$$

(4)
$$\delta^{-t}\frac{\partial\mathcal{L}_A}{\partial x_t} = -\lambda_t + (p_{At}-c_A)\left(1+\nu\beta p_B'(S-x_t)\right) - \delta\left[-\rho-\rho\nu\beta p_B'(S-x_t)\right]E_t\lambda_{t+1} = 0$$

Similarly, given $p_A(x) = p_A(S-x_B)$, firm B's problem can be solved by differentiating a similar Lagrangean expression $\mathcal{L}_B$ (with λ_{Bt} as Lagrange multiplier) with respect to p_{Bt} and x_{Bt}, yielding first-order conditions (3B) and (4B) which are identical to (3) and (4) except with subscripts A and B interchanged. The solution by our method consists of $p_A(x)$, $\lambda(x)$, $p_B(x_B)$ and $\lambda_B(x_B)$ which satisfy the four equations (3), (4), (3B) and (4B), where $\lambda_{t+1} = \lambda(x_{t+1})$ and $\lambda_{B,t+1} = \lambda_B(x_{B,t+1})$ with x_{t+1} and $x_{B,t+1}$ given by equation (2). In this model, the transition equation (2) for the state variable x_t happens to be nonstochastic. For exposition of the method which is applicable to stochastic x_t we keep the conditional expectation operator E_t in (3).

To solve these equations by the method described in Chow (1993), we assume λ and λ_B to be linear, which is equivalent to the corresponding value functions being quadratic:

$$\lambda = \ell + mx \; ; \quad \lambda_B = \ell_B + m_B x_B \; . \tag{5}$$

In (5) $\ell=h$ and $m=H$ in the notation of Chow (1993); ℓ and m agree with the notation of Beggs and Klemperer (1992) who use quadratic value functions of dynamic programming to solve this problem. To solve A's problem using equations (3), (4), and (5), we first use (2) to evaluate

$$\lambda_{t+1} = \ell + mx_{t+1} = \ell + m\{\rho x_t + \rho\nu \left[\beta\left(-p_{At} + p_B\ (S-x_t)\right) + \alpha\right]\} \, . \tag{6}$$

Assuming tentatively ℓ and m to be given, we substitute (6) for $E_t\lambda_{t+1} = \lambda_{t+1}$ in (3) and solve the resulting equation for p_{At}. Simple algebra shows that p_{At} is a linear function in x_t provided that $p_B(x_B)$ is also linear. Substituting the resulting function $p_{At} = p_A(x_t)$ into (4) and equating coefficients of $\lambda_t = \ell + mx_t$, we can find ℓ and m (h and H in the notation of Chow, 1993). Given ℓ and m, the function $p_A(x_t)$ is known. Similarly, using (3B) and (4B) we can find $p_{Bt} = p_B(x_{Bt})$ and $\lambda_{Bt} = \ell_B + m_B x_{Bt}$. Note that $p_A(\cdot)$ depends on the parameters of $p_B(\cdot)$ and $p_B(\cdot)$ depends on the parameters of $p_A(\cdot)$. Equilibrium is reached when these parameters are consistent.

To proceed with our solution, let

$$(7)\qquad \begin{aligned} p_A(x) &= d_A + e_A(x) = d_A + e_A S - e_A x_{Bt} \\ p_B(x_B) &= d_B + e_B(x_B) = d_B + e_B S - e_B x_t \end{aligned}$$

Substituting equation (7) for $p_B(x_{Bt})$ in equation (6) we find

$$(8)\qquad E_t\lambda_{t+1} = \ell + m\rho\nu\left[\beta(d_B+e_B S) + \alpha\right] + m\rho(1-\nu\beta e_B)x_t - m\rho\nu\beta p_{At}$$

Substituting equation (8) for $E_t\lambda_{t+1}$ in equation (3) and solving for p_{At} yield

$$(9)\qquad \begin{aligned} p_{At} &= \left[\nu\beta(\delta\rho^2\nu\beta m - 2)\right]^{-1} \\ &\times\left[(1-\nu\beta e_B)(\delta\rho^2\nu\beta m-1)x_t + \nu\beta\left(d_B+e_B S+\alpha/\beta\right)\left(\delta\rho^2\nu\beta m-1\right) - \nu\alpha + \delta\rho\nu\beta\ell\right] \end{aligned}$$

Equation (3) is used to solve for $E_t\lambda_{t+1}$ and the result is substituted into equation (4) to obtain

$$(10)\qquad \begin{aligned} \lambda_t &= \ell+mx_t = -c_A(1-\nu\beta e_B) + (1-\nu\beta e_B)p_{At} + \delta\rho(1-\nu\beta e_B)E_t\lambda_{t+1} \\ &= -\left[\nu\beta(\delta\rho^2\nu\beta m-2)\right]^{-1}(1-\nu\beta e_B)\left[\nu\beta(d_B + e_B S + \delta\rho\ell) + (1-\nu\beta e_B)x_t\right] \end{aligned}$$

Equating m to the coefficient of x_t on the last line of equation (10), we have a quadratic equation in m, the solution of which is

$$(11)\qquad m = (\rho^2\nu\beta\delta)^{-1}\left(1 \pm \left[1 - \rho^2\delta(1-\nu\beta e_B)^2\right]^{1/2}\right)$$

We next solve for e_A which is the coefficient of x_t in (9).

