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EXISTENCE OF STABLE PAYOFF CONFIGURATIONS FOR COOPERATIVE GAMES

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1. Introduction.

In R. J. Aumann and M. Maschler [1], a theory was developed to attack the following general question: If the players in a cooperative n-person game have decided upon a specific coalition-structure, how then will they distribute among themselves the values of the various coalitions in such a way that some stability requirements will be satisfied. Several criteria for the "stable" splits were given, centering upon the idea that a "stable" payoff should offer the players some security in the sense that each "objection" could be met by a "counter objection." A variety of concepts of objections and counter objections were suggested, and one of them was studied in more detail. This one, and some of the others, had the feature that for some coalition-structures there were no stable payoffs, and therefore these coalition-structures could not be used by those players who wished stability in this sense. (See also [6].) Moreover, cases were established in which even a coalition-structure which yields the maximum total amount to all the players had no stable outcome. In particular, an example was given in [1] of a game with a superadditive, non-negative, non-identically zero characteristic function, in which no outcome was stable unless each player received a zero amount.

It is conceivable that many would reject such an outcome on the ground that "rational" players in a superadditive game would always agree on an imputation, because otherwise they can all benefit by switching to an appropriate imputation.

We do not share this opinion, for we feel that often a desire for security is stronger than a wish to make some extra profit. In fact, many profitable coalitions in everyday life are never realized because the "players" do not consider them safe. Nevertheless, we do believe that in some cases, especially if large profits are at stake, people may be willing to relax their safety requirements in order to make more out of a game.

It is therefore of interest to develop a theory in which safety requirements are so relaxed that there always exist stable imputations in a superadditive game. We shall prove that this is indeed the case for one of the variants proposed in [1]. Moreover, we conjecture that this variant always provides stable outcomes for each choice of a coalition-structure. We are able to prove this conjecture for those coalition-structures in which each coalition does not contain more than three players.

The key theorem, very interesting in itself, states that each outcome induces a partial "order" relation among the players which is asymmetric and never intransitive (however, it is not necessarily transitive). This phenomenon, which, e.g., does not occur in the von Neumann-Morgenstern concept of domination, is "just enough" for proving various existence theorems.

The necessary definitions are stated to make the paper self-contained.

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2. Basic Definitions.

We consider an n-person cooperative game Γ, described by a set $N \equiv \{1, 2, \ldots, n\}$ of n players and a real function $v(B)$ defined for each non-empty subset B of N. B is called a coalition and $v(B)$ is its value. The function $v(B)$ is known as the characteristic function of the game.[1] It is not necessarily superadditive.

[1] The theory allows also for the possibility that some non-1-person coalitions are not permissible. If B is such a coalition, we simply agree that $v(B) = 0$, and modify slightly the permissible outcomes.

There will be no loss of generality if we assume that

(2.1) $v(B) \geq 0$ and $v(i) = 0$ for each i , $i = 1, 2, \ldots, n$.

An <u>outcome</u> of a game Γ is represented by a <u>payoff configuration</u> (p.c.)

(2.2) $$(\mathcal{X} ; \mathcal{B}) \equiv (x_1, x_2, \ldots, x_n ; B_1, B_2, \ldots, B_m) .$$

Here, $\mathcal{B} \equiv B_1, B_2, \ldots, B_m$ is the <u>coalition-structure</u>, and hence satisfies

(2.3) $$B_j \cap B_k = \emptyset \text{ for all } j, k, j \neq k , \text{ and } \bigcup_{j=1}^{m} B_j = N ,$$

and $\mathcal{X} \equiv (x_1, x_2, \ldots, x_n)$ represents the <u>payoff vector</u> according to which player i receives in the outcome the amount x_i , $i = 1, 2, \ldots, n$. We assume that each coalition makes full use of its value, and therefore $\mathcal{X}$ is required to satisfy

(2.4) $$\sum_{i \in B_j} x_i = v(B_j) , \quad j = 1, 2, \ldots, m .$$

We also require that each outcome is <u>individually rational</u>, i.e., that

(2.5) $$x_i \geq 0 \text{ for each } i , \quad i = 1, 2, \ldots, n .$$

Thus, for each fixed coalition-structure $\mathcal{B} \equiv B_1, B_2, \ldots, B_m$, the set of all possible payoff vectors consists of a cartesian product of m simplices

(2.6) $$S = S_1 \times S_2 \times \ldots \times S_m ,$$

where, in view of (2.4) and (2.5),

(2.7) $$S_j \equiv \left\{ \{x_i\}_{i \in B_j} \;\middle|\; \sum_{i \in B_j} x_i = v(B_j) , \quad x_i \geq 0 \right\} , \ j = 1, 2, \ldots, m .$$

Let $(\mathcal{X} ; \mathcal{B})$ be an individually rational payoff configuration (i.r.p.c.), (2.2) and (2.5), in a game Γ , and let k and ℓ be two distinct members of a coalition[1] B_j of $\mathcal{B}$.

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[1]This requires, of course, that B_j contains more than one player.

