THE LAGRANGE METHOD OF OPTIMIZATION IN FINANCE

GREGORY C. CHOW*
PRINCETON UNIVERSITY

Abstract

A method of Lagrange multipliers is presented for solving dynamic optimization problems involving stochastic differential equations. It is an alternative to dynamic programming. As a generalization of Pontryagin's maximum principle to stochastic models it avoids having to solve the Bellman equation for the value function. Its analytical advantages are illustrated by applications to classic problems of finance and investment. Its computational advantages are pointed out by presenting a numerical method for dynamic optimization in continuous time.

Econometric Research Program
Princeton University
203 Fisher Hall
Princeton, NJ 08544-1021, USA

THE LAGRANGE METHOD OF OPTIMIZATION IN FINANCE

Gregory C. Chow

It is generally believed that to solve dynamic optimization problems in continuous time, if the model is deterministic, one has the choice of using Pontryagin's maximum principle or the method of dynamic programming. The former is essentially an application of the method of Lagrange multipliers. An excellent exposition of both methods can be found in Dixit (1990, chapters 10 and 11). However, if the model is stochastic, dynamic programming is considered the only option. A main purpose of this paper is to show that in the stochastic case, the method of Lagrange multipliers is not only applicable but in certain circumstances analytically simpler and computationally more accurate than dynamic programming. Section I presents the method. Section provides a classic example of finance using the well-known optimum consumption and portfolio selecti... models of Merton (1969, 1970 and 1973). Section III applies the method to an example of investment decision with irreversibility as surveyed by Pindyck (1991), Dixit (1992) and Dixit and Pindyck (1994). Section IV provides a numerical method for solving dynamic optimization problem in continuous time and points out that this method would be very difficult, if not impossible, to implement in the framework of dynamic programming.

The basic idea of the method is that the dynamic model of a stochastic optimization problem can be viewed as a constraint on the state and control variables in adjacent time points and hence the optimization problem can be solved by the method of Lagrange multipliers. This idea is the basis of Pontryagin's maximum principle for deterministic models. When the model is stochastic, the appropriate Lagrange expression is a mathematical expectation of the sur ver many periods of the objective functions and the products of the (vector) Lagrange multiplier and the ector) dynamic model. To apply the method, one simply differentiates the Lagrange expression with respect to the control variables and the state variables which are subject to the constraint of the dynamic model. The differentiation yields two first-order conditions which can be solved for the optimal control function and the Lagrange multiplier, both being functions of the vector state variable. This is a natural and efficient way of solving the problem for the static case when time is not involved as well as the dynamic case when the state variables evolve through time as specified by a dynamic model. In the static case, such as the problem of a consumer maximizing a differentiable utility function of quantities of consumption goods subject to a budget constraint, one naturally applies the method of Lagrange multipliers to find the optimal demand functions for the consumption goods. Solving the Bellman equation for the value function would correspond to finding the indirect utility function in this case. Although solving for the indirect utility function is one way of obtaining the demand functions, it is often more difficult than the method of Lagrange multipliers.

1

The principle of optimality of dynamic programming has two components. The first is that when a decision for the control variable at time t is made, it is assumed that all future control variables shall be optimally chosen. This leads to solving the optimization problem backward in time, with the decision at time $t+1$ solved before the decision at time t, etc. The second component is the relation between the value function at time t and the value function at time $t+1$ as summarized by the Bellman equation. The principle of optimality in the context of the method of Lagrange multipliers has two corresponding components. The first is the same as for dynamic programming. The second is a relation between the Lagrange function at time t and the Lagrange function at time $t+1$ as summarized by the second first-order condition which is derived by differentiating the Lagrangean expression with respect to the state variables. This relation can also be obtained by differentiating the Bellman equation with respect to the state variables. If so one might ask what the advantage is in using the second first-order condition obtained by the method of Lagrange multipliers. In finding the optimal control function if one is willing to forget about the value function and works only with the vector of its derivatives, one would be applying the same method as the Lagrange method. However, if one tries to find the value function in the process of obtaining the optimal control function as is typically done in practice, one fails to exploit an important first-order condition for obtaining the optimum.

To illustrate this point let us consider the problem of finding the values of the control variables u_1 and u_2 for the periods 1 and 2 when the state variable x_1 is given and x_2 is a function $f(x_1, u_1)$ of x_1 and u_1. Let the objective function be

$$r(x_1, u_1) + \beta r(x_2, u_2)$$

where β is a discount factor. By dynamic programming one first maximizes $\beta r(x_2, u_2)$ with respect to u_2 and obtains the optimum $u_2 = g_2(x_2)$ as a function of x_2. The value function for period 2 is $r(x_2, g_2(x_2)) = V_2(x_2)$. To find u_1 one maximizes

$$r(x_1, u_1) + \beta V_2(x_2) = r(x_1, u_1) + \beta V_2(f(x_1, u_1))$$

and calls the result $V_1(x_1)$. If instead we form the Lagrange expression

$$\mathscr{L} = r(x_1, u_1) + \beta[r(x_2, u_2) - \lambda(x_2 - f(x_1, u_1))]$$

and differentiates with respect to u_2, x_2 and u_1 we obtain three first-order conditions. In seeking the value function the method of dynamic programming fails to exploit an important first-order condition $\partial \mathscr{L}/\partial x_2$

