

MARKETS WITH A CONTINUUM OF TRADERS II

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1. Introduction

In "Markets With a Continuum of Traders I"¹ (MCT I), we showed that in a continuous market model, the core coincides with the set of "equilibrium allocations" (i.e., the allocations which can be combined with some price structure to form a competitive equilibrium). The chief purpose of the current work is to establish alternative conditions for the truth of that theorem.

Reference to MCT I will be made by prefixing a "I"; thus "Section I.5" means "Section 5 of MCT I." The reader is referred to MCT I for an exposition of the basic concepts, explanatory material, and motivation (Section I.1) and for a historical review (Section I.5). In the formal development, we will repeat the definitions of MCT I, and quote the theorems that we need. An acquaintance with MCT I is, however, necessary to understand the motivation of this paper.

There are two conditions that we would like to replace in connection with the main theorem of MCT I: the assumption that each trader starts out with a positive amount of each commodity (I.2.1), and the desirability-of-commodities assumption (I.2.4). The restrictive nature of (I.2.1) was discussed in Section I.5. As for desirability (I.2.4), this seems a fairly acceptable assumption, but it does have one undesirable feature: it implies that the desire of the trader is never saturated, that no matter how large his commodity bundle is, he will always want more. It might be argued that this is a justifiable idealization, because in real life, market trading does not usually result in bundles that are sufficiently large to saturate desire; and indeed, those authors who have assumed non-saturation have probably done so on this basis. But in a way, this is putting the cart before

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the horse; such an idealization is not convincing unless it is shown that the core² does not in fact contain allocations with "very large" bundles. Without further assumptions this cannot be shown: an example will be given of a market obeying (I.2.1) through (I.2.4), whose core contains allocations with arbitrarily large bundles (see Section 6).

Actually, what is needed for the proof of the theorem of MCT I is not the non-saturation of all bundles, but only of bundles in allocations of the core. What we will do, therefore, is to choose a sufficiently large but bounded area, within this area assume a strengthened form of desirability, but outside the area assume nothing; in particular, outside the area saturation may obtain. The strengthened form of the desirability assumption will be sufficient to assure that core bundles are within the bounded area, so that non-saturation will hold for them, which is exactly what is needed. An additional consequence of the strengthened form is that the assumption of positive initial resources (I.2.1) may be dispensed with. The strengthened form combines desirability (I.2.4) with a "comparability" assumption which asserts, roughly speaking, that there is a positive minimum "rate of exchange" between any two commodities. A consequence of this assumption is that equilibrium prices are bounded away from zero, but this will not be used here.

The formal model for this paper is presented in Section 2; it differs from that of Section I.2 in that two assumptions (I.2.1 and I.2.4) do not appear here. The alternative form of desirability that we use here is introduced in Section 3, and the main theorem of this paper is stated. In Section 4 we recall two lemmas from MCT I. Section 5 contains the proof of the theorem stated in Section 3. In Section 6 we bring the counter-

²which always contains the set of equilibrium allocations —
cf. Section I.3.

example referred to above.

A secondary purpose of this paper is to discuss some points concerning the assumptions of continuity (I.2.2) and measurability (I.2.3) of the preference orders. This will be done in Section 7.

2. The Mathematical Model

The set of commodity bundles is the non-negative orthant Ω of Euclidean n -space R^n . The word "vector" will refer to members of R^n . Unless otherwise specified, relations between, and operations on vectors are to be understood term by term; thus for $x, y \in \Omega$, $x > y$ means $x^i > y^i$ for each $i = 1, \dots, n$, and if X is a vector valued function, then $\int X = \{ \int X^1, \dots, \int X^n \}$.

The set of all traders is denoted by T , and is taken to be the unit interval $[0, 1]$ with Lebesgue measure μ . "Measure" and "measurability" are to be taken in the sense of Lebesgue. A "null-set" is a set of measure 0. An assignment is an integrable function from T to Ω . There is a fixed initial assignment I ; it is assumed to satisfy the following assumption:

- (2.1) (i) For no i does $I^i(t)$ vanish almost everywhere (a.e.).
(ii) $I(t) = 0$ for almost no t .

This assumption excludes the degenerate cases in which one or more commodities are totally absent from the market, and those in which a significant set of traders comes to market with nothing to trade.³ (2.1) can be circumvented in the proof of our theorem; this is done by restricting consideration to those commodities for which (i) holds, and those traders for whom (ii) holds.

