MARKETS WITH A CONTINUUM OF TRADERS I

Robert J. Aumann

Econometric Research Program
Research Memorandum No. 39
5 July 1962

Princeton University
Econometric Research Program
92-A Nassau Street
Princeton, N. J.

1. Introduction

The market model that we consider here consists of a set of traders, each of whom starts out with an initial commodity bundle to be used for trading, and each of whom has a well-defined preference order on the set of all commodity bundles. A trade (or allocation) is a redistribution of the commodities in the initial bundles among the traders. The model is well-known in economic theory; historical references are given in section 5.

A competitive equilibrium is a state of the market arrived at via "the law of supply and demand": it consists of a price structure p (one price for each commodity) at which the total supply of each good exactly balances the total demand, and the allocation X that results from trading at these prices. More precisely, X is an allocation with the property that at the price structure p, no player can, with the value of his initial bundle, buy a bundle that he prefers to his part of X. If (p, X) is a competitive equilibrium, then X is called an equilibrium allocation.

An allocation X is said to be in the core of the market if no coalition of traders can force an outcome that is better for them than X. More precisely, X is in the core if there is no group of traders that by its own efforts alone — without help from traders not in the group — can assure each of its members of a final commodity bundle preferred to that obtained under X. What we mean by "its own efforts" is that the desired result can be obtained if the traders in the group swap the commodities in their initial bundles among themselves only, as if the other traders were not present.

We can interpret the connection between the concepts of competitive equilibrium and core as follows: If the traders allow market pressures to determine prices and then trade in accordance with these prices, then the outcome is liable to be a competitive equilibrium. If, on the other hand, they ignore the price mechanism and trade with each other directly, then the outcome is liable to be in the core. Intuitively, it is felt that money and prices are no more than a device to simplify trading; accordingly, the concepts of competitive equilibrium and core should lead to the same allocations. In one direction this is indeed correct: each equilibrium allocation is in the core. But the converse is false; a core member need not be an equilibrium allocation, as is shown by simple examples.[1]

It has long been felt[2] that in large markets the price mechanism is more efficient — i.e., prices reflect market pressures more accurately — than in small markets. This could be interpreted to mean that as the number of traders tends to infinity, the core approaches the set of equilibrium allocations. The trouble with this is that it is difficult to translate it into a meaningful, precise mathematical statement. First, as we add traders, we must necessarily add goods and specify additional preference orders, and it is not clear how this should be done to preserve the desired property. For example, we would probably want to demand that each trader play a "small" role in comparison with the whole market. Otherwise there might be a few "giants" who dominate the market, and a large number of "small fry" whose total influence is small compared with that of the giants; such a market would have properties much the same as that containing the giants only, no matter how many small fry there are. Second, the core and the set of equilibrium allocations are subsets of a Euclidean space of dimension mn, where m is the number of traders and n the number of commodities. Thus as m changes, so does the space, and in such a context it is not even clear what "approach" <u>means</u>.

---

[1] With as few as two commodities and two traders.

[2] cf. section 5.

Scarf (1961) recently gave a brilliant solution to the problems outlined in the previous paragraph. Shortly thereafter Debreu (1961) obtained a greatly simplified and very elegant version of Scarf's result. Here we describe Debreu's treatment. Instead of working with m traders and letting m approach ∞, there are denumerably infinitely many traders to start with. This raises the problem of assuring that each individual trader's role in the market is negligible as compared to the whole market; at first sight, this seems impossible with a denumerable number of traders. The problem is solved by dividing the traders into a fixed finite number of types; within each of the types, there are denumerably infinitely many traders, each of whom starts out with the same initial bundle and has the same preferences. It follows that the total sum of goods in the market necessarily diverges. The problem of defining "allocation" in such a context is solved by a device too technical to discuss here. The chief result is that in this infinite market, the core is the same as the set of equilibrium allocations.

In this paper we obtain a more general result in what we consider to be a more natural setting. Our model contains a <u>continuum</u> of traders, <u>each one of whom may have different initial bundles and different preferences</u>. The negligibility of each individual trader's role is automatic. The total amount of goods is finite, because it is obtained by integrating the initial bundles over the set of traders rather than by summing them; so there is no problem in defining allocations either. Our main result is the same as that in the Scarf-Debreu model: the core equals the set of equilibrium allocations.