2

= 0. The same point applies to the case of a stochastic model for x_t and also in continuous time as the next section demonstrates. Using dynamic programming if one finds u_1 by differentiation, the first-order condition is

$$\frac{\partial r(x_1, u_1)}{\partial u_1} + \beta \frac{dV_2(x_2)}{dx_2} \frac{\partial f(x_1, u_1)}{\partial u_1} = 0$$

Hence only knowledge of $\lambda_2 = dV_2(x_2)/dx_2$ and not $V_2(x_2)$ itself is required. Solving the Bellman equation for the value function V is not necessary and often more difficult than finding λ. Even if it were not, one always needs to find $\lambda = dV/dx$ by either method of dynamic programming or Lagrange multipliers.

I. Solution of Continuous-time Optimization Problem by Lagrange Multipliers

Let $x(t)$ be a $p \times 1$ vector of state variables and $u(t)$ be a $q \times 1$ vector of control variables at time t. The argument t is suppressed when it is understood. The stochastic model is assumed to be

$$dx = f(x, u)dt + S(x, u)dz \tag{1}$$

where $dx(t) = x(t+dt) - x(t)$, and $z(t)$ is an $n \times 1$ vector Wiener process with covariance matrix $\mathrm{Cov}(dz) = \Phi dt$. The covariance matrix $\mathrm{Cov}(Sdz) = S\Phi S' dt$ will be denoted by Σdt. Let $r(x, u)$ be the rate of flow of return or utility and the objective be to maximize the expectation

$$E \int_0^\infty e^{-\beta t} r(x, u) dt \tag{2}$$

To solve this problem by the Lagrange method, we form a Lagrangean expression based on the objective function (2) and the constraint of the stochastic differential equation (1) using a p-component vector $\lambda(x)$ of Lagrange multipliers as follows.

$$\mathcal{L} = \int_0^\infty E_t \left\{ e^{-\beta t} r(x, u) \, dt - e^{-\beta(t+dt)} \lambda'(t+dt)[x(t+dt) - x(t) - f(x, u)dt - S(x, u)dz] \right\} \tag{3}$$

where the conditional expectation E_t is justified by the statement of the problem that when the control $u(t)$ is determined the information at time t including the value $x(t)$ is given, and where we have changed the

3

$$e^{\beta t} \frac{\partial \mathcal{L}}{\partial x_i} = \frac{\partial r}{\partial x_i} dt - \lambda_i + e^{-\beta dt} \left[E_t \lambda_i(t+dt) + \frac{\partial f'}{\partial x_i} E_t \lambda(t+dt) dt + E_t dz' \frac{\partial S_i}{\partial x_i} [\lambda_i(t+dt)] \right] \tag{6}$$

$$= \frac{\partial r}{\partial x_i} dt - \lambda_i + (1-\beta dt) \left[\lambda_i + E_t d\lambda_i + \frac{\partial f'}{\partial x_i} \lambda dt + E_t dz' \frac{\partial S_i}{\partial x_i} (\lambda_i + d\lambda_i) \right] + o(dt)$$

$$= \frac{\partial r}{\partial x_i} dt - \beta \lambda_i dt + \left[\frac{\partial \lambda_i}{\partial x'} f + \frac{1}{2} tr \left(\frac{\partial^2 \lambda_i}{\partial x \partial x'} \cdot \Sigma \right) \right] dt + \frac{\partial f'}{\partial x_i} \lambda dt + tr \left[\frac{\partial S_i}{\partial x_i} \frac{\partial \lambda_i}{\partial x'} S\Phi \right] dt + o(dt) = 0 \quad i=1, ..., p$$

Equations (5) and (6) are two first-order conditions for the optimum control u and the Lagrange multiplier λ. To insure that the solution of (5) and (6) achieves a maximum, one has to examine the second-order conditions for the method of Lagrange multipliers which are discussed in Chow (1994, section 2.5) for the analogous stochastic optimization problem in discrete time.

If the model is nonstochastic S and $\Sigma = S\Phi S'$ are zero. All the trace terms in the last lines (5) and (6) vanish. Our solution reduces to the well-known solution of the corresponding nonstochastic optimal control problem in continuous time with the dynamics given by

$$dx = f(x, u)dt$$

In the above solution we assume λ to be a function of x only and not of t. If we relax this assumption, equation (4) for $d\lambda_i$ will have a term $\frac{\partial \lambda_i}{\partial t}$ inside the square brackets. The same term will appear inside the square brackets multiplying dt after the last equality sign of (6), yielding the following partial differential equation for λ.

$$\frac{\partial \lambda}{\partial t} + \frac{\partial \lambda}{\partial x'} f + \frac{\partial f'}{\partial x} \lambda + \frac{\partial r}{\partial x} - \beta \lambda = 0 \tag{7}$$

Equation (7) and equation (5) with the trace terms omitted, or

$$\frac{\partial r}{\partial u} + \frac{\partial f'}{\partial u} \lambda = 0$$

5

provide a pair of equations for u and λ. These equations can be derived from the well-known Pontryagin's maximum principle for solving nonstochastic optimal control problem in continuous time.