For a coalition C and a distribution $\{y_i\}$, $i \in C$, of its value among its members, the pair $(\{y_i\}; C)$ is called an <u>objection</u> of k against ℓ in $(\mathscr{x};\mathscr{B})$, if

(2.8) $$k \in C\ ,\quad \ell \notin C\ ,\quad k, \ell \in B_j\ ,$$

(2.9) $$\sum_{i \in C} y_i = v(C)\ ,$$

(2.10)[1] $$y_k > x_k\ ,\quad y_i \geq x_i \quad \text{for all } i\ ,\ i \in C\ .$$

Let $(\mathscr{x};\mathscr{B})$ be an i.r.p.c. (2.2) and (2.5), in a game Γ and let $(\{y_i\}; C)$ be an objection of a player k against a player ℓ in $(\mathscr{x};\mathscr{B})$, (2.8), (2.9) and (2.10). For a coalition D and a distribution $\{z_i\}$, $i \in D$, of its value among its members, the pair $(\{z_i\}; D)$ is called a <u>counter objection</u> to the above objection, if

(2.11) $$\ell \in D\ ,\quad k \notin D\ ,$$

(2.12) $$\sum_{i \in D} z_i = v(D)\ ,$$

(2.13) $$z_i \geq x_i \quad \text{for all } i\ ,\ i \in D\ ,$$

(2.14) $$z_i \geq y_i \quad \text{for all } i\ ,\ i \in D \cap C\ .$$

<u>Definition 2.1</u> An i.r.p.c. $(\mathscr{x};\mathscr{B})$ in a game Γ is called <u>stable</u> ($\mathscr{M}_1^{(i)}$ - stable), if for each objection there exists a counter objection.

The set of all the stable p.c.'s is called the <u>bargaining set</u>[2] $\mathscr{M}_1^{(i)}$.

It will be of advantage to introduce a "strength" relation among the players, which corresponds to each i.r.p.c.

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[1]No loss of generality will be caused if we assume that all the inequalities are strict.

[2]This is one of several variants mentioned in R. J. Aumann and M. Maschler [1]. Although formulated differently, it is actually the same as $\mathscr{M}_1$ of [1], with the coalitional rationality requirement being replaced by individual rationality. The definition in [1], however, "sounds" more general. (See [6].)

Definition 2.2 Let $(x;\mathcal{B})$ be an i.r.p.c., (2.2) and (2.5), for a game Γ. Let k and ℓ be two players in a coalition B_j of $\mathcal{B}$. We say that player k is stronger than player ℓ in $(x;\mathcal{B})$, and we denote this by $k \succ \ell$, if player k has an objection against player ℓ, which cannot be countered.

We say that a player k is equal to player ℓ in $(x;\mathcal{B})$, and denote this by $k \sim \ell$, if $k \not\succ \ell$ and $\ell \not\succ k$. ($\not\succ$ means "not stronger than").

Obviously, an i.r.p.c. $(x;\mathcal{B})$ is stable in a game Γ if and only if in each coalition of $\mathcal{B}$, each player is equal to each other player who belongs to the same coalition.

In the next section we shall study some properties of the relation $\succ$.

3. Weak Partial Order.

Definition 3.1 A binary relation $\mathcal{R}$ will be called a weak partial order, if it is never intransitive. I.e., if

$$(3.1) \qquad A_1 \mathcal{R} A_2 \,,\; A_2 \mathcal{R} A_3 \,,\; \ldots,\; A_{\alpha-1} \mathcal{R} A_\alpha \Longrightarrow \sim A_\alpha \mathcal{R} A_1 \,.$$

It will be shown subsequently that $\succ$ is such a relation, hence this relation may enter everyday situations in a natural way.

Certainly $\mathcal{R}$ can be imbedded in a partial order relation $\mathcal{R}^*$ by defining $A_1 \mathcal{R}^* A_\alpha$ whenever $A_1 \mathcal{R} A_\alpha$ or a sequence $A_1, A_2, \ldots, A_\alpha$ exists, which satisfies the left-hand side of (3.1). However, it is not always advisable to replace $\mathcal{R}$ by $\mathcal{R}^*$, if one wishes to derive theorems concerning $\mathcal{R}$ itself.

It follows from (3.1) that a weak partial order is an asymmetric and an irreflexive relation.

Let $\mathcal{S}$ be a binary relation defined by:

(3.2) $$A_\upsilon \mathcal{S} A_\mu \text{ if and only if } \sim A_\upsilon \mathcal{R} A_\mu \text{ and } \sim A_\mu \mathcal{R} A_\upsilon ,$$

then $\mathcal{S}$ is a reflexive and symmetric relation (but not necessarily transitive). Certainly, the relation [$\mathcal{R}$ or $\mathcal{S}$] is complete.

Let $(x;\mathcal{B})$ be an i.r.p.c., (2.2) and (2.5), for a game Γ, and let C be a coalition. Then the expression

(3.3) $$e(C) \equiv v(C) - \sum_{i \in C} x_i$$

will be called the <u>excess of the coalition</u> C in $(x;\mathcal{B})$. Clearly, this excess, if it is positive, is the supremum of the amounts with which a player in C can "manoeuvre," if he claims an objection by forming the coalition C.

<u>Lemma 3.1</u> <u>Let</u> $(x;\mathcal{B})$ <u>be an i.r.p.c.</u>, (2.2) <u>and</u> (2.5), <u>for a game</u> Γ, <u>and let</u> k <u>and</u> ℓ <u>be two distinct players in a coalition</u> B_j <u>of</u> $\mathcal{B}$. <u>Suppose that player</u> k <u>has an objection</u> $(\{y_i\} ; C)$ <u>against player</u> ℓ, <u>and that this objection cannot be countered. Under these conditions, any coalition</u> D <u>for which</u>

(3.4) $$\ell \in D , \quad e(D) \geq e(C)$$

<u>must contain player</u> k.