Denumerably infinite or continuous models are nothing new in economics or game theory, but it is usually parameters such as price or strategy that are allowed to take infinitely many values. Models with infinitely many players (traders in this instance) are a relative novelty, and it is still possible to count the references on one's fingers. The only published paper in the field is that of Kalisch and Nering (1959) on the denumerable case. Other references (unpublished) for the denumerable case are the works of Scarf (1961) and Debreu (1961) mentioned above, and a paper by Shapley (1962) on the Shapley values of such games. For the continuous case there is a paper by Milnor and Shapley (1961), and one by Shapley alone (1961), both on the Shapley values; also, there is a paper by Davis (1962) on von Neumann-Morgenstern solutions to such games. That is all we know of.

The idea of games with infinitely many players, and particularly with a continuum of players, may seem outlandish to the reader. Actually, a continuum of traders is no stranger than a continuum of prices or of strategies, or a continuum of "particles" in fluid mechanics. In all these cases the continuum can be considered an approximation to the "true" situation, in which there is a large but finite number of particles (or traders or strategies or possible prices). The purpose of adopting the continuous approximation is to make available the powerful and elegant methods of the branch of mathematics called "analysis," in a situation where treatment by finite methods would be much more difficult or even hopeless (think of trying to do fluid mechanics by solving n-body problems for large n).

The chief advantage of the continuous model for markets is that it enables us to express the condition that the impact of each individual trader on the market be negligible, without resorting to the artificial condition that each trader occur in infinitely many "copies." Also, it enables us to prove the desired theorem under conditions on the preferences of the traders that are much more general than those assumed by Scarf and Debreu; but this is of secondary importance. The proof of our main theorem follows the ideas of Debreu's proof; the chief complications and difficulties disappear, and the proof becomes even simpler and more direct than Debreu's. Indeed, over and above the specific result obtained here, what we would like to stress is the power and simplicity of the continuum-of-players method in describing mass phenomena in economics and game theory. The present work should be considered primarily as an illustration of this method as applied to an area where no other treatment seemed completely satisfactory; and we hope that it may stimulate more extensive development and use of models with a continuum of players.

The mathematical model is presented in the following section, and the proof of the identity between the core and the set of equilibrium allocations in sections 3 and 4. This concludes the essential part of the paper. Section 5 is devoted to a review of the related literature and a detailed comparison with the Scarf-Debreu model. Section 6 contains miscellaneous explanatory notes, and section 7 is an appendix which discusses in detail a point raised in section 6.

2. The Mathematical Model and the Main Theorem.

We will be working in a Euclidean space R^n ; the dimensionality n of the space represents the number of different commodities being traded in the market. Equations and inequalities between members of R^n are to be taken to hold coordinate-wise; for example $x \geq y$ means each coordinate of x is $\geq$ the corresponding coordinate of y. Similarly, the integral of a vector function is to be taken as the vector of integrals of the components. Superscripts will be used exclusively to denote coordinates. The scalar product $\Sigma_{i=1}^{n} x^i y^i$ of two members x and y of R^n is denoted $x \cdot y$.

A commodity bundle x is a point in the non-negative orthant Ω of R^n. The set of traders is the closed unit interval $[0, 1]$; it will be denoted T. An assignment (of commodity bundles to traders) is a function

X from T to Ω, each coordinate of which is assumed to be Lebesgue integrable over T. There is a fixed initial assignment I. Intuitively, $I(t)$ is the bundle with which trader t starts out; we assume

(2.1) $$I(t) > 0 \text{ for all } t .$$

An allocation (or "final assignment") is an assignment X such that

$$\int_T X(t)\,dt = \int_T I(t)\,dt .$$

For each trader t there is defined a relation $\succ_t$ on Ω, which is called the preference relation of t and is assumed to satisfy the following three conditions:

(2.2) Continuity (in the commodities): For each $y \in \Omega$, the sets $\{x : x \succ_t y\}$ and $\{x : y \succ_t x\}$ are open (relative to Ω).

(2.3) Measurability: For all assignments X and Y, the set $\{t : X(t) \succ_t Y(t)\}$ is Lebesgue measurable.

(2.4) Desirability (of the commodities): $x > y$ implies $x \succ_t y$.

Note specifically that $\succ_t$ is not assumed to be complete, nor even transitive.