To apply Pontryagin's maximum principle to our nonstochastic control problem, we form the Hamiltonian

$$H = r(x, u) + e^{-\beta dt}\lambda' f(x, u) \qquad (9)$$

and set

$$\frac{\partial H}{\partial u} = 0 \qquad (10)$$

$$\frac{\partial(e^{-\beta dt}\lambda)}{\partial t} = - \frac{\partial H}{\partial x} \qquad (11)$$

and

$$\frac{dx}{dt} = \frac{\partial H}{\partial(e^{-\beta dt}\lambda)} \qquad (12)$$

Equations (10) and (11) are identical with the first-order conditions (8) and (7) respectively, while equation (12) gives the differential equation for the dynamics of the state variable. Note that we have written this differential equation as $dx = fdt$ rather than $dx/dt = f$ because in the stochastic version of our problem, the derivative dx/dt does not exist. dx has a term dz which is of order $(dt)^{1/2}$. Hence dz/dt is of order $(dt)^{-1/2}$ which approaches infinity as dt approaches zero.

II. Optimum Consumption and Portfolio Selection Over Time

The well-known problem of this section was studied by Merton (1969, 1971), and its discrete-time version by Samuelson (1969). At time t, the individual chooses his rate of consumption $c(t)$ per unit time during period t (between t and $t+dt$) and the number $N_i(t)$ of shares to be invested in asset i during period t, given his initial wealth $W(t) = \sum_i^n N_i(t-dt)P_i(t)$ and the prices $P_i(t)$ per share of the assets. To simplify exposition we assume the prices to follow a geometric Brownian motion.

6

$$\frac{dP_i}{P_i} = \alpha_i dt + s_i dz_i \tag{13}$$

where z_i are components of a multivariate Wiener process, with $E(dz_i) = 0$, $var(dz_i) = 1$ and $E(dz_i dz_j) = \rho_{ij}$. If there is no wage income and all incomes are derived from capital gains (dividends being included in changes in asset prices), it can be shown that the change in wealth from t to $t+dt$ satisfies the budget constraint

$$dW = \sum_1^n N_i(t)dP_i - c(t)dt \tag{14}$$

Let $w_i(t) = N_i(t)P_i(t)/W(t)$ be the fraction of wealth invested in asset i, with $\Sigma_i w_i = 1$. We :itute (13) for dP_i in (14) to obtain

$$dW = \sum_1^n w_i W \alpha_i dt - cdt + \sum_1^n w_i W s_i dz_i \tag{15}$$

If we assume the nth asset to be risk-free, i.e., $s_n = 0$, and denote the instantaneous rate of return α_n of this asset by r, we can write (15) as, with $m = n-1$,

$$dW = \sum_1^m w_i(\alpha_i - r)Wdt + (rW - c)dt + \sum_1^m w_i W s_i dz_i \tag{16}$$

The state variable of this problem is W which is governed by the stochastic differential equation (15). The control variables are c and $w = (w_1,...,w_n)'$, with $\Sigma_1^n w_i = 1$. The problem is to maximize

$$E_0 \int_0^\infty e^{-\beta t} u(c)dt$$

This model assumes that assets are traded continuously in time and that there are no transaction costs in trading.

The Lagrangean expression for this optimization problem is

7

$$\mathscr{L} = \int_{0}^{\infty} E_t \left\{ e^{-\beta t} u(c) dt - e^{-\beta(t+dt)} \lambda'(t+dt)[dW - (W \sum_{1}^{n} w_i \alpha_i - c) dt - W \sum_{1}^{n} w_i s_i dz_i] + e^{-\beta t} \mu[1 - \sum_{1}^{n} w_i] dt \right\} \quad (17)$$

Noting $f(x,u) = (W \sum_{1}^{n} w_i \alpha_i - c)$ in this case, with W as the state variable and c and $w' = (w_1,...,w_n)$ as the control variables, we can write the first-order condition (5) where the matrix S is now the row vector $W(w_1 s_1 \cdots w_n s_n)$ as

$$u'(c) - \lambda = 0 \quad (18)$$

$$W \alpha_i \lambda + W^2 tr \left\{ \begin{bmatrix} 0 \\ \vdots \\ s_i \\ \vdots \\ 0 \end{bmatrix} \lambda_w [w_1 s_1 \cdots w_n s_n] \begin{bmatrix} 1 & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{21} & 1 & \cdots & \rho_{2n} \\ & & \cdots & \\ \rho_{n1} & \rho_{n2} & \cdots & 1 \end{bmatrix} \right\} - \mu \quad (19)$$

$$= W \alpha_i \lambda + W^2 \lambda_w \sum_{j=1}^{n} w_j \sigma_{ij} - \mu = 0 \qquad i=1, ..., n$$

where we have denoted $d\lambda/dW$ by λ_w and $s_i s_j \rho_{ij}$ by σ_{ij}. The first-order condition (6) implies, with λ_{ww} denoting $\partial^2 \lambda/\partial W^2$,

$$\beta \lambda = (\sum_{1}^{n} w_i \alpha_i) \lambda + \lambda_w (W \sum_{1}^{n} w_i \alpha_i - c) + \frac{1}{2} \lambda_{ww} \cdot W^2 \sum_{ij} w_i w_j \sigma_{ij} + \lambda_w W \sum_{ij} w_i w_j \sigma_{ij} \quad (20)$$