<u>Proof:</u> Certainly, by (2.9) and (2.10), $e(C) > 0$ and therefore $e(D) > 0$. If $k \notin D$, then (2.11) is satisfied. Player ℓ can then counter-object by $(\{z_i\} ; D)$, where

(3.5) $$\begin{cases} x_i & \text{for } i \in D - C , \quad i \neq \ell , \\ y_i & \text{for } i \in D \cap C \\ v(D) - \sum_{i \in D - \{\ell\}} z_i & \text{for } i = \ell . \end{cases}$$

Indeed, it remains to show that (2.13) is satisfied for $i = \ell$. Actually,

$$z_\ell - x_\ell = v(D) - \sum_{i\in D-\{\ell\}} z_i - x_\ell = v(D) - \sum_{i\in D\cap C} y_i - \sum_{i\in D-C} x_i =$$

$$v(D) - v(C) + \sum_{i\in C-D} y_i - \sum_{i\in D-C} x_i \geq v(D) - v(C) + \sum_{i\in C-D} x_i - \sum_{i\in D-C} x_i = e(D) - e(C) \geq 0 .$$

This contradicts the assumption that the objection cannot be countered.

Theorem 3.1 Let $(x;\mathcal{B})$ be an i.r.p.c. (2.2) and (2.5) for a game Γ; then the relation $\succ$ in $(x;\mathcal{B})$ (see Definition 2.2) induces a weak partial order (see Definition 3.1) among the members of each coalition in $\mathcal{B}$.

Proof: Let B_j be a coalition in $\mathcal{B}$, and suppose that the relation $\succ$ is not a weak partial order among the players in a coalition B_j of $\mathcal{B}$. Without loss of generality we can assume that B_j contains the players $1, 2, \ldots, t$, and that in $(x;\mathcal{B})$,

(3.6) $$1 \succ 2\ ,\ 2 \succ 3\ ,\ \ldots,\ t-1 \succ t\ ,\quad t \succ 1\ .$$

We know, therefore, that an objection $(\{y_i^\upsilon\}; C^\upsilon)$, of player υ against player $(\upsilon+1) \pmod t$, exists, which cannot be countered, $\upsilon = 1, 2, \ldots, t$. Let C^{υ_o} be a coalition among the C^υ's, which has the maximum excess (see (3.3)). We shall show that C^{υ_o} contains all the players $1, 2, \ldots, t$, and this will furnish the contradiction, because, by (2.8), C^{υ_o} cannot contain player $(\upsilon_o+1) \pmod t$. We proceed by induction: By (2.8), $\upsilon_o \in C^{\upsilon_o}$. Suppose that a player υ belongs to the coalition C^{υ_o}; then, by Lemma 3.1, replacing k, ℓ, C, D by $(\upsilon-1) \pmod t$, υ, $C^{(\upsilon-1) \pmod t}$, C^{υ_o}, respectively, we find that player $(\upsilon-1) \pmod t$ also belongs to C^{υ_o}. This completes the proof.

Example 3.1 Let Γ be a 5-person game with the characteristic function $v(123) = 30$, $v(14) = 40$, $v(35) = 20$, $v(245) = 30$, $v(B) = 0$ otherwise; and consider the p.c. $(10, 10, 10, 0, 0; 123, 4, 5)$. In this p.c., $1 \succ 2$, because player 1 can object against player 2 by $((11, 29); 14)$ and this objection cannot be countered. Similarly, $2 \succ 3$, the objection being $((11, 1, 18); 245)$. On the other hand $1 \sim 3$. This example shows that <u>the relation $\succ$ is not necessarily transitive</u>.

<u>Example 3.2</u> Let Γ be a 5-person game with the characteristic function: $v(123) = 30$, $v(14) = 30$, $v(34) = 20$, $v(25) = 30$, $v(B) = 0$ otherwise. Clearly, $1 \sim 2$, $2 \sim 3$, but $1 \succ 3$ in the p.c. $(10, 10, 10, 0, 0 ; 123, 4, 5)$. This shows that the relation $\sim$ is not necessarily transitive.

4. <u>Making a Coalition Stable</u>.

<u>Definition 4.1</u> Let $(\mathfrak{x} ; \mathcal{B})$ be an i.r.p.c., (2.2) and (2.5), for a game Γ, and let B_j be a coalition in $\mathcal{B}$. We shall say that the coalition B_j is <u>stable</u> with respect to $(\mathfrak{x} ; \mathcal{B})$, if each player in B_j is equal to each other player in B_j.

Clearly, <u>an i.r.p.c.</u> $(\mathfrak{x};\mathcal{B})$ <u>is stable if and only if all the coalitions in</u> $\mathcal{B}$ <u>are stable</u>.