A coalition of traders is a Lebesgue measurable subset of T. An allocation Y dominates an allocation X via a coalition S if S is of positive Lebesgue measure, $Y(t) \succ_t X(t)$ for each $t \in S$, and S is effective for Y, i.e.,

$$\int_S Y(t)\,dt = \int_S I(t)\,dt .$$

The core is the set of all allocations that are not dominated via any coalition.

A price vector p is an n-tuple of non-negative real numbers, not all of which vanish. A competitive equilibrium is a pair consisting of a price vector p and an allocation X, such that for almost every trader t, $X(t)$ is maximal with respect to $\succ_t$ in $\{x : p \cdot x \leq p \cdot I(t)\}$. An equilibrium allocation is an allocation X for which there exists a price vector p such that (p, X) is a competitive equilibrium.

Main Theorem. The core coincides with the set of equilibrium allocations.

3. Proof of the Main Theorem: First Half.

We wish to show that every equilibrium allocation is in the core. The proof follows precisely the corresponding proof when there are only finitely many traders.[3]

Let (p, X) be a competitive equilibrium. Suppose, contrary to the theorem, that X is dominated via a coalition S by an allocation Y. Then by the definition of competitive equilibrium, we have $p \cdot Y(t) > p \cdot I(t)$ for almost all $t \in S$. Hence

$$p \cdot \int_S Y(t)\,dt = \int_S p \cdot Y(t)\,dt > \int_S p \cdot I(t)\,dt = p \cdot \int_S I(t)\,dt ,$$

and this contradicts

$$\int_S Y(t)\,dt = \int_S I(t)\,dt .$$

This completes the proof.

Remark 3.1 The proof made no use of assumptions (2.1), (2.2), (2.3), or (2.4).

Remark 3.2 We can define a notion analogous to that of competitive equilibrium, but without insisting that prices be non-negative. Then this half of the theorem still goes through. The proof is unchanged.

---

[3] See Scarf (1961), section 1.

4. Proof of the Main Theorem: Second Half.

Let X be an allocation in the core. Define

$$\Phi(t) = \{x : x \succ_t X(t)\}$$
$$\psi(t) = \Phi(t) - I(t) = \{x - I(t) : x \in \Phi(t)\} .$$

Because of continuity (2.2), $\Phi(t)$ and $\psi(t)$ are open for each t. For each set U of traders, let $\Delta(U)$ denote the convex hull of the union $\cup_{t \in U} \psi(t)$. Define a set U of traders to be full if it contains almost all of T. Let 0 denote the origin.

Lemma 4.1 There is a full set U of traders, such that $0 \notin \Delta(U)$.

Proof: For each rational z in R^n (i.e. z with rational coordinates), define $R(z)$ to be the set of all traders t who prefer $z + I(t)$ to $X(t)$, i.e. $R(z) = \{t : z \in \psi(t)\}$. Because of measurability (2.3), each $R(z)$ is measurable. Let N be the set of all those rational z such that $\mu(R(z)) = 0$, where μ is Lebesgue measure. Obviously N is denumerable. Define $U = T - \cup_{z \in N} R(z)$. Then U is full.

Because the $\psi(t)$ are open relative to Ω, $\Delta(U)$ is the convex hull of the rational points in $\cup_{t \in U} \psi(t)$. Therefore, if $0 \in \psi(t)$, we may apply Caratheodory's theorem (cf., e.g., Eggleston (1958), p. 35), and obtain $n + 1$ traders $t_0, \ldots, t_n \in U$ (not necessarily distinct), rational points $z_i \in \psi(t_i)$, and non-negative numbers $q^0, \ldots, q^n$ summing to 1, such that $\Sigma_{i=0}^n q^i z_i = 0$. Then $t_i \in R(z_i)$, and since $t_i \in U$, it follows that $z_i \notin N$. Therefore $R(z_i)$ is of positive measure for each i. Therefore for a sufficiently small positive number δ, we can find disjoint sets $S_i \subset R(z_i)$ such that $\mu(S_i) = \delta q^i$. Define a coalition S by $S = \cup_{i=0}^n S_i$ and an assignment Y by

$$Y(t) = \begin{cases} z_i + I(t) & \text{for } t \in S_i \\ I(t) & \text{for } t \notin S . \end{cases}$$