Without solving for λ using (20), we can define the inverse function $G = [u'(c)]^{-1}$ and solve (18) for the optimal consumption function $\hat{c} = G(\lambda)$. To obtain the optimal portfolio w, we divide (19) by $W^2 \lambda_w$, denote $-\mu/(W^2 \lambda_w)$ by μ^*, and write the resulting equations together with $\Sigma w_i = 1$ in matrix form as

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} & 1 \\ & & & & \vdots \\ & & & & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn} & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \\ \mu^* \end{bmatrix} = -\frac{\lambda}{W \lambda_w} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \\ 1 \end{bmatrix}, \quad (21)$$

8

Partitioning the coefficient matrix of (21) into four blocks and finding the partioned inverse we obtain the first n rows of the inverse as

$$(\sigma^{ij}) - \gamma^{-1} \begin{bmatrix} (\Sigma\sigma^{1j})(\Sigma\sigma^{1j}) & (\Sigma\sigma^{1j})(\Sigma\sigma^{2j}) & \dots & (\Sigma\sigma^{1j})(\Sigma\sigma^{nj}) \\ & \dots & & \\ (\Sigma\sigma^{1j})(\Sigma\sigma^{nj}) & (\Sigma\sigma^{2j})(\Sigma\sigma^{nj}) & \dots & (\Sigma\sigma^{nj})(\Sigma\sigma^{nj}) \end{bmatrix}, \quad \gamma^{-1} \begin{bmatrix} \Sigma\sigma^{1j} \\ \vdots \\ \Sigma\sigma^{nj} \end{bmatrix} \tag{22}$$

where (σ^{ij}) denotes the matrix inverse $(\sigma_{ij})^{-1}$ and $\gamma = \Sigma_i \Sigma_j \sigma^{ij}$. Premultiplying (21) by (22) we obtain the well-known optimal portfolio rules of Merton (1969, 1971):

$$\hat{w}_k = \sum_l [\sigma^{kl} - \gamma^{-1}(\Sigma_j\sigma^{kj})(\Sigma_i\sigma^{li})] \frac{-\lambda}{W\lambda_w} \alpha_l + \gamma^{-1}\Sigma_j\sigma^{kj} \tag{23}$$

$$= \gamma^{-1}\Sigma_j\sigma^{kj} - \frac{-\lambda}{W\lambda_w} [\Sigma_l\sigma^{kl}\alpha_l - \gamma^{-1}\Sigma_j\sigma^{kj}\Sigma_l\Sigma_i\sigma^{li}\alpha_l]$$

$$= h_k + m(W,t) \cdot g_k, \qquad k=1,\dots,n,$$

where we have defined

$$h_k = \gamma^{-1}\Sigma_j\sigma^{kj} \tag{24}$$

$$m(W,t) = \frac{-\lambda}{W\lambda_w} \tag{25}$$

$$g_k = \Sigma_j\sigma^{kj} [\alpha_j - \gamma^{-1} \Sigma_l\Sigma_i\sigma^{li}\alpha_l] \tag{26}$$

implying $\Sigma_1^n h_k = 1$ and $\Sigma_1^n g_k = 0$.

In deriving the above well-known optimum consumption and portfolio selection functions $\hat{c}$ *and* $\hat{w}$, we have demonstrated that the Lagrange method is easy to apply and that the optimum control functions depend on $\lambda = \partial V/\partial x$ and not on the value function V itself. The Lagrange method would be superior in situations where the first-order condition (6) or

9

$$\beta\lambda_i = \frac{\partial r}{\partial x_i} + \frac{\partial \lambda_i}{\partial x'} f + \frac{\partial f'}{\partial x_i}\lambda + \frac{1}{2} tr\left[\frac{\partial^2\lambda_i}{\partial x \partial x'} \cdot S\Phi S'\right] + tr\left[\frac{\partial S_i}{\partial x_i} \cdot \frac{\partial \lambda_i}{\partial x'} S\Phi\right] \tag{27}$$

is easier to solve than the Bellman equation

$$\beta V = r + f' \frac{\partial V}{\partial x} + \frac{1}{2} tr\left[\frac{\partial^2 V}{\partial x \partial x'} \cdot S\Phi S'\right] \tag{28}$$

which can be differentiated to obtain (27).