<u>Theorem 4.1</u> <u>Let</u> $(\mathfrak{x} ; \mathcal{B})$ <u>be an i.r.p.c.</u> (2.2) <u>and</u> (2.5), <u>for a game</u> Γ, <u>and let</u> B_j <u>be a fixed coalition in</u> $\mathcal{B}$. <u>It is possible to modify the payoffs to the players in</u> B_j, <u>without changing the other payoffs and the coalition-structure, in such a way that</u> B_j <u>will be stable with respect to the modified p.c.</u>

<u>Proof:</u> There is no loss of generality in assuming that the coalition B_j consists of the players $1, 2, \ldots, t$. We know that all the possible payoffs to the members of B_j constitute the simplex S_j defined by (2.7), (j being fixed). To each point $\mathfrak{x}^* = (x_1^*, x_2^*, \ldots, x_t^*)$ in S_j there corresponds an i.r.p.c. $(\hat{\mathfrak{x}};\mathcal{B})$, where

$$(4.1) \qquad \hat{x}_i = \begin{cases} x_i^* & \text{for all } i, \ i \in B_j \\ x_i & \text{for all } i, \ i \notin B_j \end{cases} .$$

Let $E_\upsilon \equiv E_\upsilon(\{x_i\}_{i \notin B_j}; \mathcal{B})$, $\upsilon = 1, 2, \ldots, t$, be the set of points $\mathcal{x}^*$, $\mathcal{x}^* \in S_j$, for which player υ is stronger than or equal to ($\gtrsim$) all the players i , $i \in B_j$, in the p.c. ($\hat{\mathcal{x}}$; $\mathcal{B}$) . The theorem will be proved if we show that

$$(4.2) \qquad M_j \equiv M_j(\{x_i\}_{i \notin B_j}; \mathcal{B}) \equiv \bigcap_{\upsilon=1}^{t} E_\upsilon \neq \emptyset .$$

In order to show this, note first that the face $x_\upsilon = 0$ of the simplex S_j is contained in E_υ , $\upsilon = 1, 2, \ldots, t$. Indeed, if $x_\upsilon = 0$ in ($\hat{\mathcal{x}}$; $\mathcal{B}$) , then, by (2.1), player υ can counter object to each objection raised against him (if such exists) by ({0}; υ) .

We shall now show that

$$(4.3) \qquad \bigcup_{\upsilon=1}^{t} E_\upsilon = S_j .$$

Indeed, suppose that there exists a point $\mathcal{x}^*$ in S_j which is not in this union, then there exist players $i_1, i_2, \ldots, i_t$ in B_j such that in ($\hat{\mathcal{x}}$; $\mathcal{B}$)

$$(4.4) \qquad 1 \prec i_1,\ 2 \prec i_2,\ \ldots,\ t \prec i_t .$$ [1]

This violates the non-intransitivity property of the relation $\succ$. (See Theorem 3.1.) Thus, (4.3) holds. Applying now the lemma of B. Knaster, C. Kuratowski, and S. Mazurkiewicz [4], usually used to prove in a direct way the Brouwer fixed-point theorem (see also Kuratowski [5]), (4.2) follows immediately. This completes the proof of the theorem.

Corollary 4.1 An important consequence of this lemma is that if the characteristic function is superadditive, then there always exists an imputation $\mathcal{x}$, such that ($\mathcal{x}$; N) is stable.

We conjecture that to each coalition-structure $\mathcal{B}$, there is a payoff vector $\mathcal{x}$ such that ($\mathcal{x}$; $\mathcal{B}$) is stable. It seems that in order

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[1] We define $\prec$ in the obvious way.

to prove this, one has to know more properties of M_j. We shall state some of the properties we have in mind in the next section, and verify them in some cases.

<u>Theorem 4.2</u> <u>The set</u> M_j, <u>defined by</u> (4.2), <u>is a union of a finite number of closed convex polyhedra</u>.

<u>Proof:</u> Let $F_{\mu\upsilon} \equiv F_{\mu\upsilon}(x_{t+1}, x_{t+2}, \ldots, x_n ; \;)$ be the set of points $\mathcal{X}^*, \mathcal{X}^* \epsilon\, S_j$, for which player μ is stronger than or equal to player υ, in the p.c. $(\hat{\mathcal{X}};\mathcal{B})$ (see (4.1)). $\mu, \upsilon \in B_j$. If we prove that $F_{\mu\upsilon}$ is a union of a finite number of closed convex polyhedra, then so also will M_j be, because

$$(4.5) \qquad M_j = \bigcap_{i=1}^{t} E_i = \bigcap_{i=1}^{t} \bigcap_{\upsilon=1}^{t} F_{i\upsilon} .$$

By a well-known theorem in logic it follows (see [1], Theorem 2.1) that $F_{\mu\upsilon}$ is a union of a finite number of convex polyhedra. We shall prove that it is closed by showing that its complement is open. Indeed, if $\mathcal{X}^*$ belongs to the complement of $F_{\mu\upsilon}$, with respect to S_j, then player υ has an objection $(\{y_i\}; C)$ against player μ, which cannot be countered. Without loss of generality, we can assume that $y_i > x_i$ for all i, $i \in C$. (See footnote to (2.10).) Let z_μ be the maximum amount that player μ can assure himself by paying each other member of a coalition D, $\mu \in D$, $\upsilon \notin D$, the amount x_i if this member is in $D - C$ and y_i if he is in $D \cap C$. $0 \leq z_\mu < x_\mu$, because the objection cannot be countered. Let $\delta = \mathrm{Min}\{x_\mu - z_\mu, y_i - x_i ; i \in C\}$, then $\delta > 0$. Any point of S_j which is in a δ/n - neighborhood of $\mathcal{X}^*$ also belongs to the complement of $F_{\mu\upsilon}$, because at such a point $(\{y_i\}; C)$ is still an objection which cannot be countered. This completes the proof.

<u>Corollary 4.2</u> The set $G_{\mu\upsilon} \equiv G_{\mu\upsilon}(x_{t+1}, \ldots, x_n;\mathcal{B})$ of points $\mathcal{X}^*$ in S_j, for which player μ is stronger than player υ in $(\hat{\mathcal{X}};\mathcal{B})$ is open in S_j, $\upsilon, \mu \in B_j$.