That $Y(t) \in \Omega$ is trivial for $t \notin S$, and for $t \in S_i$ it follows from $z_i \in \psi(t)$ (which in turn follows from $S_i \subset R(z_i)$). Next,

$$\int_S Y(t)\,dt = \Sigma_{i=0}^{n} \delta q^i z_i + \int_S I(t)\,dt$$
$$= 0 + \int_S I(t)\,dt ,$$

and therefore S is effective for Y; since $Y(t) = I(t)$ for $t \notin S$, it follows that Y is an allocation. Finally, from $S_i \subset R(z_i)$ it follows that $z_i + I(t) \succ_t X(t)$ for $t \in S_i$; in other words, $Y(t) \succ_t X(t)$ for $t \in S$. Since S is of positive measure, we have shown that X is not in the core, contrary to assumption. This proves the lemma.

<u>Remark 4.2</u> The proof of Lemma 4.1 made no use of (2.1) or of (2.4).

Let U be as in the lemma. To avoid annoying repetitions, let us agree that in the remainder of the proof, statements about traders will refer to $t \in U$. This is sufficient because U is full.

From the lemma we obtain a hyperplane $p \cdot x = 0$ that supports $\Delta(U)$. Therefore it also supports each of the $\psi(t)$; thus $p \cdot x \geq 0$ for $x \in \psi(t)$, or $p \cdot x \geq p \cdot I(t)$ for $x \in \Phi(t)$. We claim

(4.3) $$p \cdot x > p \cdot I(t) \quad \text{for } x \in \Phi(t) .$$

Indeed, because of desirability (2.4), each $\psi(t)$ contains a translate of the positive orthant, and therefore

$$p \geq 0 .$$

Then because $I(t) > 0$ (by (2.1)) and $p \neq 0$, it follows that $p \cdot x \geq p \cdot I(t) > 0$. There is therefore a coordinate j such that both $p^j > 0$ and $x^j > 0$; without loss of generality $j = 1$. Because of the relative openness of $\Phi(t)$, it then follows that $y_\epsilon = x - (\epsilon, 0, \ldots, 0) \in \Phi(t)$; therefore $p \cdot I(t) \leq p \cdot y_\epsilon = p \cdot x - \epsilon p^1$, and it follows that $p \cdot x > p \cdot I(t)$.

This proves (4.3).

What (4.3) says is that $x \succ_t X(t)$ implies $p \cdot x > p \cdot I(t)$; in other words, no element of the "budget set" $\{x \in \Omega : p \cdot x \leq p \cdot I(t)\}$ is preferred to $X(t)$. To demonstrate the maximality of $X(t)$ in the budget set, it therefore remains only to show that $X(t)$ belongs to the budget set; we do this for almost all t . Indeed, because of desirability (2.4), $X(t)$ is in the closure of $\Phi(t)$, and therefore (e.g. by (4.3)) $p \cdot X(t) \geq p \cdot I(t)$; if for a t-set of positive measure we would have $p \cdot X(t) > p \cdot I(t)$, then

$$\int_T p \cdot X(t) > \int_T p \cdot I(t)\, dt ,$$

contrary to the fact that X is in the core and hence is an allocation. Therefore $X(t)$ is maximal in t's budget set for almost all t , i.e., (p, X) is a competitive equilibrium. This completes the proof of the main theorem.

5. Relations with the Literature.

The market model that we have here extended to a continuum of traders is standard in economic literature; we will not give anything like a complete survey, contenting ourselves with a few highlights. Walras (1881) described a version of the model, defined the notion of competitive equilibrium, and gave an argument for its existence. Wald (1936, 1937) gave a rigorous proof for the existence of competitive equilibria under certain somewhat restrictive conditions. Arrow and Debreu (1954) established the existence of competitive equilibria under rather more general conditions than Wald's; theirs is an integrated production-consumption model, but can easily be specialized to the "pure exchange case," which is what concerns us here.