A well-known case is when $f(x,u)$ is linear, say equal to $Ax+Cu$, S is a constant matrix and $r(x,u)$ is quadratic, say equal to $(1/2)x' R_1 x + (1/2)u' R_2 u + x' R_{12} u + a_1' x + a_2' u$. In this case V is quadratic and λ is linear, say equal to $Hx+h$. Since the second derivatives of λ_i are zero and $\partial S'/\partial x_i$ is also zero, only the first three terms on the right side of equation (27) remain. In vector form equation (27) becomes

$$\beta\lambda = Hx+h = a_1 + R_1 x + R_{12} u + H(Ax+Cu) + A'(Hx+h) \tag{29}$$

The first-order condition (5) becomes

$$a_2 + R_2 u + R_{12}' x + C'(Hx+h) = 0 \tag{30}$$

implying

$$u = -R_2^{-1}(C'H + R_{12}')x - R_2^{-1}(C'h + a_2) \equiv Gx+g \tag{31}$$

Substituting (31) for u in (29) and equating coefficients would give two equations for H and h. These are the well-known matrix Recatti equations for solving the unknown parameters of the linear function λ. Once H and h are determined, the linear function u is found by (31). Other examples favoring the Lagrange method can be found in Chow (1994). Possible computational advantages of using the Lagrange method will be discussed in section IV.

Returning to problems of finance, it is useful to point out that the capital asset pricing model of Merton (1973) introduces additional state variables $[x_1(t)...x_N(t)] = x(t)$ which follow the stochastic

10

differential equations

$$dx_i = f_i(x)dt + g_i(x)dq_i \qquad i=1, ..., N$$

with dq_i correlated with dz_j of equation (13). The optimization problem can be solved and the results of Merton (1973) obtained by adding to the Lagrange expression (17) the following constraint

$$e^{-\beta t(t+dt)} \sum_{i=1}^{N} \lambda_i(t+dt)[dx_i - f_i(x)dt - g_i(x)dq_i]$$

and differentiating a respect to these additional s   variables.

III. Investment as Exercising an irreversible Option to Invest

Let us consider a model for the investment decision of a firm which treats the decision as exercising an option to invest. Once the option is taken or exercised, it cannot be reversed. Such theories of investment are surveyed and discussed in Pindyck (1991), Dixit (1992) and Pindyck and Dixit (1993). In the simplest case assume that to exercise the option, it would cost the firm I dollars. The present value $v(t)$ of the investment project at time t is assumed to vary through time according to the stochastic differential equation

$$dv = \alpha v dt + \sigma v dz \qquad (32)$$

where z is a Wiener process h var(dz) $= dt$. The blem is to determine the optimum time T to invest or to exercise the option. When ne option is exercised, the firm gains $v(T) - I$ but loses the opportunity to invest in a future time T+s when $v(T+s)$ may be larger than $v(T)$.

We formulate this optimization problem starting from time 0 using the Lagrangean expression

$$\mathscr{L} = E_0 e^{-\beta T}(v(T)-I)u - \int_0^T E_t e^{-\beta t}\lambda(t+dt)[v(t+dt) - v(t) - \alpha v(t)(1-u)dt - \sigma v(t)(1-u)dz(t)] \qquad (33)$$

The state variable is $v(t)$. The control function $u(v)$ could be viewed as a step function, with $u(v) = 1$ meaning to understake the investment when v reaches $v(T) \equiv v^*$ and $u(v) = 0$ meaning not to undertake the investment when $v < v^*$. To maximize (33) we consider the two cases $u = 0$ and $u = 1$. Keeping u fixed, we first find a first-order condition by differentiating the Lagrangean expression with respect to the state

11

variable v. We will then find a second condition to determine v^*.

To find the function $\lambda(v)$ we set the derivative of $e^{\beta t}\mathscr{L}$ with respect to the state variable $v = v(t)$ equal to zero. To obtain $\partial\mathscr{L}/\partial v$ we need to evaluate $d\lambda$ by Ito's lemma.

$$d\lambda = \left(\frac{d\lambda}{dv}\,\alpha v + \frac{1}{2}\,\frac{d^2\lambda}{dv^2}\,\sigma^2 v^2\right)dt + \frac{d\lambda}{dv}\,\sigma v\,dz \tag{34}$$

Using (34) we find

$$e^{\beta t}\,\frac{\partial\mathscr{L}}{\partial v} = -E_t\lambda(t+dt)[-1 - \alpha dt - \sigma dz] - e^{\beta dt}\lambda(t) \tag{35}$$

$$= E_t[\lambda(t) + d\lambda(t)][1 + \alpha dt + \sigma dz] - (1 + \beta dt)\lambda(t) + o(dt)$$

$$= \lambda(t) + \alpha\lambda(t)dt + d\lambda(t) + E_t d\lambda(t)\sigma dz(t) - \lambda(t) - \beta\lambda(t)dt + o(dt)$$

$$= \left\{\frac{1}{2}\,\sigma^2 v^2\,\frac{d^2\lambda}{dv^2} + (\alpha+\sigma^2)v\,\frac{d\lambda}{dv} + (\alpha-\beta)\lambda\right\}dt + o(dt) = 0$$

Setting to zero the expression in curly brackets in (35) provides a second-order differential equation for λ.