³If we use the "intrinsic" formulation of Section I.7, then (ii) holds implicitly and need not be explicitly assumed.

However, doing this correctly occupies valuable time and space, for no worthwhile purpose; so we prefer to make this harmless assumption explicitly. Note specifically that we do not make assumption I.2.1, which says that $I(t) > 0$.

For each trader t there is defined a relation \succ_t on Ω , called a preference relation, which is assumed to satisfy the following conditions:

(2.2) Continuity. For each $y \in \Omega$, the sets $\{x : x \succ_t y\}$ and $\{x : y \succ_t x\}$ are open (relative to Ω).

(2.3) Measurability. For each pair of assignments X, Y , the set $\{t : X(t) \succ_t Y(t)\}$ is measurable.

Note that nothing in the spirit of desirability of commodities (I.2.4) is assumed at this stage. Assumptions (2.2) and (2.3) are identical with (I.2.2) and (I.2.3).

An allocation is an assignment X such that

$$\int_{\mathbb{T}} X(t) dt = \int_{\mathbb{T}} I(t) dt .$$

A coalition is a measurable set of traders. An allocation Y is said to dominate an allocation X via a coalition S , if $\mu(S) > 0$, $Y(t) \succ_t X(t)$ for all $t \in S$, and S is effective for Y , i.e.

$$\int_S Y(t) dt = \int_S I(t) dt .$$

The core is the set of undominated allocations.

A price vector is a member of Ω , not the origin. A competitive equilibrium is a pair (p, X) , where p is a price vector and X an allocation, such that for a.e. t , $X(t)$ is maximal w.r.t. \succ_t in $\{x : p \cdot x \leq p \cdot I(t)\}$. If (p, X) is a competitive equilibrium, then X is called an equilibrium allocation.

3. λ -Desirability

Let λ be a real number > 1 . For $x \in \mathbb{R}^n$ set $K(x) = \{i : x^i \geq 0\}$ and $L(x) = \{i : x^i < 0\}$. Write $x \succ_{\lambda} 0$ ("x is λ -greater than 0") if

$$\sum_{K(x)} x^i + \lambda \sum_{L(x)} x^i > 0,$$

and $x \succ_{\lambda} y$ if $x - y \succ_{\lambda} 0$. Figure 1 illustrates the relation \succ_{λ} in two dimensions, contrasting it with the relation $>$.

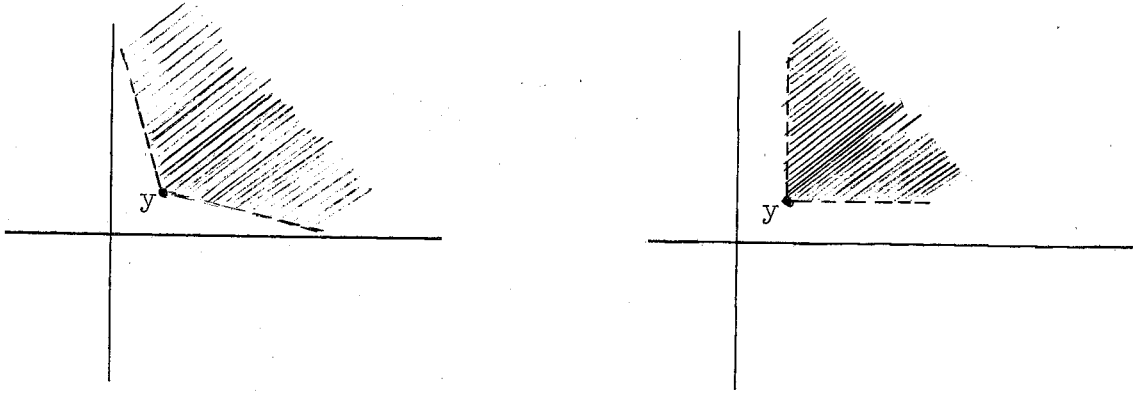


Fig. 1. The shaded areas are the sets of points that are $\succ_{\lambda} y$ (on the left), and $> y$ (on the right). The slopes of the lines on the left are $-\lambda$ and $-1/\lambda$. The dotted lines indicate that the boundary is not included.