The notion of core comes from game theory, where it is fundamental. It was used repeatedly by von Neumann and Morgenstern (1944) in their investigation of games with side payments; but they preferred that it remain anonymous, because they wanted to stress solutions (and also because the core is empty for zero-sum games,[4] in which they were most interested). The name is due to Gillies (1953, 1959), who also investigated the properties of the core. An extension of the core notion to games without side payments — of which the market game under consideration is an example — was made by Aumann and Peleg (1960); Aumann (1961) investigated the properties of this extension. Returning to economics, Edgeworth (1881) discussed what corresponds to the core for certain special markets. The notion of Pareto optimality is related to that of core, but does not usually coincide with it: Whereas the core consists of all those outcomes of a game with the property that no coalition of players can do better through its own efforts alone, Pareto optimality demands this only for the all-player coalition and for the individual player. Pareto optimality is of particular interest in connection with integrated production-consumption models (rather than with pure exchange models); but we do not wish to get sidetracked into this area (cf. Debreu (1954), also Karlin (1959), pp. 293-301).

All those writers who discussed competitive equilibrium stressed that the economic validity of their considerations is based on the assumption of "perfect competition." This means that there is no price discrimination, and that "each of the participants is of the opinion that his own transactions do not influence the prevailing prices" (Wald, 1936). The phrase in quotes is, of course, an idealization; it is intuitively valid only when the market has "many" traders and each individual trader is negligible. Mathematically, the individual traders cannot be negligible as long as there are only finitely many of them. This is what led to the search for models that

---

[4]Market games are not zero-sum, because trading generally is beneficial to <u>all</u> traders.

rigorize the vague notion of "large" markets. Two approaches are possible: one can seek limit theorems for n-person markets as n tends to infinity; and one can consider markets that, to start with, have infinitely many traders.

The first approach has been "in the air" ever since the beginning of game theory — not only for markets, but for economic models in general. Von Neumann and Morgenstern (1944) wrote that "when the number of participants becomes really great, some hope emerges that the influence of every particular participant will become negligible." But they stressed the need for a rigorous formulation of the model for finitely many participants, "before anything can be proved about the ... limiting case of large numbers, such as free competition" (pp. 13-14 of the third edition). Shubik (1959) was the first actually to carry out a mathematical investigation of the asymptotic properties of the core of n-trader markets as $n \longrightarrow \infty$. Following Edgeworth (1881), he restricted his considerations to markets with two commodities and two types of traders, which we may arbitrarily designate "buyers" and "sellers"; unlike Edgeworth, he assumed side payments. Two limit theorems were obtained: one in which there is only one seller and the number of buyers tends to ∞ (monopoly), the other in which the number of buyers and the number of sellers tend simultaneously to ∞ but maintain a constant ratio. In both cases he showed (under appropriate conditions) that in a certain sense the core "shrinks to a single point" as the number of traders tends to ∞. The precise statement of his theorems involves the difficulties mentioned in the introduction in connection with the precise statement of any asymptotic theorem in this area; Shubik overcame these difficulties by exploiting the special nature of his investigations. Very recently, Debreu and Scarf succeeded in obtaining an asymptotic theorem in a fairly general setting, by an ingenious method which enabled them to by-pass the above-mentioned difficulties.[5] They considered markets (without side payments) in which there are an arbitrary but fixed finite number of types of traders (rather than just two types, as in Shubik's analysis); within each type there are k traders, all identical copies of one another (as concerns both preference order and initial resources). Now, when $k \to \infty$, Debreu and Scarf showed that in a certain sense the core tends to a limiting set, which can be identified with the set of equilibrium allocations for $k = 1$.

The second approach to a mathematical rigorization of the notion of large markets — that involving an infinity of traders to start with — was first carried out in the papers of Scarf (1961) and Debreu (1961). In the introduction we briefly described their model and discussed the chief difference between it and the model proposed here. We would now like to compare the two models in somewhat greater detail, especially as far as preference assumptions are concerned. Scarf's and Debreu's assumptions are essentially the same; for definiteness, we follow Debreu's treatment.

Debreu assumes that each trader has a transitive and complete[6] preference-or-indifference order $\succsim$ (from which a preference order $\succ$ can be derived) which satisfies: a continuity assumption similar to our (2.2); a local non-saturation assumption which asserts that to every commodity bundle there is a bundle preferred to it, and in fact the preferred bundle can be chosen arbitrarily close to the given one[7]; a convexity assumption which asserts that the indifference levels are convex (more precisely, that

---

[5]Private correspondence with Professor Debreu.

[6]He does not state these properties explicitly, but they are implicit in his use of utility functions to represent the preferences.