$$(12)\qquad \begin{aligned} e_A &= \left[\nu\beta(\delta\rho^2\nu\beta m - 2)\right]^{-1}\left[(1-\nu\beta e_B)(\delta\rho^2\nu\beta m-1)\right] \\ &= (\nu\beta)^{-1}\left[(1-\nu\beta e_B) - (1-\nu\beta e_B)^{-1}\nu\beta m\right] \end{aligned}$$

Note that both m and e_A are functions of e_B. Substituting (11) for m in (12) one obtains the following quadratic equation in e_A:

$$\text{(13)} \quad (\nu\beta)^2(1-\nu\beta e_B)e_A^2 - 2\nu\beta\left[1-\rho^2\delta(1-\nu\beta e_B)^2\right]e_A + (1-\nu\beta e_B)\left[\rho^2\delta(1-\nu\beta e_B)^2 - 2\right] = 0$$

The identical solution to firm B's problem yields equation (11B) for m_B, which is the same as equation (11) with e_A replacing e_B, equation (12B) for e_B, which is the same as (12) with e_A and m_B replacing e_B and m, and a quadratic equation (13B) in e_B, which is the same as (13) with e_A and e_B interchanged. (13) and (13B) provide a pair of equations for e_A and e_B.

For the remaining parameters, we solve for ℓ by equating it to the intercept term of (10), yielding a linear function in ℓ, given m. d_A is set equal to the intercept term in equation (9), which depends on d_B and e_B. Given e_A and e_B, the intercepts of (9) and (9B) provide a pair of equations for d_A and d_B. Thus the equilibrium price functions $p_A(x) = d_A + e_A x$ and $p_B(x) = d_B + e_B x$ can be obtained.

To allow for the fact that consumers take future prices into consideration in choosing product i, Beggs and Klemperer (BK) assume that the sum W_i of the expected discounted utilities of firm i's old customers is linear in x_i, i.e.,

$$\text{(14)} \qquad W_i(x_i) = g_i + h_i x_i$$

Hence the marginal new consumer's distance from i, $Z_i(p_i,p_j,x_i)$, satisfies BK's equation (A5), with time subscript t suppressed,

$$\text{(A5)} \quad -\tau Z_i(p_i,p_j,x_i) - p_i + \rho\,\delta_c W_i(x_{i,t+1}) = -\tau\left(1 - Z_i(p_i,p_j,x_i)\right) - p_j + \rho\,\delta_c W_j(S - x_{i,t+1})$$

and the evolution of x_{it} follows

(15)
$$x_{i,t+1} = \rho x_{it} + \rho v Z_i(p_{it}, p_{jt}, x_{it})$$

Profit of firm i at time t is, with Z_{it} denoting $Z_i(p_{it}, p_{jt}, x_{it})$,

(16)
$$\pi_{it} = (p_{it} - c_i)(x_{it} + vZ_{it})$$

Assuming that firm i maximizes the sum of expected discounted profits subject to the constraint (15) we form the Lagrangian

(17)
$$\mathcal{L}_i = \sum_{t=0}^{\infty} E_t\{\delta^t \pi_{it} - \delta^{t+1}\lambda_{i,t+1}[x_{i,t+1} - \rho x_{it} - \rho v Z_{it}]\}$$

and obtain the first-order conditions

(18)
$$\delta^{-t}\frac{\partial \mathcal{L}_i}{\partial p_{it}} = x_{it} + vZ_{it} + (p_{it} - c_i)v\frac{\partial Z_{it}}{\partial p_{it}} + \delta\rho v\frac{\partial Z_{it}}{\partial p_{it}}E_t\lambda_{i,t+1} = 0$$

(19)
$$\delta^{-t}\frac{\partial \mathcal{L}_i}{\partial x_{it}} = -\lambda_{it} + (p_{it} - c_i)(1 + v\frac{\partial Z_{it}}{\partial x_{it}}) + \delta\rho(1 + v\frac{\partial Z_{it}}{\partial x_{it}})E_t\lambda_{i,t+1} = 0$$

We differentiate (A5) using (14) and (15) to obtain:

(20)
$$\frac{\partial Z_{it}}{\partial p_{it}} = [\rho^2\delta_c v(h_A + h_B) - 2\tau]^{-1}$$

In equilibrium $x_{it} + vZ_{it} = \rho^{-1}x_{i,t+1} = \rho^{-1}(\eta_i + \mu x_{it})$ where η_i and μ are defined by BK's equations (A1) and (A2). As before we assume $\lambda_i(x_i) = l_i + m_i x_i$ as given by equation (5). Substituting (5), the above equilibrium condition and (20) into (18) we can obtain p_{it} as a linear function of x_{it} with parameters e_i and d_i as given by BK's equation (A7) -- with the factor 2 in front of $\rho\delta m_i$ for both e_i and d_i missing. Given e_i and d_i, we substitute (5), the above equilibrium condition and its implication $1 + v\,\partial Z_{it}/\partial x_{it} = \mu/\rho$ into (19) and equate coefficients to obtain the parameters l_i and m_i of $\lambda_i(x_i)$. As compared with the method of dynamic programming (BK, p. 663), our method saves the trouble of finding the constants k_i in the quadratic value functions and, having found them, the trouble of differentiating the value functions and ignoring these constants to solve for the parameters of $p_i(x_i)$ using the first-order condition (18). For both cases of myopic and forward looking consumers, this note has demonstrated the usefulness and simplicity of the method of Lagrange multipliers in solving problems of dynamic games.

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