For fixed trader t and $y \in \Omega$, we shall say that commodities are λ -desirable for t at y , or that \succ_t obeys the λ -desirability assumption at y , if

$$(3.1) \quad \text{For all } x \in \Omega, \quad x \succ_{\lambda} y \text{ implies } x \succ_t y.$$

Intuitively, λ -desirability at y involves two separate concepts: First, that the trader's desire for goods is not saturated; second, that he considers no commodity more than λ times as valuable as any other commodity, i.e., that he will always be willing to trade a given commodity bundle for any other bundle which is more than λ times as large in toto⁴ (for example, weighs

⁴Of course, this depends on the units being used for the various commodities. But it is the existence of a λ for which λ -desirability holds that is important, and this is independent of the units.

more than λ times as much). We call the second intuitive assertion "comparability." Both non-saturation and comparability are asserted only when the trader currently holds y . When λ is large, comparability seems a quite acceptable assumption, and indeed we believe that it holds in many economic situations; rarely does one good become infinitely more valuable than another. But it does have its limitations; when the usefulness of a commodity is intimately tied up with that of another, the assumption may fail. Thus in a market in which the commodities are salami and mustard, the desire for mustard given a fixed quantity of salami will quickly become saturated; even for small current holdings, our trader may be unwilling to trade as much as a single slice of salami for untold tons of mustard. In spite of this limitation, we believe that λ -desirability is of sufficiently wide applicability to make the theorem stated below interesting and useful.

For each trader t , let $\sigma(t) = \sum_{j=1}^n I^j(t)$, and let $M(t) = (\sigma(t), \dots, \sigma(t))$.

Theorem. Suppose that there is a λ such that for each trader t , the preference relation \succsim_t obeys λ -desirability at all $y \leq \lambda M(t)$. Then the core coincides with the set of equilibrium allocations.

The advantages of this theorem over that of MCT I (Section I.2) are that it allows saturation for sufficiently large bundles, and does not assume that initial assignments are positive. The disadvantage is that it insists on λ -desirability (which is stronger than desirability) for "sufficiently small" bundles. Though the present theorem is more complicated both to state and to prove than that of MCT I, we believe that it is at least as interesting from the economic viewpoint.

The particular form of the function $M(t)$ is of no economic significance. It is the existence of such a function satisfying our theorem that is important.

4. Lemmas from MCT I.

A coalition is called full if its complement is null. For a fixed assignment X , define

$$\psi(t) = \{x - I(t) : x \succ_t X(t)\} .$$

If U is a coalition, define $\Delta(U)$ to be the convex hull of $\bigcup_{t \in U} \psi(t)$.

Lemma 4.1 There is a full coalition U such that $0 \notin \Delta(U)$.

Proof: This is Lemma I.4.1. As was remarked (I.4.2), the proof there given depended neither on I.2.1 nor on I.2.4. Therefore the lemma applies here.

Lemma 4.2 Every equilibrium allocation is in the core.

Proof: This is the contents of Section I.3. There, too, it was remarked (I.3.1) that the proof depended on none of the assumptions (I.2.1) - (I.2.4). Hence it applies here as well.

5. Proof of the Theorem

The proof appears long, but the principle is simple. The key tool is Lemma 5.5. If he wishes, the reader may get some perspective on the proof as follows: First read the statement of Lemma 5.5; then, taking its proof on faith, skip straight to the proof of the theorem, at the end of the section; finally, return to the beginning of the section and read systematically.

Write $N = \{1, \dots, n\}$. For a given partition $\{K, L\}$ of N , we will denote by λ^K the vector whose i^{th} component is 1 when $i \in K$, and λ when $i \in L$ (the notation is justified because L is determined by K). Thus $x \succ_{\lambda} 0$ if and only if $\lambda^{K(x)} \cdot x > 0$.

Lemma 5.1 A necessary and sufficient condition that $x \succ_{\lambda} 0$ is that for all partitions of N into disjoint sets K and L , $\lambda^K \cdot x > 0$.