[7]Debreu does not state this explicitly, but it is a consequence of his assumptions 2 and 3.

they are boundaries of convex sets) — this is commonly called "the law of diminishing returns" in economics; and finally, an assumption which says that a bundle from which any one of the commodities is missing is indifferent with the empty bundle. Debreu does not assume desirability of the goods in any form; as a result, the prices he obtains may be negative.

The model presented here does not assume transitivity, completeness, or convexity; nor does it assume Debreu's rather anti-intuitive fourth assumption concerning bundles from which one commodity is missing. It does assume local (and hence also global) non-saturation; this follows from desirability (2.4). We remark that if desirability is dropped in the current model and the less restrictive assumption of local non-saturation is substituted, our theorem remains true, except that the prices may be negative, as in Debreu's model; the proof needs only slight modifications. In addition to desirability, the only assumption that is made here but not by Debreu is measurability (2.3); this is of technical significance only, and it would be difficult to raise intuitive economic objections against it.[8]

Debreu's definition of domination differs slightly from ours. It is impossible to state his definition in our model, because it involves preference-or-indifference, which is absent from our model. If, however, we would add this concept to our model together with the appropriate assumptions, then a definition of domination analogous to Debreu's would demand that $Y(t) \succsim_t X(t)$ for each $t \in S$, and $Y(t) \succ_t X(t)$ for a set of t in S that is of positive measure; whereas ours demands that $Y(t) \succ_t X(t)$ for each $t \in S$. However, it is easily seen that the two definitions lead to the same core.

---

[8] Because of the extremely "pathological" nature of non-measurable sets, it is unlikely that sets occurring in the context of an economic model would be non-measurable. The matter will be discussed in more detail in a subsequent paper.

One other assumption, common to Debreu's model and ours, deserves mention. This is assumption (2.1), which asserts that each trader starts out with a positive quantity of each commodity. Although not anti-intuitive, it does constitute a considerable restriction. Wald (1936) did not assume it, and it excludes the cases considered by Edgeworth (1881) and Shubik (1959); in fact, any pure buyer-seller market is excluded. In a subsequent paper we will consider a model that does not make this assumption but still enables us to obtain our theorem.

Much of the above comparison between our model and Debreu's also applies, with appropriate modifications, to the newer Debreu-Scarf asymptotic model. In particular this is so for the chief advantage of our model, namely that it does not require that each trader appear in many identical copies.

In this paper we prove only the identity between the core and the set of equilibrium allocations, but do not show that either set is non-empty. In both of the Scarf-Debreu models (the infinite player model and the asymptotic model) the non-emptiness follows easily from the existence of competitive equilibria in markets with finitely many traders (but it should be noted that this requires much more stringent conditions than are needed for the theorem of this paper). In our model, however, the question is more delicate. It will be discussed further in a subsequent paper.

6. Notes.

To Section 1.

1) See Scarf (1961, section 3) for an excellent discussion of the difficulties involved in the asymptotic approach (i.e. the approach in which the number of traders is allowed to tend to ∞).

2) Shapley (1961) has applied a model with a continuum of players to an analysis — via Shapley values — of the power relationships in a

corporation with two large stockholders.

To Section 2.

1) Our model deals with commodities that are available in arbitrary quantities, not in discrete multiples of a given unit.

2) The main theorem can also be proved when R^n is substituted for Ω, i.e., when negative quantities of commodities are permitted. This even simplifies the proof, and makes assumption (2.1) unnecessary.

3) The main theorem remains valid if the space T of traders is any totally finite measure space without atoms, rather than the unit interval with Lebesgue measure. The condition that the space be without atoms is the mathematical translation of the intuitive requirement that each individual's role be negligible. The proof remains unchanged.

This remark throws into sharp focus a somewhat disturbing feature of our model: The measure on the set of traders appears arbitrary and devoid of economic significance; but it affects in an essential manner all the concepts subsequently defined. Thus an assignment that is an allocation for a particular choice of a measure on T need not and generally will not be an allocation for other choices of the measure; and therefore the core and the set of equilibrium allocations depend on an apparently arbitrary choice.

This difficulty can be overcome by a method described in the appendix (section 7).

4) Why do we demand that the "blocking coalitions" S appearing in the definition of core have positive measure? If not, then every allocation would be dominated, because a coalition of measure 0 is effective for every allocation (an integral over a set of measure 0 always vanishes).