A solution to this differential equation is

$$\lambda = av^\gamma \tag{36}$$

Substituting (36) and its first and second derivatives in the above differential equation yields a second-degree equation in the unknown parameter γ, the solution of which is

$$\gamma = -\alpha/\sigma^2 - \frac{1}{2} \pm \left\{\left[\alpha/\sigma^2 + \frac{1}{2}\right]^2 - 2(\alpha-\beta)/\sigma^2\right\}^{1/2} \tag{37}$$

Since at time T when the value of the option equals to $v(T)-I$, the rate of change of the option value $\mathscr{L}$ with respect to v, or $\lambda(v(t))$, equals 1. We have $\lambda(v^*) = av^{*\gamma} = 1$, yielding $a = v^{*-\gamma}$ and

$$\lambda = (v/v^*)^\gamma \tag{38}$$

The solution is complete once we can find v^*. To do so we observe that the value of the option at time T evaluated by the Lagrangean expression is v^*-I. Since the value of the option is $\int \lambda(v)dv$ which equals $(\gamma+1)^{-1}v^{*-\gamma}v^{\gamma+1}$ by (38). This value when $v = v^*$ is $(\gamma+1)^{-1}v^*$. Equating $(\gamma+1)^{-1}v^*$ to v^*-I we obtain

$$v^* = (\gamma+1)I/\gamma \tag{39}$$

The decision rule is to undertake the investment project when $v(t)$ reaches v^* given by (39). Since a negative v^* cannot be the optimum γ cannot be negative and we take the larger of the two roots of γ given by (37).

The above example is the most basic example in the theory of investment when undertaking an investment project is considered an irreversible decision as surveyed by Pindyck (1991). Pindyck (pp. 1122 and 1145-1146) solves this problem by dynamic programming. The value function F(v) for the option to invest is solved by using the Bellman equation which is a second-order differential equation in F:

$$\frac{1}{2} \sigma^2 v^2 \frac{\partial^2 F}{\partial v^2} + \alpha v \frac{\partial F}{\partial v} - \beta F = 0 \tag{40}$$

Differentiating (40) with respect to v and setting $\partial F/\partial v = \lambda$ would yield our differential equation from (35). In this example the differential equation for $\lambda = v$ is of the same form as the different equation for F. In circumstances where the value function is easy to find by solving the Bellman equation, one should by all means find it as it contains much useful information. Dixit and Pindyck (1993) provide examples of such circumstances.

IV. A Numerical Method by Locally Quadratic Approximations to the Lagrange Function

As numerical solution of the Bellman equation for the value function V has been an active area of research for over three decades, numerical solution of the first-order conditions (6) or (27) for the Lagrange multiplier λ is an important area of research if one wishes to solve dynamic optimization problems numerically. A well-known but crude numerical method is to approximate the function f by a linear function and the function r by a quadratic function and to solve the resulting linear-quadratic control problem, yielding a linear control function and a quadratic value function. Since the Lagrange function is the

vector of derivatives of the value function, it is linear. Below equations (29) and (30) we have discussed the numerical solution for λ as a linear function.

In Chow (1993a, 1993b), locally rather than globally linear functions for the optimal control u and the Lagrange multiplier λ have been suggested and implemented. Since u and λ need to be evaluated for any given x, we can solve the approximate linear-quadratic optimal control problem by approximately f, $\partial r/\partial x$ and $\partial r/\partial u$ linearly about any given x, yielding locally linear control function u and Lagrange function λ. In this section we recommend locally quadratic approximations for λ.

If one compares the first-order condition (6) or (27) for λ with the Bellman equation (28) for the value function V, one realizes that because a quadratic approximation of V using (28) will yield only a linear approximation of $\lambda = \partial V/\partial x$, a quadratic approximation of λ using (27) will improve upon this linear approximation and provide a more accurate approximation of u by solving equation (5) which requires both λ and $\partial\lambda/\partial x'$. In other words, solving the Bellman equation (28) by a locally quadratic approximation of the value function V at any required value of x amounts to a locally linear approximation of λ and is therefore inferior to a quadratic approximation of λ for the purpose of solving equation (5).

To solve equations (5) and (6) for any given x_t, let $\lambda(x)$ be approximated locally about x_t by a quadratic function. Denoting $\partial^2\lambda_i/\partial x\partial x'$ or $\partial^2\lambda_i/\partial x_i\partial x'$ evaluated at x_t by Q_{it}, we can write

$$\lambda(x) = \lambda(x_t) + \frac{\partial\lambda}{\partial x'}(x-x_t) + \frac{1}{2}\begin{bmatrix}(x-x_t)'\ Q_{1t}\ (x-x_t)\\ \vdots \\ (x-x_t)'\ Q_{pt}\ (x-x_t)\end{bmatrix} \qquad (41)$$

$$= \lambda(x_t) - \frac{\partial\lambda}{\partial x'}x_t + \frac{1}{2}\begin{bmatrix}x_t'Q_{1t}\\ \vdots \\ x_t'Q_{pt}\end{bmatrix}x_t + \left\{\frac{\partial\lambda}{\partial x'} - \begin{bmatrix}x_t'Q_{1t}\\ \vdots \\ x_t'Q_{pt}\end{bmatrix}\right\}x + \frac{1}{2}\begin{bmatrix}x'Q_{1t}\\ \vdots \\ x'Q_{pt}\end{bmatrix}x$$

$$\equiv h_t + H_t x + \frac{1}{2}\begin{bmatrix}x'Q_{1t}\\ \vdots \\ x'Q_{pt}\end{bmatrix}x = h + Hx + \frac{1}{2}\begin{bmatrix}x'Q_1\\ \vdots \\ x'Q_p\end{bmatrix}x$$