Proof: Necessity: let $x \succ_{\lambda} 0$, $\{K, L\}$ an arbitrary partition. If $i \in K(x)$ then $x^i \geq 0$, so $\lambda x^i \geq x^i$; if $i \in L(x)$ then $x^i < 0$, so $x^i > \lambda x^i$. Hence

$$\begin{aligned} & \lambda^K \cdot x \\ &= \lambda \sum_{L \cap L(x)} x^i + \lambda \sum_{L \cap K(x)} x^i + \sum_{K \cap L(x)} x^i + \sum_{K \cap K(x)} x^i \\ &\geq \lambda \sum_{L \cap L(x)} x^i + \lambda \sum_{K \cap L(x)} x^i + \sum_{L \cap K(x)} x^i + \sum_{K \cap K(x)} x^i \\ &= \lambda^{K(x)} \cdot x \\ &> 0. \end{aligned}$$

Sufficiency: If $\lambda^K \cdot x > 0$ for all partitions, then in particular it holds when $K = K(x)$ and $L = L(x)$; hence $x \succ_{\lambda} 0$.

Lemma 5.2 If $y \in \Omega$ and not $y \leq \lambda M(t)$, then $y \succ_{\lambda} I(t)$.

Proof: There is at least one j such that $y^j > \lambda M^j(t)$. Setting $K = K(y - I(t))$, $L = L(y - I(t))$, we have

$$\lambda^K \cdot y \geq y^j > \lambda \sum_{i=1}^n I^j(t) \geq \lambda^K \cdot I(t);$$

hence $y \succ_{\lambda} I(t)$, as was to be proved.

If for all partitions $\{K, L\}$ of N , we have $\lambda^K \cdot x \geq 0$, then we will write $x \succeq_{\lambda} 0$; if $x - y \succeq_{\lambda} 0$, then we will write $x \succeq_{\lambda} y$.

Lemma 5.3 Let Y and X be allocations, S a coalition such that S is effective for Y . Assume that $Y(t) \succeq_{\lambda} X(t)$ for all $t \in S$, that $Y(t) \succ_{\lambda} X(t)$ for all t in a non-null subset U of S , and that for all $t \in S$, \succ_t obeys λ -desirability at $X(t)$. Then X is not in the core.

Proof: If $\mu(U) = \mu(S)$, then Y dominates X via U , and the conclusion of the lemma follows; assume therefore that $\mu(S - U) > 0$. We will prove the lemma by constructing an allocation Z that dominates X via S .

Since $X(t) \geq 0$, it follows that not $0 >_{\lambda} X(t)$. Therefore, for all $t \in U$, not $Y(t) = 0$, i.e., there is a j and a non-null subset V of U such that $Y^j(t) > 0$ for all $t \in V$. Let K range over all subsets of N , and set

$$P(t) = \frac{1}{\lambda} \min_K (\lambda^K \cdot (Y(t) - X(t))) .$$

$P(t)$ is measurable, and because $V \subset U$ it is positive for all $t \in V$.

Hence if $Q(t) = \min (P(t), Y^j(t))$, then $Q(t) > 0$ for all $t \in V$. Hence

$$V = \bigcup_{k=1}^{\infty} \{t \in V : Q(t) > \frac{1}{k}\} .$$

Since V is non-null, at least one of the terms in this union is non-null, say the k_0 th; denote it by W , and set $\delta = 1/k_0$. Then

$$(5.4) \quad \left\{ \begin{array}{l} W \text{ is non-null} \\ Y^j(t) > \delta > 0 \text{ for } t \in W \\ P(t) > \delta > 0 \text{ for } t \in W . \end{array} \right.$$

Define Z by

$$Z^i(t) = \left\{ \begin{array}{ll} Y^j(t) - \delta; & \text{when } i = j \text{ and } t \in W \\ Y^j(t) + \delta \frac{\mu(W)}{\mu(S-U)} & \text{when } i = j \text{ and } t \in S - U \\ Y^i(t) & \text{otherwise .} \end{array} \right.$$

Then $Z^i(t) \geq Y^i(t) \geq 0$ except in the first case; and then $Z^i(t) > 0$ by (5.4). Hence $Z(t) \geq 0$ for all t , i.e., Z is an assignment. Next we show that Z is an allocation and that S is effective for Z . Since both these statements are true for Y , we need only show that

$$\int_S (Z^i(t) - Y^i(t)) dt = 0$$

for all i . This is trivial unless $i = j$; and then, too, we have

$$\begin{aligned} \int_S (Z^j(t) - Y^j(t)) &= \int_W (-\delta) dt + \int_{S-U} (\delta \mu(W)/\mu(S-U)) dt \\ &= -\delta \mu(W) + \delta \mu(W) = 0 . \end{aligned}$$