Similarly, in the definition of competitive equilibrium, $X(t)$ need only be maximal for almost every trader. A set of traders of measure 0 has no influence on the market and can always be ignored.

7. Appendix: Invariance under Changes in the Underlying Measure.

In order to formulate our model and to state and prove our results we must work with an underlying measure on the space T of traders; but the choice of a particular measure for this purpose is arbitrary and devoid of economic significance. In some sense, therefore, the theory should remain essentially unaffected — "invariant" — if one underlying measure is chosen rather than another. The underlying measure here plays much the same role as the coordinate system in physics and analytic geometry — it is needed as a tool, but has no applied significance. In physics and geometry we have "invariance under coordinate changes," and the analogous property should hold here.

We start with an intuitive description of this approach, then give the formal model.

The basic objects of investigation in our theory are assignments. Analogous to a point in analytic geometry, we should think of an assignment as an "invariant object" whose representation depends on the choice of an underlying measure. The transformation formulas connecting the various representations should be chosen in such a way so that for a fixed assignment and a fixed set S of traders, the total amount of commodities held by S under the given assignment — given by the integral of the assignment over S — should be invariant under changes in the underlying measure. As we shall see, this leads to transformation formulas involving Radon-Nikodym derivatives (cf. Halmos (1950), pp. 132-135). Better still, it leads to a "measure-free" (analogous to "coordinate-free") representation for assignments: Rather than starting out with a commodity bundle for each trader and obtaining the total bundle of a coalition by integration, we <u>start</u> with the function that associates with each coalition S its total bundle. In other words, an assignment is simply an (n-dimensional) vector measure on T .

It is also possible to give a "measure-free" representation for the preference orders, but this leads to complexities, and does not seem worthwhile. Therefore, we will think of the set of preference orders of the traders as an "invariant object," but will work with different representations depending on the underlying measure chosen for T . Transformation formulas will be given; these again will depend on Radon-Nikodym derivatives.

We proceed to the formal model. To start with, T is given as a measurable space, i.e., a space on which there is defined a σ-field of subsets called measurable sets or coalitions (but no measure as yet). For example, we may think of T as being the unit interval together with its Borel subsets. On T there is imposed a totally finite vector measure ι of dimension n , which is assumed to be non-atomic in all its components; ι is called the initial assignment, and is fixed once and for all.

Though we are not yet ready to introduce "underlying measures" on T , we do define "measure zero." A coalition is said to be null or of measure zero if it is of measure zero w.r.t. each of the coordinates. The terms "almost all," etc., will be used accordingly; in particular, if we speak of a measure μ as being "absolutely continuous" without specifying w.r.t. what measure, then this will mean that $\mu(S) = 0$ whenever S is null. A measure μ will be called strictly positive if $\mu(S) > 0$ whenever S is non-null. Any two absolutely continuous strictly positive measures are absolutely continuous w.r.t. each other, and their mutual Radon-Nikodym derivatives are almost everywhere positive. A vector measure is called absolutely continuous, etc., if each component is absolutely continuous, etc.

An assignment ξ is an absolutely continuous strictly positive vector measure of dimension n (the number of commodities); it is called an allocation if $\xi(T) = \iota(T)$. An underlying measure μ is an absolutely continuous strictly positive scalar measure. Because of the strict positivity, every assignment ξ must be absolutely continuous w.r.t. every underlying measure μ . We define the representation of ξ w.r.t. μ to be the vector-valued function on T given by the Radon-Nikodym derivative $d\xi/d\mu$. The representations X_μ and X_υ of an assignment w.r.t. the underlying measures μ and υ respectively are connected by the transformation formula

$$X_\upsilon(t) = X_\mu(t) \left.\frac{d\mu}{d\upsilon}\right|_t .$$

This follows from the chain rule for Radon-Nikodym derivatives.

Corresponding to (2.1) we make the following assumption on the initial assignments:

(7.1) Each coordinate of ι is strictly positive.

Assumption (2.1) for each representation of ι follows from assumption (7.1) (but we must substitute "almost all t " for "all t ").