5. The Existence Problem.

We shall now generalize somewhat a theorem due to von Neumann [7]. We shall employ Kakutani's method of proof [3], but we shall make use of the S. Eilenberg and D. Montgomery sharper fixed-point theorem [2]:

Lemma 5.1 (von Neumann theorem for $m = 2$).

Let $S_1, S_2, \ldots, S_m$ be m bounded closed acyclic polyhedra[1,2] in the euclidean spaces $R^{n_1}, R^{n_2}, \ldots, R^{n_m}$, respectively. Let us consider their cartesian product $T \equiv S_1 \times S_2 \times \ldots \times S_m$ in $R^{n_1 + n_2 + \ldots + n_m}$, and let $T_i \equiv S_1 \times S_2 \times \ldots \times S_{i-1} \times S_{i+1} \times \ldots \times S_m$ be the respective cartesian product in $R^{n_1 + n_2 + \ldots + n_{i-1} + n_{i+1} + \ldots + n_m}$, $i = 1, 2, \ldots, m$. Let $U_1, U_2, \ldots, U_m$ be m closed subsets of T such that for each point $x^{(i)} \equiv \{x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\}$ in T_i, the set $V^{(i)}(x^{(i)})$ of all the points x_i, $x_i \in S_i$, such that $\{x_1, x_2, \ldots, x_m\} \in U_i$ is a non-empty closed acyclic polyhedron, $i = 1, 2, \ldots, m$. Under these assumptions, the sets $U_1, U_2, \ldots, U_m$ have a non-empty intersection.

Proof: We define a point-to-set mapping $x \to \phi(x)$, of T into itself, as follows:

$$(5.1) \quad \phi(x) \equiv \phi(x_1, x_2, \ldots, x_m) = V^{(1)}(x^{(1)}) \times V^{(2)}(x^{(2)}) \times \ldots \times V^{(m)}(x^{(m)}) .$$

This mapping is upper-semi-continuous because the sets $U_1, U_2, \ldots, U_m$ are closed. The image of each point is a cartesian product of acyclic closed polyhedra; hence it is an acylic polyhedron, and so is T itself.

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[1] I.e., connected polyhedra whose homology groups of order ≥ 1 vanish.

[2] This lemma can further be applied for absolute neighborhood retracts.

Therefore, by the Eilenberg and Montgomery fixed-point theorem [2], there exists a point $\xi \equiv \{\xi_1, \xi_2, \ldots, \xi_m\}$ in T, such that $\xi \in \emptyset(\xi)$. In other words, the components $\xi_1, \xi_2, \ldots, \xi_m$ satisfy $\xi_i \in S_i$, $\xi_i \in V^{(i)}(\xi^{(i)})$, $i = 1, 2, \ldots, m$; therefore, $\xi \in U_1 \cap U_2 \cap \ldots \cap U_m$. This completes the proof.

Theorem 5.1 Let $\mathcal{B} \equiv B_1, B_2, \ldots, B_m$ be a fixed coalition-structure (2.3) for a game Γ, and let $(x;\mathcal{B})$ be an arbitrary i.r.p.c., (2.2) and (2.5). Let $M_j \equiv M_j(\{x_i\}_{i \notin B_j};\mathcal{B})$ be the set of points x^*, $x^* \in S_j$, defined by (2.7), for which B_j is stable with respect to $(\hat{x};\mathcal{B})$, defined by (4.1). If, for each choice of j, $j = 1, 2, \ldots, m$, and for each choice of $(x;\mathcal{B})$, the set M_j is acyclic,[1] then there exists a stable p.c. $(\xi_1, \xi_2, \ldots, \xi_n;\mathcal{B})$ having $\mathcal{B}$ as a coalition-structure.

Proof: Let U_j, $j = 1, 2, \ldots, m$ be the set of points x in $S = S_1 \times S_2 \times \ldots \times S_m$ for which B_j is stable. Clearly, the sets $V^{(j)}$ defined in Lemma 5.1 are now the sets M_j, $j = 1, 2, \ldots, m$. Thus, the sets U_j, $j = 1, 2, \ldots, m$, have a non-empty intersection. This intersection is precisely the set of points x in S such that $(x;\mathcal{B})$ is stable.

In some cases we are able to show that M_j is indeed acyclic. The following lemma will be of much use.

Lemma 5.2 Let $\mathcal{B} \equiv B_1, B_2, \ldots, B_m$ be a fixed coalition-structure for a game Γ, and suppose that B_1 consists of the players $1, 2, \ldots, t$. Let $(x;\mathcal{B}) \equiv (x_1, x_2, \ldots, x_t, x_{t+1}, \ldots, x_n; B_1, B_2, \ldots, B_m)$ and $(\xi;\mathcal{B}) \equiv (\xi_1, \xi_2, \ldots, \xi_t, x_{t+1}, \ldots, x_n; B_1, B_2, \ldots, B_m)$ be two i.r.p.c.'s. Denote by P the set of players i, different from player 2, for which $\xi_i > x_i$. If

[1] By Theorem 4.2, we know that it is a closed polyhedron.