It remains only to show that $Z(t) \underset{\lambda}{>} X(t)$ for all $t \in S$. For this it is sufficient, because of λ -desirability, to show that $Z(t) \underset{\lambda}{>} X(t)$ for all $t \in S$. When $t \in S-U$ we have for all K that

$$\begin{aligned} \lambda^K \cdot (Z(t) - Y(t)) &= \begin{cases} \delta\mu(W)/\mu(S-U) & \text{if } j \in K \\ \lambda\delta\mu(W)/\mu(S-U) & \text{if } j \in L \end{cases} \\ &> 0 ; \end{aligned}$$

and from $Y(t) \underset{\lambda}{\geq} X(t)$ it follows that $\lambda^K \cdot (Y(t) - X(t)) \geq 0$. Hence $\lambda^K \cdot (Z(t) - X(t)) > 0$, and thus $Z(t) \underset{\lambda}{>} X(t)$ when $t \in S-U$. When $t \in W$, then because of (5.4) we have

$$\begin{aligned} \lambda^K \cdot (Z(t) - X(t)) &\geq \lambda^K \cdot (Y(t) - X(t)) - \lambda\delta \geq \lambda P(t) - \lambda\delta \\ &> \lambda\delta - \lambda\delta = 0 . \end{aligned}$$

Therefore $Z(t) \underset{\lambda}{>} X(t)$ again. Otherwise $Z(t) = Y(t)$, and $t \in U$, so $Z(t) \underset{\lambda}{>} X(t)$ again. This completes the proof of the lemma.

Lemma 5.5 Under the hypotheses of the Theorem, if X is in the core, then $X(t) \leq \lambda M(t)$ for almost every t .

Proof: Let $S = \{t : X(t) \leq \lambda M(t)\}$; assume the lemma is false, i.e., $\mu(S) < 1$, i.e., $\mu(T-S) > 0$. From Lemma 5.2 it follows that $X(t) - I(t) \underset{\lambda}{>} 0$ for $t \in T-S$. Let us write $z = \int_{T-S} (X(t) - I(t)) dt$; then from $\mu(T-S) > 0$ and Lemma 5.1 it follows that $z \underset{\lambda}{>} 0$. Set $K = K(z)$ and $L = L(z)$.

Now $\mu(S) > 0$; otherwise, since X is an allocation, $z = 0$, and hence $0 \underset{\lambda}{>} 0$, an absurdity. We distinguish two cases, according as L is non-empty or empty.

Suppose first that L is non-empty. Then because X is an allocation,

$$\begin{aligned}
 & \int_S X(t) dt + z \\
 = & \int_S (X(t) - I(t)) dt + \int_S I(t) dt + \int_{T-S} (X(t) - I(t)) dt \\
 = & \int_T (X(t) - I(t)) dt + \int_S I(t) dt \\
 = & \int_S I(t) dt \geq 0 .
 \end{aligned}$$

Hence

$$(5.6) \quad \int_S X^i(t) dt \geq -z^i > 0 \text{ for } i \in L .$$

Now define

$$\begin{aligned}
 Y(t) &= I(t) \quad \text{for } t \in T-S \\
 Y^i(t) &= \begin{cases} X^i(t) \left[1 + \frac{z^i}{\int_S X^i(t) dt} \right] & \text{for } t \in S, i \in L \\
 X^i(t) + z^i \left[\frac{\sum_L (X^j(t) z^j / \int_S X^j(t) dt)}{\sum_L z^j} \right] & \text{for } t \in S, i \in K ; \end{cases}
 \end{aligned}$$

the definition is legitimate because of (5.6). We wish to arrive at a contradiction by showing, via Lemma 5.3, that X is not in the core. For this we must show that Y is an allocation, i.e., that

- (i) $Y(t) \in \Omega$ for all t , and
- (ii) $\int_T Y(t) dt = \int_T I(t) dt$;

also that S is effective for Y , i.e., that

- (iii) $\int_S Y(t) dt = \int_S I(t) dt$;

also that

- (iv) $Y(t) \succeq_\lambda X(t)$ for all $t \in S$;

and finally that

$$(v) \quad Y(t) >_{\lambda} X(t) \quad \text{for } t \text{ in a non-null subset } U \text{ of } S .$$