For each underlying measure μ and almost each trader t there is defined a relation $\succ_t^\mu$ on Ω , which is called the representation of the preference order of t w.r.t. μ , is assumed to satisfy (2.2), (2.3), (2.4) (with "measurable" instead of "Lebesgue measurable" in (2.3)), and the transformation formula

$$x \succ_t^\upsilon y \text{ if and only if } x \left.\frac{d\mu}{d\upsilon}\right|_t \succ_t^\mu y \left.\frac{d\mu}{d\upsilon}\right|_t .$$

In fact, since $\frac{d\mu}{d\upsilon}$ is positive (almost everywhere), the validity of (2.2), (2.3) and (2.4) for all μ follows from the transformation formula and their validity for a single μ .

The notions of core and competitive equilibrium are easily seen to be invariant. The main theorem can therefore be proved in one representation (i.e., w.r.t. a fixed underlying measure), and follows at once for all representations.

Because Radon-Nikodym derivatives are defined only up to a null set, all our representations — both of assignments and of preference orders — are defined only up to a null set of traders. This causes no difficulties because null sets of traders "don't matter" for the concepts (core and competitive equilibrium) defined here. Strictly speaking, each different representative of a Radon-Nikodym derivative yields a different representation. We always work with a fixed representation, or when transforming, with a fixed pair of representations. Whether or not a given assignment is in the core or is an equilibrium allocation does not depend on the choice of a representative of a Radon-Nikodym derivative.

REFERENCES

Arrow, K. J., and G. Debreu, "Existence of an equilibrium for a competitive economy," Econometrica 22 (1954), pp. 265-290.

Aumann, R. J., "The core of a cooperative game without side payments," Trans. Amer. Math. Soc. 98 (1961), pp. 539-552.

Aumann, R. J., and B. Peleg, "Von Neumann-Morgenstern solutions to cooperative games without side payments," Bull. Amer. Math. Soc. 66 (1960), pp. 173-179.

Davis, M., "Symmetric solutions to symmetric games with a continuum of players," see Maschler (1962), pp. 119-126.

Debreu, G., "Valuation equilibrium and Pareto optimum," Proc. Nat. Acad. Sci. 40 (1954), pp. 588-592.

———, "On a theorem of Scarf," Cowles Foundation Discussion Paper No. 130, Nov. 14, 1961 (multilithed).

Edgeworth, F. Y., Mathematical Psychics, London, 1881.

Eggleston, H. G., Convexity, Cambridge University Press, 1958.

Gillies, D. B., Some Theorems on n-Person Games, Ph.D. Thesis, Department of Mathematics, Princeton University, 1953.

———, "Solutions to general non-zero sum games," see Luce and Tucker (1959), pp. 47-85.

Halmos, P. R., Measure Theory, Van Nostrand, 1950.

Kalisch, G. K , and E. D. Nering, "Countably infinitely many person games," see Luce and Tucker (1959), pp. 43-45.

Karlin, S., Mathematical Methods and Theory in Games, Programming, and Economics, Vol. I, Addison-Wesley, 1959.

Luce, R. D., and A. W. Tucker, editors, Contributions to the Theory of Games IV, Annals of Mathematics Studies No. 40, Princeton University Press, 1959.

Maschler, M., editor, Recent Advances in Game Theory, Proceedings of the Princeton University Conference of October 1961, privately printed for members of the conference, May 1962.

Milnor, J. W., and L. S. Shapley, "Values of large games II: Oceanic games," The RAND Corporation, RM 2649, February 28, 1961 (multilithed).

von Neumann, J., and O. Morgenstern, Theory of Games and Economic Behavior, Princeton University Press, 1944 (Third edition, 1953).

Scarf, H., "An analysis of markets with a large number of participants," Institute for Mathematical Studies in the Social Sciences, Stanford University, 1961 (mimeographed); see also Maschler (1962), pp. 127-155.

Shapley, L. S., "Values of large games III: A corporation with two large stockholders," The RAND Corporation, RM 2650 PR, December 1961 (multilithed).

Shubik, M., "Edgeworth market games," see Luce and Tucker (1959), pp. 267-278.

Wald, A., "Über einige Gleichungssysteme der mathematischen Ökonomie," Zeitschrift fur Nationalökonomie 7 (1936), pp. 637-670 (translated as "On some systems of equations of mathematical economics," Econometrica 19 (1951), pp. 368-403).

——, in Ergebnisse eines mathematischen Kolloquiums No. 8 (Vienna: Deuticke, 1937).

Walras, L., Mathematische Theorie der Preisbestimmung der wirtschaftlichen Güter, Stuttgart, Ferdinand Enke, 1881.