(5.2) $\xi_1 \leq x_1 , \ \xi_2 \geq x_2$,

(5.3) $x_1 - \xi_1 \geq \sum_{i \in P} (\xi_i - x_i)$,

(5.4) $1 \succ 2$ in the p.c. $(\mathfrak{x} ; \mathcal{B})$,

then

(5.5) $1 \succ 2$ also in the p.c. $(\xi ; \mathcal{B})$.

Proof: Since $\xi_1 + \xi_2 + \ldots + \xi_t = x_1 + x_2 + \ldots + x_t = v(B_1)$ (see (2.4)), (5.3) is equivalent to

(5.6) $\xi_2 - x_2 \geq \sum_{i \in Q} (x_i - \xi_i)$,

where Q is the set of players i , different from player 1 , for which $x_i > \xi_i$. Intuitively, (5.3) and (5.6) will make it "easier" for player 1 to object against player 2, and "more difficult" to counter object.

Let $(\{y_i\}; C)$ be an objection of player 1 against player 2 in $(\mathfrak{x} ; \mathcal{B})$, which cannot be countered. Thus, (2.8), (2.9) and (2.10) hold for $k = 1, \ell = 2$. We shall form an objection $(\{\eta_i\}; C)$ of player 1 against player 2 in $(\xi ; \mathcal{B})$ as follows:

(5.7) $$\eta_i = \begin{cases} \operatorname{Max} (y_i, \xi_i) & \text{for all } i , i \neq 1 , i \in C \\ v(C) - \sum_{i \in C - \{1\}} \eta_i & \text{for } i = 1 . \end{cases}$$

Clearly, (2.8) and (2.9) are satisfied for $k = 1, \ell = 2$, and so is (2.10) for $i \neq 1$. Checking the case $i = 1$, we find, by (5.7), that

$$\eta_1 = v(C) - \sum_{i \in C - \{1\}} \eta_i = v(C) - \sum_{i \in E} y_i - \sum_{i \in F} \xi_i ,$$

where E (F) is the set of players $i, i \neq 1, i \in C$, for which $y_i \geq \xi_i$ $(\xi_i > y_i)$. Certainly $F \subset P$, hence, by (2.9), (2.10) and (5.3),

$$\eta_1 - \xi_1 = y_1 + \sum_{i \in F} y_i - \sum_{i \in F} \xi_i - \xi_1 > x_1 - \xi_1 - \sum_{i \in F} (\xi_i - x_i) \geq 0 .$$

This objection cannot be countered. Indeed, if $(\{\zeta_i\}; D)$ is a counter objection, then (2.11) - (2.14) are satisfied for $k = 1$, $\ell = 2$, x_i, y_i, z_i being respectively replaced by ξ_i, η_i, ζ_i, $i \in D$. Consider the payoff $\{z_i\}$, $i \in D$, defined by

$$(5.8) \qquad z_i = \begin{cases} x_i & \text{for all } i, \ i \neq 2, \ i \in D - C \\ y_i & \text{for all } i, \ i \in D \cap C \\ v(D) - \sum_{i \in D - \{2\}} z_i . \end{cases}$$

We shall arrive at a contradiction by showing that $(\{z_i\}; D)$ is a counter objection to the objection $(\{y_i\}; C)$ in $(\mathcal{X}; \mathcal{B})$. Indeed, (2.11), (2.12) and (2.14) are satisfied and so is (2.13) for $i \neq 2$. Checking for $i = 2$, we find that, by (5.8), (2.11)-(2.14) applied to $(\{\zeta_i\}; D)$, and by (5.6),

$$z_2 - x_2 = v(D) - \sum_{i \in D-C} x_i - \sum_{i \in D \cap C} y_i = \sum_{i \in D} \zeta_i - \sum_{i \in D-C} x_i - \sum_{i \in D \cap C} y_i \geq$$

$$\geq \sum_{i \in D-C} \xi_i + \sum_{i \in D \cap C} \eta_i - \sum_{i \in D-C} x_i - \sum_{i \in D \cap C} y_i = \sum_{i \in D-C} (\xi_i - x_i) + \sum_{i \in D \cap C \cap F} (\xi_i - y_i) \geq$$

$$\geq \sum_{i \in D-C} (\xi_i - x_i) \geq (\xi_2 - x_2) - \sum_{i \in Q} (x_i - \xi_i) \geq 0 .$$

This completes the proof.

<u>Corollary 5.1</u> Let $(\mathcal{X}; \mathcal{B})$ be an arbitrary i.r.p.c. for a game Γ, and let B_j be a coalition in $\mathcal{B}$ which contains 2 players. Then, the set $M_j = M_j(\{x_i\}_{i \in B_j}; \mathcal{B})$ of the points $\mathcal{X}^*$, $\mathcal{X}^* \in S_j$, which make B_j stable in $(\hat{\mathcal{X}}; \mathcal{B})$, defined by (4.1), is a closed interval.[1]

<u>Proof:</u> We may assume that B_j consists of the players 1, 2. Let $\mathcal{ab}$ be the simplex S_j, where $x_1 = 0$ at $\mathcal{a}$ and $x_2 = 0$ at $\mathcal{b}$. If $\mathcal{c}^* \equiv (c_1^*, c_2^*)$ is a point in $\mathcal{ab}$ having the property that $1 \succ 2$ in $(\hat{\mathcal{c}}; \mathcal{B})$, then, by Lemma 5.2, all the points $\mathcal{X}^* \equiv (x_1^*, x_2^*)$ of the closed interval $\mathcal{ac}$ have

[1] Possibly a point. See (2.7), (4.2), and Definition 4.1.

the same property. Thus, the set of points x^* with this property is either empty or consists of an interval with a being one of its end points. By Corollary 4.2, this interval is open with respect to ab. Similarly, the set of points x^*, having the property that $2 \succ 1$ in $(\hat{x};\mathcal{B})$ is either empty or consists of an open interval with b as an end point. (See Figure 1.) Since the relation $\succ$ is asymmetric, this implies that M_j is a non-empty closed interval; and therefore it is acyclic.