From (5.6) we obtain

$$1 + \frac{z^i}{\int_S X^i(t) dt} > 0$$

for $i \in L$, and hence $Y^i(t) \geq 0$ for $t \in S$ and $i \in L$; for other t and i this is trivial, and (i) is established. To demonstrate (ii), we note

$$\begin{aligned} \int_{\mathbb{T}} Y(t) dt &= \int_{\mathbb{T}-S} I(t) dt + \int_S [Y(t) - X(t)] dt + \int_S X(t) dt \\ &= \int_{\mathbb{T}-S} [I(t) - X(t)] dt + \int_S [Y(t) - X(t)] dt + \int_{\mathbb{T}} X(t) dt \\ &= -z + \int_S [Y(t) - X(t)] dt + \int_{\mathbb{T}} I(t) dt . \end{aligned}$$

For $i \in L$ we have

$$\int_S [Y^i(t) - X^i(t)] dt = \int_S z^i X^i(t) dt / \int_S X^i(t) dt = z^i ;$$

hence

$$(5.7) \quad \int_{\mathbb{T}} Y^i(t) dt = \int_{\mathbb{T}} I^i(t) dt .$$

For $i \in K$ we have

$$\begin{aligned} &\int_S [Y^i(t) - X^i(t)] dt \\ &= z^i \left[\frac{\sum_L (z^j \int_S X^j(t) dt / \int_S X^j(t) dt)}{\sum_L z^j} \right] \\ &= z^i , \end{aligned}$$

and again we have (5.7); this proves (ii). (iii) follows at once from (ii)

if we note that

$$\int_{\mathbb{T}-S} Y(t) dt = \int_{\mathbb{T}-S} I(t) .$$

Finally, we demonstrate (iv) and (v). For $t \in S$, the sign of $Y^i(t) - X^i(t)$ coincides with that of z^i , or at the worst the former is zero whereas the latter is not. Hence $Y(t) \geq_{\lambda} X(t)$ if and only if $\lambda^K \cdot [Y(t) - X(t)] \geq 0$, and similarly $Y(t) >_{\lambda} X(t)$ if and only if $\lambda^K \cdot [Y(t) - X(t)] > 0$. Now

$$\begin{aligned} & \lambda^K \cdot [Y(t) - X(t)] \\ &= \frac{\sum_{i \in K} z^i}{\sum_{j \in L} z^j} \sum_{j \in L} [X^j(t) z^j / \int_S X^j(t) dt] + \lambda \sum_{j \in L} (X^j(t) z^j / \int_S X^j(t) dt) \\ &= \left(\frac{\sum_{j \in L} [X^j(t) z^j / \int_S X^j(t) dt]}{\sum_{j \in L} z^j} \right) (\sum_{i \in K} z^i + \lambda \sum_{j \in L} z^j) . \end{aligned}$$

The left factor has non-positive numerator and negative denominator; hence it is non-negative. The right factor is positive, and so the whole expression is non-negative. This demonstrates (iv). To show (v), pick an arbitrary $j \in L$. From (5.6) it follows that $X^j(t)$ does not vanish for almost all $t \in S$, i.e., there is a non-null subset U of S such that $X^j(t) > 0$ when $t \in U$. Then for $t \in U$, the numerator of the left factor is negative and so the whole expression is positive, i.e., $Y(t) >_{\lambda} X(t)$. This completes the proof when L is non-empty.

If L is empty, then $z^i \geq 0$ for all i . Define

$$Y(t) = \begin{cases} X(t) + \frac{z}{\mu(S)} & \text{for } t \in S \\ I(t) & \text{for } t \in T-S \end{cases}$$

We easily verify that $Y(t) \geq 0$, that it is an allocation, and that S is effective for Y . Also, for $t \in S$ we obtain from $z >_{\lambda} 0$ that

$$Y(t) - X(t) = z/\mu(S) >_{\lambda} 0 ;$$

hence $Y(t) >_{\lambda} X(t)$. Since $X(t) \leq_{\lambda} M(t)$ for $t \in S$, λ -desirability holds at $X(t)$ for $>_t$, and therefore X is dominated by Y via S . Hence X

is not in the core, a contradiction. The proof of Lemma 5.5 is complete.

Completion of the proof of the Theorem. Let X be in the core. Applying the supporting hyperplane theorem to Lemma 4.1, we obtain a full coalition U and a non-zero vector p such that $\Delta(U) \subset \{x : p \cdot x \geq 0\}$. By Lemma 5.5, λ -desirability holds a.e. for \succ_t at $X(t)$, and therefore almost all the $\psi(t)$ contain a translate of the non-negative orthant. Therefore also $\Delta(U)$ contains a translate of the non-negative orthant, and hence $p \geq 0$.