where the subscript t of h, H and Q_i ($i=1, ..., p$) is omitted when understood. In view of the fact that the ith component $\lambda_i(x)$ of $\lambda(x)$ is the partial derivative of the value function with respect to x_i, many elements of the matrices Q_i ($i=1, ..., p$) are equal. Q_{ijk}, the j-k element of the matrix Q_i, is the third

partial of the value function with respect to x_i, x_j and x_k; it is equal to all elements with the same three subscripts regardless of order. One way to list the distinct elements of the Q matrices is to start with a symmetric Q_1 matrix, omit the first row of a symmetric Q_2 matrix as all derivatives of the value function with respect to x_1 have been counted, omit the first two rows of a symmetric Q_3, etc., until only Q_{ppp} of Q_p is specified. The suggested iterative method for solving (5) and (6) consists of two steps. First, given h, H, Q_i ($i=1, ..., p$), solve (5) for u as a linear function $g+Gx$ of x. Second, given g, G, h, H and Q_i from Step 1, revise h, H and Q_i by equating coefficients on both sides of equation (6) or (27).

The approximation of $\lambda(x)$ near x_t by the quadratic function (41) implies

$$\frac{\partial \lambda}{\partial x'} = H + \begin{bmatrix} x_t' Q_1 \\ \vdots \\ x_t' Q_p \end{bmatrix} \quad ; \quad \frac{\partial^2 \lambda}{\partial x_i \partial x'} = Q_i \tag{42}$$

As in (41), we further approximate (with the subscripts t and $*$ of all coefficients omitted as all derivatives are understood to be evaluated at x_t and some tentative value u_* for u)

$$\frac{\partial r}{\partial x} = k_1 + K_1 x + K_{12} u + \frac{1}{2} \begin{bmatrix} x' P_{1x} x + u' P_{1u} u \\ \vdots \\ x' P_{px} x + u' P_{pu} u \end{bmatrix} \tag{43}$$

$$\frac{\partial r}{\partial u} = k_2 + K_2 u + K_{21} x + \frac{1}{2} \begin{bmatrix} x' S_{1x} x + u' S_{1u} u \\ \vdots \\ x' S_{qx} x + u' S_{qu} u \end{bmatrix} \tag{44}$$

$$f = Ax + Cu + b \tag{45}$$

We denote by d_2 the $q \times 1$ vector with its ith element equal to the trace term of (5) and by d_1 the $p \times 1$ vector with its ith element equal to the trace term of (6). In the first step we substitute (44), (45), (41) and d_2 into (5) and solve for u.

15

$$K_2 + K_{21}x + K_2 u + \frac{1}{2}\begin{bmatrix} x'S_{1x}x + u'S_{1u}u \\ \vdots \\ x'S_{qx}x + u'S_{qu}u \end{bmatrix} + C'h + C'Hx + \frac{1}{2}C'\begin{bmatrix} x'Q_1 x \\ \vdots \\ x'Q_p x \end{bmatrix} + d_2 = 0 \qquad (46)$$

Linearizing the quadratic terms in (46) about x_t and u_* and solving the resulting equation for u, we obtain

$$u = Gx + g \qquad (47)$$

where

$$G = -\left\{K_2 + \begin{bmatrix} u'_* S_{1u} \\ \vdots \\ u'_* S_{qu} \end{bmatrix}\right\}^{-1}\left\{K_{21} + \begin{bmatrix} x'_t S_{1x} \\ \vdots \\ x'_t S_{qx} \end{bmatrix} + C'\left(H + \begin{bmatrix} x'_t Q_1 \\ \vdots \\ x'_t Q_p \end{bmatrix}\right)\right\} \qquad (48)$$

$$g = -\left\{K_2 + \begin{bmatrix} u'_* S_{1u} \\ \vdots \\ u'_* S_{qu} \end{bmatrix}\right\}^{-1}\left\{k_2 - \frac{1}{2}\begin{bmatrix} x'_t S_{1x}x_t + u'_* S_{1u}u_* \\ \vdots \\ x'_t S_{qx}x_t \quad u'_* S_{qu}u_* \end{bmatrix} - \frac{1}{2}C'\begin{bmatrix} x'_t Q_1 x_t \\ \vdots \\ x'_t Q_p x_t \end{bmatrix} + C'h + d_2\right\} \qquad (49)$$

In the second step, given G and g, we can revise the parameters of λ by substituting (43), (45), (41), and d_1 into (6) or (27) to yield, , letting A+CG = R and b+Cg = c,

$$(\beta I - A')\left(h + Hx + \frac{1}{2}\begin{bmatrix} x'Q_1 x \\ \vdots \\ x'Q_p x \end{bmatrix}\right) = k_1 + K_{12}g + (K_1 + K_{12}G)x \qquad (50)$$

$$+ \frac{1}{2}\begin{bmatrix} x'(P_{1x} + G'P_{1u}G)x \\ \vdots \\ x'(P_{px} + G'P_{pu}G)x \end{bmatrix} + \begin{bmatrix} g'P_{1u} \\ \vdots \\ g'P_{pu} \end{bmatrix}Gx + \frac{1}{2}\begin{bmatrix} g'P_{1u}g \\ \vdots \\ g'P_{pu}g \end{bmatrix}$$

$$+ \left(H + \begin{bmatrix} x'Q_1 \\ \vdots \\ x'Q_p \end{bmatrix} \right) (Rx + c) + d_1 + \frac{1}{2} \begin{bmatrix} tr(Q_1\Sigma) \\ \vdots \\ tr(Q_p\Sigma) \end{bmatrix}$$