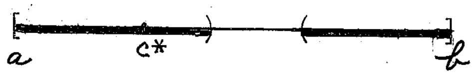

Figure 1.

If B_j contains more than 2 players, M_j is not necessarily a convex set.

Example 5.1 Let Γ be a 5-person game with the characteristic function $v(123) = 10$, $v(15) = 100$, $v(24) = 100$, $v(34) = 98$, $v(B) = 0$ otherwise. It is easy to verify that the coalition 123 is stable both in (10, 0, 0, 0, 0; 123, 4, 5) and in (0, 6, 4, 0, 0; 123, 4, 5), but not in (5, 3, 2, 0, 0; 123, 4, 5), where $2 \succ 3$. Thus, (10, 0, 0) and (0, 6, 4) belong to $M \equiv M(0, 0; 123, 4, 5)$ but (5, 3, 2) does not. (See Figure 2, where the points of M are marked.)

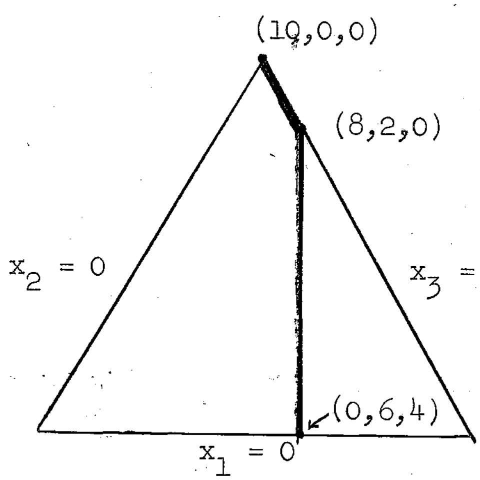

Figure 2.

Corollary 5.2 Let $(x;\mathcal{B})$ be an arbitrary i.r.p.c. for a game Γ, and let B_j be a coalition in $\mathcal{B}$ which contains 3 players. Then the set $M_j \equiv M_j(\{x_i\}_{i \notin B_j};\mathcal{B})$ of the points x^*, $x^* \in S_j$, which make B_j stable in $(\hat{x};\mathcal{B})$, defined by (4.1), is an acyclic closed polygon.

Proof: We know by Theorem 4.2 that M_j is a closed polygon.

(i) Let $B_j = (1, 2, 3)$. Let abc be the simplex S_j, where $x_1 = 0$ on the face bc, $x_2 = 0$ on ac and $x_3 = 0$ on ab. If $d^* = (d_1^*, d_2^*, d_3^*)$ is a point in S_j having the property that $1 \succ 2$ in $(\hat{d};\mathcal{B})$, draw parallels through d^* to the faces ac and bc. By Lemma 5.2, all the points $x^* = (x_1^*, x_2^*, x_3^*)$ in the shaded region[1] of Figure 3 have the same property. (Actually, by Corollary 4.2, there exists a neighborhood of this region whose points have the same property.)

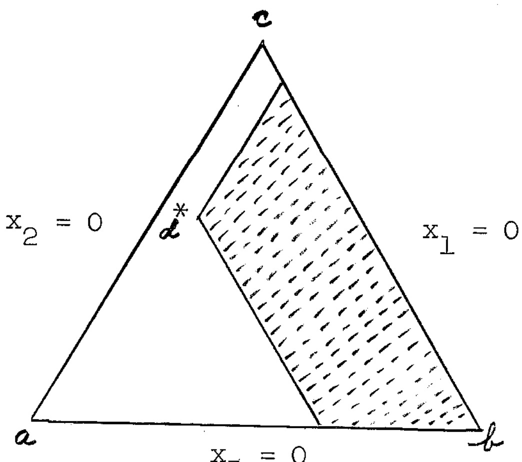

Figure 3.

(ii) We shall first show that M_j is always a connected set. Indeed, if this is not the case, let e^* and f^* be the two nearest points in two nearest distinct components of M_j. By definition, $1 \sim 2$, $1 \sim 3$, $2 \sim 3$ hold both in $(\hat{e};\mathcal{B})$ and in $(\hat{f};\mathcal{B})$.

[1] Characterized by $x_1^* \leq d_1^*$, $x_2^* \geq d_2^*$.

Case A. Suppose that e^* and f^* lie on a line parallel to a 1-face, say bc, and let x^* be any point of the segment e^*, f^*. (Figure 4.) If $1 \succ 2$ in $(\hat{x};\mathcal{B})$, then in view of (i), $1 \succ 2$ also in $(\hat{f};\mathcal{B})$, contrary to our assumption. If $2 \succ 1$ in $(\hat{x};\mathcal{B})$ then $2 \succ 1$ also in $(\hat{e};\mathcal{B})$, contrary to our assumption. In a similar fashion one proves that no strong relation holds between any other pair among the players 1, 2, and 3. Thus the segment e^*, f^* belongs to M_j, contrary to the assumption that e^* and f^* belong to distinct components of M_j.