Since λ -desirability holds almost everywhere for \succ_t at $X(t)$, it follows that $X(t) - I(t)$ is a limit point of $\psi(t)$ a.e. Hence, $p \cdot X(t) \geq p \cdot I(t)$ a.e. If for a non-null set of t we would have $p \cdot X(t) > p \cdot I(t)$, then it would follow that

$$\int_{\mathbb{T}} p \cdot X(t) dt > \int_{\mathbb{T}} p \cdot I(t) dt ,$$

which is impossible because X is an allocation. Hence

$$(5.8) \quad p \cdot X(t) = p \cdot I(t) \quad \text{a.e.}$$

We next show that $p > 0$. Suppose, on the contrary, that $p^j = 0$ for some j . Choose i so that $p^i > 0$; this is possible because p is not zero. From (2.1) (i) we obtain that

$$\int_{\mathbb{T}} I^i(t) dt > 0 ,$$

and hence, because X is an allocation,

$$\int_{\mathbb{T}} X^i(t) dt > 0 .$$

Therefore $X^i(t) > 0$ for t in a non-null subset of U . Hence there is a $t_0 \in U$ such that $X^i(t_0) > 0$, $p \cdot X(t_0) = p \cdot I(t_0)$ (because of (5.8)), and λ -desirability holds for \succ_{t_0} at $X(t_0)$ (because of Lemma 5.5). Now let z be a vector whose i^{th} coordinate is -1 , whose j^{th} coordinate is

$\lambda + 1$, and whose other coordinates are 0. Because $X^i(t_0) > 0$, there is a sufficiently small $\delta > 0$ such that $X(t_0) + \delta z \in \Omega$. Because of λ -desirability at $X(t_0)$ it follows that $X(t_0) - I(t_0) + \delta z \in \psi(t_0)$. But $p \cdot (X(t_0) - I(t_0) + \delta z) = p \cdot (X(t_0) - I(t_0)) - \delta p^i = -\delta p^i < 0$, contradicting $t_0 \in U$. This shows that $p > 0$.

The price vector p was chosen so that $p \cdot y \geq p \cdot I(t)$ whenever $y \succ_t X(t)$ (a.e.); we now show that

$$(5.9) \quad p \cdot y > p \cdot I(t) \text{ whenever } y \succ_t X(t) \text{ (a.e.)}.$$

Because of (2.1) (ii), we may assume that not $I(t) = 0$. Hence, since $p \cdot y \geq p \cdot I(t)$ and $p > 0$, it follows that not $y = 0$, say $y^1 > 0$. Then by continuity (2.2), for sufficiently small δ we still have $y - (\delta, 0, \dots, 0) \succ_t X(t)$. Hence $p \cdot I(t) \leq p \cdot (y - (\delta, 0, \dots, 0)) = p \cdot y - p^1 \delta < p \cdot y$, and this establishes (5.9).

(5.8) and (5.9) together assert that (p, X) is a competitive equilibrium, and hence X is an equilibrium allocation. Thus the core is included in the set of equilibrium allocations. The converse is Lemma 4.2. The proof of our theorem is complete.

6. An Example.

We wish to give an example of a market satisfying (I.2.1) through (I.2.4), in which there are members of the core — or equivalently equilibrium allocations — that involve arbitrarily large vectors. More precisely, we will show that for each non-null coalition S and for each vector x_0 , there is a non-null subset U of S and a member X of the core of our example, such that for all $t \in U$, not $X(t) \leq x_0$.

Let $I(t) = (1, \dots, 1)$ identically for all t . Define $x \succ_t y$ if and only if $x > y$. Then for a given price vector p , which we will normalize so that $\sum p^i = 1$, y is maximal in $\{x : p \cdot x \leq p \cdot I(t)\}$ w.r.t.