Equating coefficients of (50) one obtains

$$h = (\beta I - A')^{-1} \left(k_1 + K_{12}g + \frac{1}{2} \begin{bmatrix} g'P_{1u}g \\ \vdots \\ g'P_{pu}g \end{bmatrix} + Hc + d_1 + \frac{1}{2} \begin{bmatrix} tr(Q_1\Sigma) \\ \vdots \\ tr(Q_p\Sigma) \end{bmatrix} \right) \tag{51}$$

$$H = (\beta I - A')^{-1} \left(K_1 + K_{12}G + HR + \begin{bmatrix} c'Q_1 \\ \vdots \\ c'Q_p \end{bmatrix} + \begin{bmatrix} g'P_{1u} \\ \vdots \\ g'P_{pu} \end{bmatrix} G \right) \tag{52}$$

$$\beta Q_i = P_{ix} + G'P_{iu}G + 2Q_iR + \sum_{j=1}^{p} a_{ji}Q_j \qquad (i=1, ..., p) \tag{53}$$

where a_{ji} denotes the j-i element of $A = \partial f/\partial x'$.

The parameters of λ as revised by (51)-(53) can be used to solve equation (5) for u in step 1 of the iterative method. This new value of u will replace u_* in the evaluation of all required derivatives in the next round of steps 1 and 2 until convergence. When the function λ is assumed to be linear Chow (1993b) has reported on computational experience with fairly rapidly converging iterations based on the discrete-time version of equations (5) and (6). To start the two-step iterative procedure, I recommend using as u_* the solution $\hat{u}(x_t)$ of the optimization problem based on locally linear approximations of $\partial r/\partial x$, $\partial r/\partial u$, f and λ, i.e., on the above algorithm with all matrices Q_i, P_{ix}, P_{iu}, S_{ix} and S_{iu} set equal to zero. Such a value of $u_*(x_t)$ will be used together with x_t to obtain all required derivatives in (41)-(45) to start the iterations. We await further research to report on computational experience of this numerical method.

V. Conclusion

In this paper, the method of Lagrange multipliers is presented for solving dynamic optimization problems where the state variables follow a system of stochastic differential equations. The method

17

presented is an alternative to dynamic programming. It has been applied to solve the well-known problems of optimum consumption and portfolio selection of Merton (1969, 1970) and of irreversible investment as surveyed by Pindyck (1991) and Dixit (1992). The method is a generalization of Pontryagin's maximum principle to stochastic models in continuous time. It avoids solving the Bellman equation for the value function. In certain circumstances, including the common example of solving optimal control problems with a linear dynamic model and a quadratic objective function, it is analytically simpler than dynamic programming. Other such examples for nonlinear models in discrete time can be founded in Chow (1994). It is numerically more accurate to the extent that a quadratic approximation of the Lagrangean function is better than a linear approximation. The latter amounts to a quadratic approximation the value function, whereas a cubic approximation of the value function of many state variables is very difficult if not impossible. In conclusion, the Lagrange method is recommended to students and researchers in finance. Just as Pontryagin's maximum principle is widely used to solve dynamic optimization problems under certainty, the Lagrange method will be found convenient to use for solving dynamic optimization problems under uncertainty.

References

Chow, G. C., 1993, Optimal control without solving the Bellman equation, *Journal of Economic Dynamics and Control*, **17**, 621-630. (a)

_____, 1993, Computation of optimum control functions by Lagrange multipliers, in David A. Belsley, ed., *Computational Techniques for Econometrics and Economic Analysis*, (Boston, Kluwer Academic Publishers), 65-72. (b)

_____, 1994, *Dynamic Economics*, (Princeton, Princeton University, mimeo).

Dixit, A. K., 1990, *Optimization in Economic Theory* (Oxford, Oxford University Press).

_____, 1992, Investment and hysteresis, *The Journal of Economic Perspectives*, **6**, 107-132.

_____, and R. S. Pindyck, 1994, *Investment Under Uncertainty*, (Princeton, N.J., Princeton University Press).

Merton, Robert C., 1969, Lifetime portfolio selection under uncertainty, the continuous-time case, *Review of Economics and Statistics*, **51**, 247-257.

_____, 1970, Optimum consumption and portfolio rules in a continuous-time model," *J. of Economic Theory*, **3**, 373-413.

_____, 1973, An intertemporal capital asset pricing model, *Econometrica*, **41**, 867-887.

Pindyck, R. S., 1991, Irreversibility, uncertainty, and investment, *Journal of Economic Literature*, **29**, 1110-1148.

Samuelson, P. A., 1969, Lifetime portfolio selection by dynamic stochastic programming, *Review of Economics and Statistics*, **51**, 234-246.