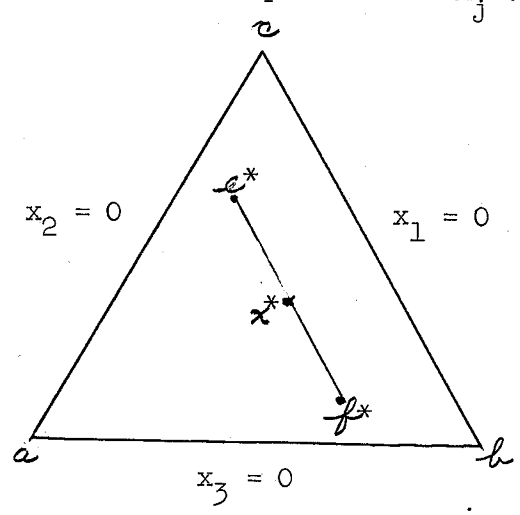

Figure 4.

Case B. Draw the straight line joining e^* and f^*, now assuming that it is not parallel to any of the faces of the triangle. There exists exactly one side, say ℓ, of the triangle, which forms both angles greater than 60° with this line. From each of the points e^* and f^* we draw lines parallel to the 2 sides other than ℓ, and consider the parallelogram formed by them. We may assume that the situation is as shown in Figure 5. Obviously, the parallelogram $e^* g^* f^* h^*$ belongs to S_j. By applying the results stated in (i), one observes immediately that $1 \sim 2$ and $1 \sim 3$ in $(\hat{x};\mathcal{B})$, if x^* lies in this closed parallelogram. Moreover, $2 \gtrsim 3$ in $(\hat{g};\mathcal{B})$ and $3 \gtrsim 2$ in $(\hat{h};\mathcal{B})$. Take any closed path which lies in the parallelogram and joins the points g^* and h^*. Then, in view of Corollary 4.2, and the fact that $\succ$ is an asymmetric relation, it follows that there exists a point x^* on the path with the property that $2 \sim 3$ in $(\hat{x};\mathcal{B})$. Therefore, $x^* \in M_j$. Obviously, x^* is closer to e^* than f^*, and this contradicts our assumption. Therefore, M_j is connected.

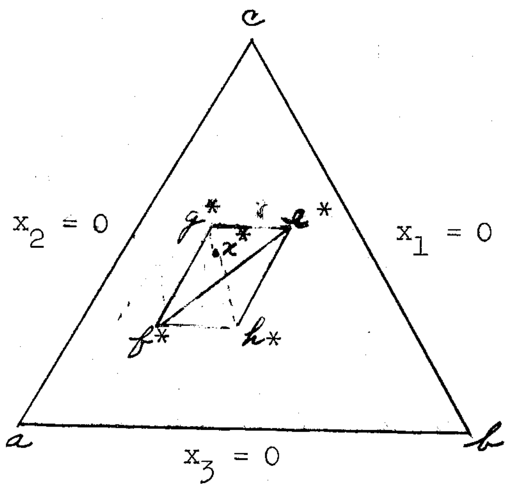

Figure 5.

(iii) We shall now show that any 1-cycle in M_j bounds.[1] Indeed, if α is a 1-cycle of M_j which does not bound, then there exists a point d^* in S_j, which is surrounded by the carrier α^* of α, and $d^* \notin M_j$. If, say, $1 \succ 2$ in $(\hat{d};\mathcal{B})$, then, by the result stated in (i), there is a region of points x^* having the property that $1 \succ 2$ in $(\hat{x};\mathcal{B})$. This region connects d^* to the face $x_1 = 0$ and hence it intersects α^*. This is impossible, since $\alpha^* \in M_j$, and we have arrived at a contradiction. This completes the proof of the Corollary.

From Theorem 5.1, Corollaries 5.1 and 5.2, we deduce:

<u>Theorem 5.2</u> <u>Let</u> $\mathcal{B} \equiv B_1, B_2, \ldots, B_m$ <u>be a coalition structure (2.3) for a game</u> Γ, <u>such that each</u> B_j, $j = 1, 2, \ldots, m$, <u>does not contain more than 3 players. Then, there exists a payoff</u> $x = x_1, x_2, \ldots, x_n$ <u>such that</u> $(x;\mathcal{B})$ <u>is</u> $\mathcal{M}_1^{(i)}$<u>-stable</u>.

[1]Assuming that M_j is now triangulated.

6. Miscellaneous.

Let $\mathcal{B}$ be a fixed coalition-structure for a game Γ, and let $(x;\mathcal{B})$ be an i.r.p.c. We shall show that any intersection of the form $H \equiv \bigcap_{s=1}^{r} F_{\mu\upsilon_s}$, μ fixed, $\mu\upsilon_s \in B_j$, $B_j \in \mathcal{B}$, $r \geq 1$, (see Theorem 4.2), is acyclic. In particular, E_μ is acyclic. Indeed, if a point x^* belongs to H, then, by decreasing x_μ and increasing the other components of x^* in any arbitrary way, we always get points of H, because, by Lemma 5.2, μ will remain stronger than or equal to each υ_s, $s = 1,2, \ldots, r$. Moreover, the face $x_\mu = 0$ obviously belongs to H. Hence H is contractible over itself to a point, and therefore it is acyclic. By similar considerations, one can prove that the set I of points x^* having the property that a player μ is weaker than or equal to the players $\upsilon_1, \upsilon_2, \ldots, \upsilon_r$ is acylic.

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