\succ_t if and only if $p \cdot y = p \cdot I(t) = 1$. Hence any allocation all of whose values lie on the hyperplane $p \cdot x = 1$ forms a competitive equilibrium with p and is therefore an equilibrium allocation. Obviously, as p varies the set of all such allocations contains vectors as large as we please (though $X(t) > I(t)$ is impossible, as indeed it obviously is for all equilibrium allocations). For example, if $n = 2$, then for fixed p define

$$X(t) = \begin{cases} (\frac{1}{p^1}, 0) & \text{if } t < p^1 \\ (0, \frac{1}{p^2}) & \text{if } t \geq p^1 \end{cases}$$

Then (p, X) is a competitive equilibrium, and as p varies, X is unbounded.

7. Continuity and Measurability

The continuity assumption (2.2) (and (I.2.2)) is stronger than it need be; only half of it, the half that says that $\{x : x \succ_t y\}$ is open, has been used — both in MCT I and here. We stated the stronger version because it is the traditional one and it did not seem worthwhile, especially in the less technical context of MCT I, to make an issue of the matter.

The two versions are not equivalent. In Figure 2 we show indifference curves for a transitive and complete preference relation that satisfies the condition that $\{x : x \succ_t y\}$ be open, but not that $\{x : y \succ_t x\}$

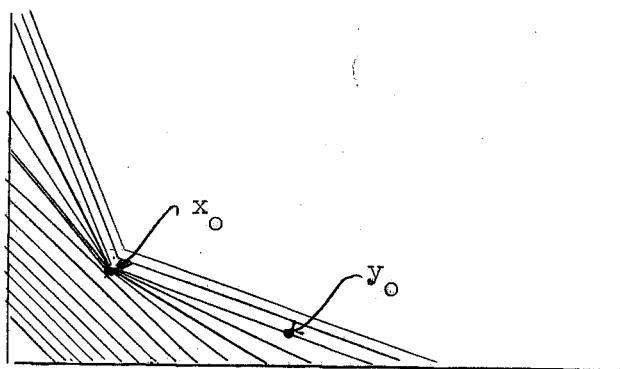


Figure 2

be open. The "singular" point x_0 is to be taken as indifferent with the points on the lowest line on which it lies (the straight line), and then the second condition is violated when y is any other point on one of the other curves containing x_0 (say $y = y_0$).

Some readers may have a vague feeling of uneasiness in connection with the measurability assumption ((2.3) and (I.2.3)). This can probably be traced to the fact that it is not clear how one would verify, in a particular context, whether or not the assumption holds. One feels that a statement concerning assignments should be a theorem and not an assumption, as an assignment is a fairly complicated mathematical object; an assumption should deal with more "basic" objects. Suppose, for example, that the preference relation is represented by a real utility function $u(t, x)$, i.e., that $x \succ_t y$ if and only if $u(t, x) > u(t, y)$. Then a sufficient condition for assumption (2.2) is that u be measurable in both variables t and x (simultaneously). This assumption of measurability of u is the kind of thing we are seeking. Can it be formulated in our context, when representation by a utility function in general is impossible?

The answer is yes; such a formulation is possible in terms of the product space $\Omega \times T \times \Omega$. On this space we impose the product structure, rather than the Lebesgue structure; that is, we define a subset A of $\Omega \times T \times \Omega$ to be measurable if it is in the minimal σ -field generated by all the "measurable rectangles," i.e., the sets of the form $\Xi \times S \times \Theta$, where Ξ , S , Θ are (Lebesgue) measurable in Ω , T , and Ω respectively. We now formulate our condition as follows:

(7.1) The set $\{(x, t, y) : x \succ_t y\}$ is measurable in the product structure on $\Omega \times T \times \Omega$.

We wish to demonstrate that (7.1) implies (2.3). Let X and Y be assignments. Define a mapping $f : T \rightarrow \Omega \times T \times \Omega$ by $f(t) = (X(t), t, Y(t))$.

If $A = \Xi \times S \times \Theta$, then $f^{-1}(A) = X^{-1}(\Xi) \cap S \cap Y^{-1}(\Theta)$; by the measurability of X and Y , it follows that $f^{-1}(A)$ is measurable in T . Now f^{-1} preserves unions and complements, and therefore the set of all $A \subset \Omega \times T \times \Omega$ such that $f^{-1}(A)$ is measurable in T is a σ -field. We have seen that this σ -field contains all measurable rectangles, and therefore it contains the product structure. In particular, applying (7.1), we conclude that $f^{-1}\{(x,t,y) : x \succ_t y\}$ is measurable in T ; but this is exactly the set $\{t : X(t) \succ_t Y(t)\}$ that appears in (2.3). The demonstration is complete.